Shortlisted Problems

 $25^{\mathrm{th}} \mathrm{~ELMO}$

Pittsburgh, PA, 2023

Note of Confidentiality

The shortlisted problems should be kept strictly confidential until disclosed publicly by the committee on the ELMO.

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Problems

A1. Find all polynomials P(x) with real coefficients such that for all nonzero real numbers x,

$$P(x) + P\left(\frac{1}{x}\right) = \frac{P\left(x + \frac{1}{x}\right) + P\left(x - \frac{1}{x}\right)}{2}.$$
(Holden Mui)

A2. Let $\mathbb{R}_{>0}$ denote the set of positive real numbers. Find all functions $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that for all positive real numbers x and y,

$$f(xy+1) = f(x)f\left(\frac{1}{x} + f\left(\frac{1}{y}\right)\right).$$

(Luke Robitaille)

A3. Does there exist an infinite sequence of integers a_0, a_1, a_2, \ldots such that $a_0 \neq 0$ and, for any integer $n \geq 0$, the polynomial

$$P_n(x) = \sum_{k=0}^n a_k x^k$$

has n distinct real roots?

(Amol Rama, Espen Slettnes)

A4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that for all real numbers $x \neq 1$,

$$f(x - f(x)) + f(x) = \frac{x^2 - x + 1}{x - 1}.$$

Find all possible values of f(2023).

(Linus Tang)

A5. Find the minimum positive integer M for which there exist an integer n and n threevariable polynomials $P_1(x, y, z)$, $P_2(x, y, z)$, ..., $P_n(x, y, z)$ with integer coefficients satisfying

$$Mxyz = P_1(x, y, z)^3 + P_2(x, y, z)^3 + \dots + P_n(x, y, z)^3.$$

(Karthik Vedula)

A6. Let $\mathbb{R}_{>0}$ denote the set of positive real numbers and $\mathbb{R}_{\geq 0}$ the set of nonnegative real numbers. Find all functions $f : \mathbb{R} \times \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ such that for all real numbers a, b, x, y with x, y > 0, we have

$$f(a, x) + f(b, y) = f(a + b, x + y) + f(ay - bx, xy(x + y)).$$

(Luke Robitaille)

C1. Elmo has 2023 cookie jars, all initially empty. Every day, he chooses two distinct jars and places a cookie in each. Every night, Cookie Monster finds a jar with the most cookies and eats all of them. If this process continues indefinitely, what is the maximum possible number of cookies that the Cookie Monster could eat in one night?

(Espen Slettnes)

C2. Alice is performing a magic trick. She has a standard deck of 52 cards, which she may order beforehand. She invites a volunteer to pick an integer $0 \le n \le 52$, and cuts the deck into a pile with the top n cards and a pile with the remaining 52 - n. She then gives both piles to the volunteer, who riffles them together and hands the deck back to her face down. (Thus, in the resulting deck, the cards that were in the deck of size n appear in order, as do the cards that were in the deck of size 52 - n.)

Alice then flips the cards over one-by-one from the top. Before flipping over each card, she may choose to guess the color of the card she is about to flip over. She stops if she guesses incorrectly. What is the maximum number of correct guesses she can guarantee?

(Espen Slettnes)

C3. Find all pairs of positive integers (a, b) with the following property: there exists an integer N such that for any integers $m \ge N$ and $n \ge N$, every $m \times n$ grid of unit squares may be partitioned into $a \times b$ rectangles and fewer than ab unit squares.

(Holden Mui)

C4. Let *n* be a positive integer and consider an $n \times n$ square grid. For $1 \le k \le n$, a *python* of length *k* is a snake that occupies *k* consecutive cells in a single row, and no other cells. Similarly, an *anaconda* of length *k* is a snake that occupies *k* consecutive cells in a single column, and no other cells.

The grid contains at least one python or anaconda, and it satisfies the following properties:

- No cell is occupied by multiple snakes.
- If a cell in the grid is immediately to the left or immediately to the right of a python, then that cell must be occupied by an anaconda.
- If a cell in the grid is immediately to above or immediately below an anaconda, then that cell must be occupied by a python.

Prove that the sum of the squares of the lengths of the snakes is at least n^2 .

(Linus Tang)

C5. Define the *mexth* of k sets as the kth smallest positive integer that none of them contain, if it exists. Does there exist a family \mathcal{F} of sets of positive integers such that

- for any nonempty finite subset \mathcal{G} of \mathcal{F} , the mexth of \mathcal{G} exists, and
- for any positive integer n, there is exactly one nonempty finite subset \mathcal{G} of \mathcal{F} such that n is the mexth of \mathcal{G} .

(Espen Slettnes)

C6. For a set S of positive integers and positive integer n, consider the game of (n, S)-nim, which is as follows. A pile starts with n stones. Two players, Anthony and Brandon, alternate turns removing stones from the pile, with Anthony going first. On any turn, the number of stones removed must be an element of S. The last player to move wins. Let f(S) denote the set of positive integers n for which Anthony has a winning strategy in (n, S)-nim.

Let T be a set of positive integers. Must the sequence

$$T, f(T), f(f(T)), \ldots$$

be eventually constant?

(Brandon Wang, Edward Wan)

C7. A discrete hexagon with center (a, b, c) (where a, b, c are integers) and radius r (a nonnegative integer) is the set of lattice points (x, y, z) such that x + y + z = a + b + c and $\max(|x-a|, |y-b|, |z-c|) \leq r$.

Let n be a nonnegative integer and S be the set of triples (x, y, z) of nonnegative integers such that x + y + z = n. If S is partitioned into discrete hexagons, show that at least n + 1 hexagons are needed.

(Linus Tang)

C8. Let $n \ge 3$ be a fixed integer, and let α be a fixed positive real number. There are n numbers written around a circle such that there is exactly one 1 and the rest are 0's. An *operation* consists of picking a number a in the circle, subtracting some positive real $x \le a$ from it, and adding αx to each of its neighbors.

Find all pairs (n, α) such that all the numbers in the circle can be made equal after a finite number of operations.

(Anthony Wang)

G1. Let *ABCDE* be a regular pentagon. Let *P* be a variable point on the interior of segment *AB* such that $PA \neq PB$. The circumcircles of $\triangle PAE$ and $\triangle PBC$ meet again at *Q*. Let *R* be the circumcenter of $\triangle DPQ$. Show that as *P* varies, *R* lies on a fixed line.

(Karthik Vedula)

G2. Let ABC be an acute scalene triangle with orthocenter H. Line BH intersects AC at E and line CH intersects \overline{AB} at F. Let X be the foot of the perpendicular from H to the line through A parallel to \overline{EF} . Point B_1 lies on line XF such that $\overline{BB_1}$ is parallel to \overline{AC} , and point C_1 lies on line XE such that $\overline{CC_1}$ is parallel to \overline{AB} . Prove that points B, C, B_1, C_1 are concyclic.

(Luke Robitaille)

G3. Two triangles intersect to form seven finite disjoint regions, six of which are triangles with area 1. The last region is a hexagon with area A. Compute the minimum possible value of A.

(Karthik Vedula)

G4. Let *D* be a point on segment *PQ*. Let ω be a fixed circle passing through *D*, and let *A* be a variable point on ω . Let *X* be the intersection of the tangent to the circumcircle of $\triangle ADP$ at *P* and the tangent to the circumcircle of $\triangle ADQ$ at *Q*. Show that as *A* varies, *X* lies on a fixed line.

(Elliott Liu, Anthony Wang)

G5. Let *ABC* be an acute triangle with circumcircle ω . Let *P* be a variable point on the arc *BC* of ω not containing *A*. Squares *BPDE* and *PCFG* are constructed such that *A*, *D*, *E* lie on the same side of line *BP* and *A*, *F*, *G* lie on the same side of line *CP*. Let *H* be the intersection of lines *DE* and *FG*. Show that as *P* varies, *H* lies on a fixed circle.

(Karthik Vedula)

G6. Let *ABCDEF* be a convex cyclic hexagon such that quadrilateral *ABDF* is a square, and the incenter of $\triangle ACE$ lines on \overline{BF} . Diagonal *CE* intersects diagonals *BD* and *DF* at points *P* and *Q*, respectively. Prove that the circumcircle of $\triangle DPQ$ is tangent to \overline{BF} .

(Elliott Liu)

G7. Let \mathcal{E} be an ellipse with foci F_1 and F_2 , and let P be a point on \mathcal{E} . Suppose lines PF_1 and PF_2 intersect \mathcal{E} again at distinct points A and B, and the tangents to \mathcal{E} at A and B intersect at point Q. Show that the midpoint of \overline{PQ} lies on the circumcircle of $\triangle PF_1F_2$.

(Karthik Vedula)

G8. Similar quadrilaterals $ABCD \sim A_1B_1C_1D_1 \sim A_2B_2C_2D_2$ lie in the plane such that points A, A_1 , B_2 , B are collinear, points B, B_1 , C_2 , C are collinear, points C, C_1 , D_2 , D are collinear, and points D, D_1 , A_2 , A are collinear. Prove that the intersections $\overline{AC} \cap \overline{BD}$, $\overline{A_1C_1} \cap \overline{B_1D_1}$, and $\overline{A_2C_2} \cap \overline{B_2D_2}$ are collinear.



(Holden Mui)

N1. Let *m* be a positive integer. Find all polynomials P(x) with integer coefficients such that for every integer *n*, there exists an integer *k* such that $P(k) = n^m$.

(Raymond Feng)

N2. Determine the greatest positive integer n for which there exists a sequence of distinct positive integers s_1, s_2, \ldots, s_n satisfying

$$s_1^{s_2} = s_2^{s_3} = \dots = s_{n-1}^{s_n}$$
.
(Holden Mui)

N3. Let a and b be positive integers and let $k \leq b$ be a nonnegative integer. A lemonade stand owns $n \geq k$ cups, of which k are initially full and n - k are initially empty. The lemonade stand also has a filling machine and an emptying machine, which operate according to the following rules:

- If at any moment, *a* completely empty cups are available, the filling machine spends the next *a* minutes filling those *a* cups simultaneously and doing nothing else.
- If at any moment, b completely full cups are available, the emptying machine spends the next b minutes emptying those b cups simultaneously and doing nothing else.

Suppose that after a sufficiently long time has passed, both the filling machine and emptying machine work without pausing. In terms of a, b, and k, what is the least possible value of n?

(Raymond Feng)

N4. Let d(n) denote the number of positive divisors of n. The sequence a_0, a_1, a_2, \ldots is defined as follows: $a_0 = 1$, and for all integers $n \ge 1$,

$$a_n = d(a_{n-1}) + d(d(a_{n-2})) + \dots + \underbrace{d(d(\dots d(a_0)\dots))}_{n \text{ times}}.$$

Show that for all integers $n \ge 1$, we have $a_n \le 3n$.

(Karthik Vedula)

N5. An ordered pair (k, n) of positive integers is *good* if there exists an ordered quadruple (a, b, c, d) of positive integers such that $a^3 + b^k = c^3 + d^k$ and abcd = n. Prove that there exist infinitely many positive integers n such that (2022, n) is not good but (2023, n) is good.

(Luke Robitaille)

Solutions

A1. Find all polynomials P(x) with real coefficients such that for all nonzero real numbers x,

$$P(x) + P\left(\frac{1}{x}\right) = \frac{P\left(x + \frac{1}{x}\right) + P\left(x - \frac{1}{x}\right)}{2}.$$

(Holden Mui)

We begin by noticing that the substitution $x \mapsto \frac{1}{x}$ gives

$$\frac{P(x+\frac{1}{x})+P(x-\frac{1}{x})}{2} = P(x)+P\left(\frac{1}{x}\right) = \frac{P(x+\frac{1}{x})+P(\frac{1}{x}-x)}{2},$$

implying $P(x - \frac{1}{x}) = P(\frac{1}{x} - x)$ for all x. This readily implies P is even.

Now note that for even $n \ge 6$,

$$x^{n} + \left(\frac{1}{x}\right)^{n} - \frac{(x + \frac{1}{x})^{n} + (x - \frac{1}{x})^{n}}{2} = \binom{n}{2}x^{n-4} + o(x^{n-4}),$$

so if deg $P \ge 6$, the difference between the two sides of the functional equation has a term of degree deg P - 4 > 0. Thus deg $P \le 4$.

It is then routine to check that $P(x) \equiv x^2$ is a solution, and that $P(x) \equiv x^4 + r$ is a solution if and only if r = 6. Hence the solution set is

$$P(x) \equiv a(x^4 + 6) + bx^2 \text{ where } a, b \in \mathbb{R}.$$

A2. Let $\mathbb{R}_{>0}$ denote the set of positive real numbers. Find all functions $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that for all positive real numbers x and y,

$$f(xy+1) = f(x)f\left(\frac{1}{x} + f\left(\frac{1}{y}\right)\right)$$

(Luke Robitaille)

The answers are $f(x) \equiv 1$ and $f(x) \equiv \frac{1}{x}$, which work. Now we show they are the only solutions.

Let P(x, y) denote the assertion.

Claim 1.
$$f(1) = 1$$
.

Proof. By taking any xy + 1 = x and $c = \frac{1}{x} + f(\frac{1}{y})$, we have f(c) = 1. By $P(1, \frac{1}{c})$ and $P(c, \frac{1}{c})$, we have

$$f(1)f(2) = f\left(1 + \frac{1}{c}\right) = f(2),$$

implying f(1) = 1.

Claim 2. If f(a) = f(b), then f(ax + 1) = f(bx + 1) for all x.

Proof. By $P(x, \frac{1}{a})$ and $P(x, \frac{1}{b})$, we have

$$f\left(\frac{x}{a}+1\right) = f(x)f\left(\frac{1}{x}+f(a)\right) = f(x)f\left(\frac{1}{x}+f(b)\right) = f\left(\frac{x}{b}+1\right).$$

Claim 3. If f(a) = f(b), then f(ax) = f(bx) for all x.

Proof. By P(ax, 1) and P(bx, 1), we have

$$f(ax) = \frac{f(ax+1)}{f(\frac{1}{ax}+1)} = \frac{f(bx+1)}{f(\frac{1}{bx}+1)} = f(bx).$$

Now if f is injective, $P(1, \frac{1}{x})$ gives $f(1 + \frac{1}{x})f = f(f(x) + 1)$, implying $f(x) = \frac{1}{x}$ for all x, so we are done. Otherwise assume f is not injective, so f(a) = f(b) for some a < b.

Let c = b/a > 1. By Claim 3, we have f(c) = 1. Then Claim 2 implies f(x+1) = f(cx+1) for all x, and Claim 3 then implies

$$f\left(\frac{cx+1}{x+1}\right) = 1$$
 for all x .

As x varies, $\frac{cx+1}{x+1}$ ranges across (1, c), so we have f(x) = 1 for all $x \in (1, c)$.

But Claim 3 simply gives f(x) = f(cx) for all x, so we readily have f(x) = 1 everywhere.

A3. Does there exist an infinite sequence of integers a_0, a_1, a_2, \ldots such that $a_0 \neq 0$ and, for any integer $n \geq 0$, the polynomial

$$P_n(x) = \sum_{k=0}^n a_k x^k$$

has n distinct real roots?

(Amol Rama, Espen Slettnes)

By scaling, assume for convenience $a_0 = 1$. The key is the following estimate:

Lemma

For a polynomial $P(x) = 1 + b_1 x + \dots + b_n x^n$ with only real roots, we have

$$|b_n| \le \left(\frac{b_1^2 - 2b_2}{n}\right)^{n/2}$$

Proof. Let the roots be r_1, \ldots, r_n . Then by AM-GM,

$$b_1^2 - 2b_2 = \sum_{i=1}^n \frac{1}{r_i^2} \ge \frac{n}{(r_1 \cdots r_n)^{2/n}} = nb_n^{2/n}.$$

Let $C = b_1^2 - 2b_2$. Then the lemma gives

$$|a_n| \le \left(\frac{C}{n}\right)^{n/2} \to 0$$

as $n \to \infty$, contradiction since $|a_n| > 1$ for all n.

A4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that for all real numbers $x \neq 1$,

$$f(x - f(x)) + f(x) = \frac{x^2 - x + 1}{x - 1}.$$

Find all possible values of f(2023).

(Linus Tang)

The answer is $\mathbb{R} \setminus \{0, 2022, 2023, 2023 + \frac{1}{2022}\}.$

We begin with the substitution g(x) = x - f(x), which transforms the functional equation into

$$g(g(x)) = \frac{1}{1-x}$$
 for all $x \neq 1$.

However note that $h(x) = \frac{1}{1-x}$ satisfies $h^3(x) = x$ for all x for $x \notin \{0, 1\}$. Therefore:

• For $c \notin \{0, 1, 2023, -\frac{1}{2022}, \frac{2022}{2023}\}$, we can let g(0) = g(1) = 1, construct the cycle

$$2023 \mapsto c \mapsto -\frac{1}{2022} \mapsto h(c) \mapsto \frac{2022}{2023} \mapsto h^2(c) \mapsto 2023,$$

and let g(x) = h(h(x)) for all other x, so this value of c works.

- For $c = \frac{2022}{2023}$, let g(0) = g(1) = 1 and let g(x) = h(h(x)) for all x, so this value of c works.
- For c = 2023, if g(2023) = c, we have $g(g(2023)) = 2023 \neq h(2023)$, so this value of c fails.
- For $c = -\frac{1}{2022}$, if g(2023) = c, we have g(c) = g(g(2023)) = h(2023) = c implying $g(g(c)) = c \neq h(c)$.
- For $c \in \{0,1\}$, if g(2023) = c, since $g^6(2023) = 2023$ we have $g^6(c) = c$. Note that $g^2(0) = h(0) = 1$ and if $g^2(1) = t$ with $t \notin \{0,1\}$, then g^2 will send $0 \mapsto 1 \mapsto t \mapsto h(t) \mapsto h(h(t)) \mapsto t \mapsto \cdots$, contradicting $g^6(c) = c$. But if t = 0 then g^2 sends $0 \mapsto 1 \mapsto 0 \mapsto \cdots$ so $g^6(c) \neq c$, and if t = 1 then $g^2(1) = 1$ hence $g^2(2023) = 2023 \neq h(2023)$. Hence this value of c fails.

A5. Find the minimum positive integer M for which there exist an integer n and n threevariable polynomials $P_1(x, y, z), P_2(x, y, z), \ldots, P_n(x, y, z)$ with integer coefficients satisfying

$$Mxyz = P_1(x, y, z)^3 + P_2(x, y, z)^3 + \dots + P_n(x, y, z)^3.$$

(Karthik Vedula)

We first reduce the problem to the one-variable case:

Lemma

Mxyz is the sum of cubes of three-variable polynomials if and only if Mx is the sum of cubes of one-variable polynomials.

Proof. If $Mxyz = \sum P_k(x, y, z)^3$, then $Mx = \sum P_k(x, 1, 1)^3$. Moreover if $Mx = \sum Q_k(x)^3$ then $Mxyz = \sum Q_k(xyz)^3$.

Now we show the answer is M = 6, achieved by

$$6x = (x+1)^3 + (x-1)^3 + (-x)^3 + (-x)^3.$$

For the lower bound, write

$$Mx = \sum_{k=1}^{n} P_k(x)^3.$$

We will show $6 \mid M$, which suffices.

Let ω be a primitive third root of unity, so $\omega^2 = -\omega - 1$. Then for each k, there are integers a_k and b_k with $P_k(\omega) = a_k \omega + b_k$. Hence

$$M\omega = \sum_{k=1}^{n} (a_k\omega + b_k)^3$$

= $\sum_{i=1}^{n} \left[a_k^3 + b_k^3 + 3a_k^2 b_k \omega + 3a_k b_k^2 (-\omega - 1) \right]$
= $\sum_{i=1}^{n} \left[a_k^3 + b_k^3 - 3a_k b_k^2 + 3a_k b_k (a_k - b_k) \omega \right]$

Since 1 and ω are linearly independent, we must have

$$M = \sum_{k=1}^{n} 3a_k b_k (a_k - b_k)$$

However for all integers a_k and b_k , we have $6 \mid 3a_kb_k(a_k - b_k)$, so $6 \mid M$.

A6. Let $\mathbb{R}_{>0}$ denote the set of positive real numbers and $\mathbb{R}_{\geq 0}$ the set of nonnegative real numbers. Find all functions $f : \mathbb{R} \times \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ such that for all real numbers a, b, x, y with x, y > 0, we have

$$f(a, x) + f(b, y) = f(a + b, x + y) + f(ay - bx, xy(x + y)).$$

(Luke Robitaille)

The following solution is provided by Justin Lee and Linus Tang.

The answer is

$$f(a,x) \equiv k \frac{a^2}{x} + \ell$$
, where $k, \ell \ge 0$.

These work, so we check they are the only solutions.

Let P(a, x, b, y) denote the assertion.

Claim 1. f(0, x) is a constant function.

To show Claim 1, let $g : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ by $g(x) \equiv f(0, x)$ and let Q(x, y) denote P(0, x, 0, y), which gives

$$g(x) + g(y) = g(x + y) + g(xy(x + y))$$

for all x, y > 0.

Claim 2. For x > 0 we have $g(x) = g(\frac{1}{x})$.

Proof of Claim 2. For x < y, we have from Q(y - x, x) that

$$g(y - x) + g(x) = g(y) + g((y - x)xy).$$

When xy = 1 this gives g(x) = g(y).

Proof of Claim 1. For all x > 0, we have that

$$\begin{split} g(x) + g(1) &= g\left(x+1\right) + g(x(x+1)) & (Q(x,1)) \\ &= g\left(\frac{1}{x+1}\right) + g\left(\frac{1}{x(x+1)}\right) & (\text{Claim 2}) \\ &= g\left(\frac{1}{x}\right) + g\left(\frac{1}{x^2(x+1)^2}\right) & (Q(\frac{1}{x+1},\frac{1}{x^2+x})) \\ &= g(x) + g(x^2(x+1)^2) & (\text{Claim 2}), \end{split}$$

hence $g(x^2(x+1)^2) = g(1)$ for all x > 0. But $x \mapsto x^2(x+1)^2$ on $\mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is surjective, so g(x) = g(1) for all x > 0.

Now we may shift f by constants, so shift so that f(0, x) = 0 for all x. Then $f : \mathbb{R} \times \mathbb{R}_{>0} \to \mathbb{R}$ is bounded below.

Claim 3. For c > 0, we have f(ca, cx) = cf(a, x) for all a and x > 0.

Proof. For any d, we have from P(dx, x, dy, y) that

$$f(dx, x) + f(dy, y) = f(d(x+y), x+y),$$

so $x \mapsto f(dx, x)$ is Cauchy and bounded below, so it is linear (with $0 \mapsto 0$). The claim readily follows.

Claim 4. For c > 0, we have $f(a, cx) = \frac{1}{c}f(a, x)$ for all a and x > 0.

Proof. By P(ca, cx, cb, cy) and Claim 3, we have

$$\begin{split} cf(a+b,x+y) + cf(ay-bx,xy(x+y)) &= cf(a,x) + cf(b,y) \\ &= f(ca,cx) + f(cb,cy) \\ &= f(c(a+b),c(x+y)) + f(c^2(ay-bx),c^3xy(x+y)) \\ &= cf(a+b,x+y) + c^2f(ay-bx,cxy(x+y)), \end{split}$$

implying

$$f(ay - bx, xy(x + y)) = cf(ay - bx, cxy(x + y))$$

for all a, b, x > 0, y > 0.

But as (a, x) and (b, y) vary across $\mathbb{R} \times \mathbb{R}_{>0}$, (ay - bx, xy(x + y)) hits all values in $\mathbb{R} \times \mathbb{R}_{>0}$, so the claim follows.

Claim 5. f(1,1) = f(-1,1).

Proof. By P(1, 1, -1, 1) and P(1, 1, 1, 1), we have

$$f(1,1) + f(-1,1) = f(2,2) = 2f(1,1),$$

so f(1,1) = f(-1,1).

Then for each (x, y), we have by Claims 3 and 4 that

$$f(x,y) = \frac{|x|}{y} f(x,|x|) = \frac{|x|^2}{y} f(\pm 1,1),$$

implying the desired since f(1,1) = f(-1,1).

C1. Elmo has 2023 cookie jars, all initially empty. Every day, he chooses two distinct jars and places a cookie in each. Every night, Cookie Monster finds a jar with the most cookies and eats all of them. If this process continues indefinitely, what is the maximum possible number of cookies that the Cookie Monster could eat in one night?

(Espen Slettnes)

The answer is 12 cookies.

Construction: By selecting two empty jars 1024 days in a row, Elmo may ensure that 1024 jars each have 1 cookie. By selecting two jars with 1 cookie 512 days in a row, Elmo may ensure that 256 jars each have 2 cookies. By repeating this process, Elmo may ensure that there are 2^{11-n} jars with n cookies for each n.

Ultimately, Elmo can guarantee there is 1 jar with 11 cookies. The night after, the Cookie Monster may eat 12 cookies.

Upper bound: Let jar *i* contain a_i cookies, and consider

$$I = \sum_{i=1}^{2023} \begin{cases} 2^{a_i} & a_i > 0\\ 0 & a_i = 0. \end{cases}$$

Claim. Before Elmo's move, we always have $I \leq 4044$.

Proof. Assume $I \leq 4044$, and suppose Elmo puts cookies in jars containing a and b cookies initially, with $a \leq b$. We show that after the Cookie Monster's operation, the resulting value of I, say I', is also at most 4044.

If a > 0, then Elmo's operation increases I by $2^a + 2^b$, whereas the Cookie Monster's operation decreases I by at least $2^{b+1} \ge 2^a + 2^b$, so $I' \le I \le 4044$.

If a = 0 but b > 0, then Elmo's operation increases I by $1 + 2^b$, whereas the Cookie Monster's operation decreases I by at least $2^{b+1} \ge 2^b + 1$, so $I' \le I \le 4044$.

Finally if a = b = 0, then Elmo's operation increases I by 2, and the Cookie Monster's operation decreases I by at least $\frac{I+2}{2023}$ so

$$I' \le \frac{2022}{2023}(I+2) \le 4044.$$

Thus before Elmo's move, there is no jar with at least 12 cookies, so the maximum number of cookies the Cookie Monster may eat is 12.

C2. Alice is performing a magic trick. She has a standard deck of 52 cards, which she may order beforehand. She invites a volunteer to pick an integer $0 \le n \le 52$, and cuts the deck into a pile with the top n cards and a pile with the remaining 52 - n. She then gives both piles to the volunteer, who riffles them together and hands the deck back to her face down. (Thus, in the resulting deck, the cards that were in the deck of size n appear in order, as do the cards that were in the deck of size 52 - n.)

Alice then flips the cards over one-by-one from the top. Before flipping over each card, she may choose to guess the color of the card she is about to flip over. She stops if she guesses incorrectly. What is the maximum number of correct guesses she can guarantee?

(Espen Slettnes)

The maximum is 26 guesses.

Construction: Alice arranges the cards in alternating order of color. Consider the two decks D_1 and D_2 that were riffled together. Then throughout the process, Alice may keep track of the multiset

{color of the top card of D_1 , color of the top card of D_2 }.

Now whenever the top cards of D_1 and D_2 are the same color, Alice may successfully guess the color of the next card dealt. As the process proceeds, whether the top cards of D_1 and D_2 have the same color toggles after each move, so she may successfully guess the color of the drawn card half the time.

Upper bound: The volunteer picks n = 51. Assume without loss of generality the bottom card is black. Then Alice can never confidently guess the color of any red card, so she may successfully guess at most 26 cards.

C3. Find all pairs of positive integers (a, b) with the following property: there exists an integer N such that for any integers $m \ge N$ and $n \ge N$, every $m \times n$ grid of unit squares may be partitioned into $a \times b$ rectangles and fewer than ab unit squares.

(Holden Mui)

The answers are (1, 1), (1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2). Below is the author's solution, unedited.

Constructions: The constructions for 1×1 rectangles and 1×2 rectangles are obvious. For 1×3 rectangles, a construction for (m+3, n+3) can be obtained from a construction for (m, n) as shown in Figure 2.1,



Figure 2.1: Induction for 1×3 rectangles

so it suffices to give constructions for (1, 1), (1, 2), and (5, 5), by induction. The first two cases are vacuous, and a 5×5 grid can be tiled as shown in Figure 2.2.



Figure 2.2: Partitioning a 5×5 rectangle into 1×3 rectangles

Lastly, for 2×3 rectangles, a construction for (m + 6, n + 6) can be obtained from (m, n) for $m, n \ge 2$ using a method similar to the 1×3 case, since any $k \times 6$ rectangle can be tiled with 2×3 rectangles for $k \ge 2$, by writing k as the sum of a multiple of 2 and a multiple of 3. Hence, it suffices to give constructions for all pairs $(m, n) \in \{2, 3, 4, 5, 6, 7\}^2$, which is given in Figure 2.3.



Figure 2.3: Constructions for (a, b) = (2, 3)

Completeness of solution set: To show that no other pairs (a, b) work, consider the following claim.

Claim. For $c, d \in \mathbb{Z}^+$ and $k \ge 4$, any tiling of an $(ck+2) \times (dk + \lfloor \frac{k}{2} \rfloor)$ rectangle with $1 \times k$ rectangles must leave at least k cells empty.

Proof. Consider a $(ck + 2) \times (dk + \lceil \frac{k}{2} \rceil)$ with its bottom-left cell at the origin, and label each cell with the sum of its coordinates modulo k.

1	2	3	4	0	1	2	3	4	0	1	2	3
0	1	2	3	4	0	1	2	3	4	0	1	2
4	0	1	2	3	4	0	1	2	3	4	0	1
3	4	0	1	2	3	4	0	1	2	3	4	0
2	3	4	0	1	2	3	4	0	1	2	3	4
1	2	3	4	0	1	2	3	4	0	1	2	3
0	1	2	3	4	0	1	2	3	4	0	1	2

Figure 2.4: Example for k = 5, c = 1, d = 2

By construction, every $1 \times k$ rectangle must cover one cell whose label is k - 1. However, when $k \geq 4$, no cell in the top-right-most $2 \times \left\lceil \frac{k}{2} \right\rceil$ rectangle can contain an k - 1, since the top-right cell's label is

$$(2-1) + \left(\left\lceil \frac{k}{2} \right\rceil - 1 \right) = \left\lceil \frac{k}{2} \right\rceil < k-1.$$

Therefore, any tiling of $1 \times k$ rectangles must leave at least $2 \times \left\lceil \frac{k}{2} \right\rceil \ge k$ cells empty. \Box

To show that no pairs (a, b) with $\max(a, b) \ge 4$ work, assume $b \ge 4$. Since $a \times b$ rectangles can be partitioned into a separate $1 \times b$ rectangles, it suffices to find values for c and d for which

$$a \mid \underbrace{\left\lfloor \frac{(bc+2)\left(bd + \left\lceil \frac{b}{2} \right\rceil\right)}{b} \right\rfloor}_{\text{number of } 1 \times b \text{ rectangles}} = bcd + \left\lceil \frac{b}{2} \right\rceil c + 2d + 1$$

by the Claim. (For example, a 7×13 rectangle cannot contain six 3×5 rectangles since it cannot contain eighteen 1×5 rectangles.)

If b is even, then a must be odd or else large odd-sized grids cannot be partitioned. Since

$$bcd + \left\lceil \frac{b}{2} \right\rceil c + 2d + 1 = \left(\frac{b}{2}c + 1\right)(2d + 1)$$

choosing $d = \frac{a-1}{2}$ works.

If b is odd, then choosing c = k(2d + 1) gives

$$bcd + \left\lceil \frac{b}{2} \right\rceil c + 2d + 1 = (2d+1)\left(b\left(\frac{c}{2}\right) + 1\right) + \frac{c}{2} = (2d+1)\frac{bk(2d+1) + 3}{2}$$

Then d can be chosen such that the first term contains all odd prime factors of a and k can be chosen so that the second term contains all even prime factors of a.

Therefore, it suffices to check that (2,2) and (3,3) don't work. (2,2) doesn't work because each row in a partitioning of an odd-sized grid must contain a unit square, and (3,3) doesn't work for similar reasons. C5. Define the *mexth* of k sets as the kth smallest positive integer that none of them contain, if it exists. Does there exist a family \mathcal{F} of sets of positive integers such that

- for any nonempty finite subset \mathcal{G} of \mathcal{F} , the mexth of \mathcal{G} exists, and
- for any positive integer n, there is exactly one nonempty finite subset \mathcal{G} of \mathcal{F} such that n is the mexth of \mathcal{G} .

(Espen Slettnes)

The answer is yes.

First solution by explicit construction We let $\mathcal{F} = \{S_0, S_1, S_2, \ldots\}$, where $S_0 = \emptyset$ and for all $k \ge 1$,

$$S_k = \{n : 0 \le (n \mod 2^{k+1}) \le 2^k - 2\} \cup \{2^k - 1\}.$$

That is, S_k excludes every number containing a 1 in the kth bit in binary, or with 0 in the kth bit with only 1's after, except $2^k - 1$.

For each set of nonnegative indices $J = \{j_1 < \cdots < j_n\}$, we claim that the mexth of the *n* sets S_{j_1}, \ldots, S_{j_n} is the positive integer *N* containing a 1 in bits j_1, \ldots, j_n and 0's elsewhere. Of course *N* is excluded from each set, along with the number obtained by replacing bit j_i with 0 and all smaller bits with 1, for each $i = 1, \ldots, n-1$.

It is easy to check there are no other numbers smaller than N excluded by all these sets, so N is the mexth.

Second solution by freedom (Derek Liu, unedited) We use the following process to construct such sets. We keep track of counters a, b, and n, where n is the number of sets we've created, a will keep track of the mexth of the next set we create by itself, and b keeps track of the mexth of the next subset of the first n sets that we want to address. The idea is that we will simply have b increase really fast (e.g. powers of 10), in a "greedy"-esque algorithm; the counter a will fill in the remaining numbers so that every number is the mexth of some collection of sets. Start at n = 0, a = 1, b = 10.

Every time we create a new set, we obviously increment n by 1 (and let's index this set as set n). This new set will have mexth a, which we do by excluding a from this set. (There are clearly no problems with this step.) Then, we increment a by 1; if it is now a power of 10, we increment it again (as we will hit the powers of 10 using collections of multiple sets rather than single sets).

Consider the subsets of the first n sets that contain the nth set, and order them by size (so $\{n\}, \{1, n\}, \{2, n\}, \ldots, \{1, 2, n\}, \ldots$). We've addressed the mexth of $\{n\}$ already; we will now go down this list in order. Each time, we take the next set in the list and let the mexth of the corresponding collection of sets be b; we do this by removing $b, b - 1, b - 2, \ldots$ (as many numbers as needed) from all of the sets involved. Then, we multiply b by 10.

To verify that this construction is possible, we must ensure that for any collection G of at least 2 sets, when we try to "create" the mexth of G, it is never the case that |G| or more numbers are excluded from every set in G already. In fact, we claim at most one number can be excluded from every set in G. Let k be the highest-numbered set in G.

Let M be the maximum element excluded from any set before we constructed set k. Notice that M < b, so the only number less than M not in k is a. Furthermore, since we address subsets of the first n sets in order of size, we address G before any superset of G.

Hence, we could not have excluded any element above M from every set of G yet (because the only way to do so would be by doing this for some superset of G, which hasn't occurred yet). Thus, at this point, at most one element is excluded from every set in G, so we can create the mexth of G.

Therefore, this construction works.

Remark. Here is an example:

- Creating the 1st set: it excludes 1. Now a = 2 and b = 10.
- Creating the 2nd set: it excludes 2. Now a = 3.
 - Creating the mexth of $\{1, 2\}$: exclude 9 and 10 from both set 1 and set 2. Now b = 100.
 - Currently, the 1st set excludes $\{1, 9, 10\}$, while the 2nd set excludes $\{2, 9, 10\}$.
- Creating the 3rd set: it excludes 3. Now a = 4.
 - Creating the mexth of $\{1,3\}$: exclude 99 and 100 from both set 1 and set 3. Now b = 1000.
 - Creating the mexth of $\{2,3\}$: exclude 999 and 1000 from both set 2 and set 3. Now b = 10000.
 - Creating the mexth of $\{1, 2, 3\}$: exclude 9998, 9999, and 10000 from sets 1, 2, and 3. Now b = 100000.

Currently, set 1 excludes $\{1, 9, 10, 99, 100, 9998, 9999, 10000\}$, set 2 excludes $\{2, 9, 10, 999, 1000, 9998, 9999, 10000\}$ and set 3 excludes $\{3, 99, 100, 999, 1000, 9998, 9999, 10000\}$.

Note that when we create the 9th set, we exclude 9 from it. This might seem like an issue because 9 is also excluded from sets 1 and 2. But this is resolved by simply excluding b only, not b and b-1, when creating the mexths of $\{1,9\}$ and $\{2,9\}$ (and excluding b and b-1, but not b-2, from $\{1,2,9\}$).

[•] etc.

C6. For a set S of positive integers and positive integer n, consider the game of (n, S)-nim, which is as follows. A pile starts with n stones. Two players, Anthony and Brandon, alternate turns removing stones from the pile, with Anthony going first. On any turn, the number of stones removed must be an element of S. The last player to move wins. Let f(S) denote the set of positive integers n for which Anthony has a winning strategy in (n, S)-nim.

Let T be a set of positive integers. Must the sequence

$$T, f(T), f(f(T)), \ldots$$

be eventually constant?

(Brandon Wang, Edward Wan)

Yes, the sequence must be eventually constant. In what follows, let $\overline{S} = \mathbb{Z}_{\geq 0} \setminus S$, so $f(S) = S + \overline{f(S)}$ for all S. Note that $S \subseteq f(S)$ always, so the limit $f^{\infty}(T)$ is well-defined.

We take two cases.

First case: $\overline{f^{\infty}(T)}$ is closed under addition. Let $m = \min(\overline{T} \setminus 0)$, so $m \in \overline{f^{\infty}(T)}$. Then all multiples of m are in $\overline{T} \supseteq \overline{f^{\infty}(T)}$, but 1, 2, ..., m-1 are in T, so f(T) is exactly the set of non-multiples of m, implying $f(T) = f^{\infty}(T)$.

Second case: There are a and b with $a, b \in \overline{f^{\infty}(T)}$ and $a + b \in f^{\infty}(T)$. Then there must be an index j for which $n \in f^j(T) \iff n \in f^{\infty}(T)$ for $n \le a + b$. For n > a + b, note:

- If $n a b \notin f^{j+1}(T)$ then $n = (n a b) + (a + b) \in f^{j+1}(T)$.
- If $n a b \in f^{j+1}(T)$, then $n b = (n a b) + a \in f^{j+2}(T)$ so $(n b) + b \in f^{j+3}(T)$.

Thus all n > a + b are in $f^{j+3}(T) \subseteq f^{\infty}(T)$, implying $f^{j+3}(T) = f^{\infty}(T)$.

Remark. Espen Slettnes claims that a modification of the above argument shows that $f^3(T) = f^4(T)$ for all T.

C7. A discrete hexagon with center (a, b, c) (where a, b, c are integers) and radius r (a nonnegative integer) is the set of lattice points (x, y, z) such that x + y + z = a + b + c and $\max(|x-a|, |y-b|, |z-c|) \leq r$.

Let n be a nonnegative integer and S be the set of triples (x, y, z) of nonnegative integers such that x + y + z = n. If S is partitioned into discrete hexagons, show that at least n + 1 hexagons are needed.

(Linus Tang)

We present two solutions.

First solution (author) Let *B* denote the set of points that lie on the bottom row of a hexagonal tile. Consider the function $f: B \to B$ defined as follows:

- If b is the bottom-left point of its hexagonal tile, then f(b) = b.
- If b is the (i + 1)th point from the left of the bottom row of a hexagonal tile, then f(b) is the *i*th point from the left among the points lying immediately above the tile.



Draw a directed graph G connecting $b \to f(b)$ for each b. Then each connected component of G is a *chain* of the form $a_1 \to a_2 \to a_3 \to \cdots \to a_{k-1} \to a_k \to a_k$, where a_k is the bottom-left corner of a hexagon.

Note that each point on the bottom row has an indegree of 0, and thus determines one such chain. At the end of this chain is the bottom-left corner of a hexagonal tile. Thus the number of hexagonal tiles is at least the number of points in the bottom row, which is n + 1.

Second solution (Derek Liu) Consider the triangular grid T obtained by taking the centers of all upside-down unit equilateral triangles in S and extending this grid one unit in each direction. The grid T is shown in the figure below in green. (Then each point in S is the center of a right-side-up triangle in T.) Each hexagonal tile in S induces a large hexagonal tile in T containing it.



In T, assign each unit equilateral triangle (with green vertices) a charge of +1 if it is rightside-up and a charge of -1 if it is upside-down.

We may check that the entire grid T has total charge n + 1. Each point in S corresponds to a right-side-up triangle in T, so the union of all hexagonal tiles contains all positively-charged triangles in T (but not necessarily all negatively-charged triangles). Thus the union of the hexagonal tiles has total charge at least n + 1.

But we may also check that each hexagonal tile in T has total charge 1, so at least n+1 tiles are needed.

C8. Let $n \ge 3$ be a fixed integer, and let α be a fixed positive real number. There are n numbers written around a circle such that there is exactly one 1 and the rest are 0's. An *operation* consists of picking a number a in the circle, subtracting some positive real $x \le a$ from it, and adding αx to each of its neighbors.

Find all pairs (n, α) such that all the numbers in the circle can be made equal after a finite number of operations.

(Anthony Wang)

Below is the author's solution, unedited.

The answer is all (n, α) except when $\alpha = \frac{1}{2} \sec\left(\frac{\tau k}{n}\right)$ for some integer k, where τ denotes the ratio of a circle's circumference to its radius.

Call the numbers a_1, a_2, \ldots, a_n in counterclockwise order.

First, we show that such (n, α) cannot work. To do this, note that if $\zeta = e^{\tau i k/n}$, then, $\zeta \neq 1$, and

$$\alpha z^2 - z + \alpha = \alpha \left(z^2 - 2 \cos \left(\frac{\tau k}{n} \right) + 1 \right) = \alpha (z - \zeta) (z - \zeta^{-1}).$$

Therefore, the sum

$$Z := \sum_{k=1}^{n} \zeta^k a_k,$$

is invariant under the operations. But this sum starts nonzero and must be zero if all the numbers in the circle are equal, contradiction.

Now we show that all other such (n, α) work. The general idea is to first make the a_i roughly uniform and then smooth out the differences.

We present one possible way to do this, using the following two claims:

Claim 1. Let $\varepsilon > 0$ be fixed. Let $S = a_1 + a_2 + \cdots + a_n$. We can make the numbers around the circle satisfy

$$\left|\frac{a_i}{S} - \frac{1}{n}\right| < \varepsilon,$$

for all i = 1, 2, ..., n in finitely many moves.

Proof. WLOG assume a_n is the one that starts at 1 (so the rest start at 0).

The procedure is simple: apply the operation to every number in the circle at the same time, where we pick $x = a_i$ on each a_i .

Using indices modulo n, the numbers on the board evolve as follows

 a_{-3}	a_{-2}	a_{-1}	a_0	a_1	a_2	a_3	
 0	0	0	1	0	0	0	
 0	0	α	0	α	0	0	
 0	α^2	0	$2\alpha^2$	0	α^2	0	
 α^3	0	$3\alpha^3$	0	$3\alpha^3$	0	α^3	

After each number has undergone N operations, we can prove

$$a_i = \alpha^N \sum_{n|2k-N-i} \binom{N}{k},$$

by induction, where $\binom{N}{m}$ is zero if m < 0 or m > N. Notably,

- if n is odd, then each number around the circle is α^N times the sum of every nth binomial coefficient with upper term N.
- if n is even, then every other number is zero, and the rest are α^N times the sum of every (n/2)th binomial coefficient with upper term N.

We claim that for any fixed integers d and i,

$$\frac{1}{2^N} \sum_{d|k-i} \binom{N}{k} = \frac{1}{d} + o(1),$$

as $N \to \infty$. Indeed by Root of Unity Filter, we have

$$\sum_{d|k-i} \binom{N}{k} = \frac{1}{d} \sum_{\zeta^d=1} \zeta^{-i} (1+\zeta)^N.$$

But for $\zeta = 1$, the summand is 2^N , and for each $\zeta^d = 1$ with $\zeta \neq 1$, we have $|1 + \zeta| < 2$ by the Triangle Inequality. Thus,

$$\sum_{d|k-i} \binom{N}{k} = \frac{1}{d} \left(2^N + (N-1)O(c^N) \right),$$

for some c < 2, from which the conclusion follows.

Using this claim, the conclusion follows for odd n by taking sufficiently large N. Furthermore, if n is even, then every other number can be arbitrarily close to $\frac{2S}{n}$, so (without updating the value of S) we can make each number arbitrarily close to $\frac{4\alpha S}{n(2\alpha+1)}$ by subtracting $\frac{2S}{n(2\alpha+1)}$ from each. The conclusion follows.

Claim 2. If $\alpha = 2$ or $\alpha \neq \frac{1}{2} \sec\left(\frac{\tau k}{n}\right)$ for any integer k, then we can always make all the numbers equal in finitely many operations if subtracting any real x is allowed.

Proof. Write $\frac{1}{\alpha} = t + \frac{1}{t}$ for some complex $t \neq 0$. By assumption, $t^n \neq 1$. Scale so that $a_n = \alpha$ and $a_1 = a_2 = \cdots = 0$, and take indices modulo n. It suffices to find real $b_1, b_2, \ldots, b_{n-1}, b_n$ such that

$$\alpha(b_n + b_2) - b_1 = \alpha(b_1 + b_3) - b_2 = \dots = \alpha(b_{n-1} + b_1) - b_n + \alpha,$$

because then we can simply apply the operation to a_i with $x = b_i$ for each *i*. Rewrite the above as,

$$b_n + b_2 - \left(t + \frac{1}{t}\right)b_1 = b_1 + b_2 - \left(t + \frac{1}{t}\right)b_2 = \dots = b_{n-1} + b_1 - \left(t + \frac{1}{t}\right)b_n + 1.$$

If we let $c_i = b_{i-1} - tb_i$, then

$$c_1 - \frac{1}{t}c_2 = c_2 - \frac{1}{t}c_3 = \dots = c_n - \frac{1}{t}c_1 + 1.$$

Suppose this common value is k. We take two cases:

• Case 1: if t = 1, then

$$c_1 - c_2 = c_2 - c_3 = \dots = c_n - c_1 + 1 = k_1$$

 \mathbf{SO}

$$0 = (c_1 - c_2) + (c_2 - c_3) + \dots + (c_n - c_1) = nk - 1,$$

i.e. k = 1/n. From here, we can assign

$$c_i = -\frac{i}{n} + \frac{n+1}{2n},$$

and it will satisfy all the conditions along with $c_1 + c_2 + \cdots + c_n = 0$. Finally, we can extract real b_i according to $b_i = c_{i+1} + c_{i+2} + \cdots + c_n$.

• Case 2: if $t \neq 1$, we claim that we can just pick k = 0. Indeed, we simply need to satisfy, $c_n - \frac{1}{t}c_1 + 1$, and $0 = c_i - \frac{1}{t}c_{i+1}$ for each i = 1, 2, ..., n-1.

For the latter, we set, $c_i = C \cdot t^i$ for all *i* and some constant *C*. From which the latter implies $C = \frac{1}{t^n - 1}$ which is well defined since $t^n \neq 1$. Thus, we can assign

$$c_i = \frac{t^i}{t^n - 1},$$

and extract b_i according to

$$b_i = -\frac{t^{n-i}c_1 + t^{n-i+1}c_2 + \dots + t^{n-1}c_i + c_{i+1} + tc_{i+2} + \dots + t^{n-i-1}c_n}{t^n - 1}$$

which can be checked to satisfy all conditions¹.

Finally, if t is real, then the b_i are real. Otherwise, |t| = 1 since $t + \frac{1}{t} = \frac{1}{\alpha}$ is real, so $\overline{t} = \frac{1}{t}$, and $\overline{c_i} = \frac{t^{n-i}}{1-t^n} = -\overline{c_{n-i}}$. It follows that

$$\overline{b_i} = \frac{t^i c_{n-1} + t^{i-1} c_{n-2} + \dots + t c_{n-i} + t^n e c_{n-i-1} + t^{n-1} c_{n-i-2} + \dots + t^{i+1} c_0}{1 - t^n}$$

where $c_0 = \frac{1}{t^{n-1}}$. But $t^i c_{n-1} = t^{n-1} c_i$, $t^{i-1} c_{n-2} = t^{n-2} c_{i-1}$, and so on, so b_i is real, as desired.

We are done.

To finish, let A be the smallest value attained by a number around the circle in the procedure described in Claim 2. Then by repeatedly applying the claim, if $\alpha = 2$ or $\alpha \neq 2 \sec\left(\frac{\tau k}{n}\right)$ for any integer k, we can turn any sequence of reals x_1, x_2, \ldots, x_n around the circle to all the same number in finitely many moves such that the numbers around the circle are never less than $(x_1 + x_2 + \cdots + x_n)A$. Note that A only depends on n.

Furthermore, if the numbers around the circle are instead $\lambda + x_1, \lambda + x_2, \ldots, \lambda + x_n$, the same procedure allows us to make all the numbers the same such that the numbers around the circle are never less than $\lambda + (x_1 + x_2 + \cdots + x_n)A$.

But by Claim 1, if we scale so that the sum of the numbers around the circle is 1 and we set $\lambda = \frac{1}{n}$, we can ensure $|x_1 + x_2 + \cdots + x_n| < n\varepsilon$ for any $\varepsilon > 0$. Picking $\varepsilon < \frac{1}{n|A|}$, we have $\lambda + (x_1 + x_2 + \cdots + x_n)A > 0$, so it follows that we can keep all the numbers around the circle positive, as desired.

Remark. The procedure used in the proof of Claim 1 can be seen as a Markov Chain with half chance of moving to each neighbor. Then, the conclusion is a consequence of the Fundamental Limit Theorem for regular Markov Chains. Although this is not elementary, the general intuition that it just *has* to become nearly uniform if you spam the operation enough should be.

Additionally, there are likely many ways to prove this claim, but we have selected this proof as the most obvious one, although not necessarily the simplest one.

¹This is motivated by first setting $b_i = c_{i+1} + tc_{i+2} + \dots + t^{n-i-1}c_n + t^{n-i}b_n$ for each *i*, and then solving for b_n in $b_n = c_1 + tb_1$.

Remark. The $t \neq 1$ case of Claim 2 is equivalent to showing that a linear transformation is nonsingular, or equivalently that $(\alpha x^2 - x + \alpha, x^n - 1)$ is the whole ring $\mathbb{R}[x]$ (which is true because $\mathbb{R}[x]$ is a PID and the two share no roots).

Indeed, we can prove the case very easily by noting that if a(x) and b(x) are polynomials such that deg b < n and,

$$a(x)(x^{n} - 1) + b(x)(\alpha x^{2} - x + \alpha) + 1 = 0,$$

then the b_i such that $b(x) = b_n + b_{n-1}x + \dots + b_1x^{n-1}$, suffice.

We decided against putting this proof in the solution since the one presented seems more elementary and more natural, although the above proof is certainly more beautiful. **G1.** Let *ABCDE* be a regular pentagon. Let *P* be a variable point on the interior of segment *AB* such that $PA \neq PB$. The circumcircles of $\triangle PAE$ and $\triangle PBC$ meet again at *Q*. Let *R* be the circumcenter of $\triangle DPQ$. Show that as *P* varies, *R* lies on a fixed line.

(Karthik Vedula)

Let L be the point opposite D on (ABCDE), and let $X = \overline{EA} \cap \overline{BC}$. Note D, X, L are all collinear along the perpendicular bisector of \overline{AB} .



By the radical axis theorem on (ABCDE), (PAEQ), (PBCQ), point X lies on line PQ. Then by power of a point, $XD \cdot XL = XP \cdot DQ$, so DLPQ is cyclic.

It follows that R always lies on the perpendicular bisector of \overline{DL} .

G2. Let ABC be an acute scalene triangle with orthocenter H. Line BH intersects \overline{AC} at E and line CH intersects \overline{AB} at F. Let X be the foot of the perpendicular from H to the line through A parallel to \overline{EF} . Point B_1 lies on line XF such that $\overline{BB_1}$ is parallel to \overline{AC} , and point C_1 lies on line XE such that $\overline{CC_1}$ is parallel to \overline{AB} . Prove that points B, C, B_1, C_1 are concyclic.

(Luke Robitaille)

We present a few solutions.

First solution (mine) Let M be the midpoint of \overline{BC} and let $A' = \overline{BB_1} \cap \overline{CC_1}$ so that ABA'C is a parallelogram.



Since AEFX is an isosceles trapezoid and ME = MF,

$$\measuredangle B_1 F M = \measuredangle X F M = \measuredangle M E A = \measuredangle E C M = \measuredangle B_1 B M,$$

so $B_1 \in (BMF)$. Similarly $C_1 \in (CME)$.

But since $AB \cdot AF = AC \cdot AE$, line AM is the radical axis of (BMF) and (CME). In particular, A' lies on this radical axis, so $A'B \cdot A'B_1 = A'C \cdot A'C_1$ as needed.

Second solution (author) Let $A' = \overline{BB_1} \cap \overline{CC_1}$ so ABA'C is a parallelogram. Since $\overline{A'H} \perp \overline{EF}$, we have X, H, A' collinear. But

$$\measuredangle FXH = \measuredangle FEH = \measuredangle FCB = \measuredangle HA'B,$$

implying $B_1X = B_1A'$. Similarly $C_1X = C_1A'$, so $\overline{B_1C_1} \perp \overline{XHA'}$.

This means \overline{BC} and $\overline{B_1C_1}$ are antiparallel in $\angle A'$, so BB_1CC_1 is indeed cyclic.

Third solution (author) Let M be the midpoint of \overline{BC} and let $A' = \overline{BB_1} \cap \overline{CC_1}$ so that ABA'C is a parallelogram.

Let ℓ be the perpendicular bisector of \overline{EF} . Let B_2 is the reflection of B_1 in ℓ and let $M' \in \ell$ be the midpoint of $\overline{B_1B_2}$. Since \overline{XF} and \overline{AE} are reflections in ℓ , we know B_2 lies on \overline{AC} . If $M \neq M'$, this implies $\ell = \overline{MM'} \parallel \overline{AC}$, which is absurd. Hence M is the midpoint of $\overline{B_1B_2}$, i.e. $\overline{B_1M} \perp \ell$. Similarly $\overline{C_1M} \perp \ell$.

Then $\overline{B_1C_1} \parallel \overline{EF}$, implying \overline{BC} and $\overline{B_1C_1}$ are antiparallel in $\angle A'$, which gives the desired.

G3. Two triangles intersect to form seven finite disjoint regions, six of which are triangles with area 1. The last region is a hexagon with area A. Compute the minimum possible value of A.

(Karthik Vedula)

The minimum value of A is 6, achieved by taking two equilteral triangles from the vertices of a regular hexagon. Now we show $A \ge 6$.

Consider the convex hull of the two triangles, shown below, let a, b, c, d, e, f denote the areas of the "ears" as shown below.



It is easy to verify the following statement:

Claim. In a quadrilateral *ABCD* with $E = \overline{AC} \cap \overline{BD}$, we have

$$\operatorname{Area}(\triangle EAB) \cdot \operatorname{Area}(\triangle ECD) = \operatorname{Area}(\triangle EBC) \cdot \operatorname{Area}(\triangle EDA).$$

From the claim, we deduce that the area of the red triangle highlighted in the above diagram is

Area(red triangle) =
$$\frac{(b+1)(f+1)}{a} \ge \frac{2\sqrt{b} \cdot 2\sqrt{f}}{a} = \frac{4\sqrt{bf}}{a}$$

We may similarly determine the areas of the remaining five analogous red triangles.

We may check that the sum of areas of the six red triangles is simply 3A + 6, so we have

$$3A + 6 = \sum_{\text{cyc}} \text{Area}(\text{red triangle}) = \sum_{\text{cyc}} \frac{4\sqrt{bf}}{a} \ge 24,$$

which gives the desired.

Remark. There are easier solutions involving taking an affine transformation sending one of the triangles to an equilateral triangle.

G4. Let *D* be a point on segment *PQ*. Let ω be a fixed circle passing through *D*, and let *A* be a variable point on ω . Let *X* be the intersection of the tangent to the circumcircle of $\triangle ADP$ at *P* and the tangent to the circumcircle of $\triangle ADQ$ at *Q*. Show that as *A* varies, *X* lies on a fixed line.

(Elliott Liu, Anthony Wang)

Note that

$$\measuredangle PXQ = \measuredangle XPQ + \measuredangle PQX = \measuredangle PAD + \measuredangle DAQ = \measuredangle PAQ,$$

so APXQ is cyclic. Since $\angle XPQ = \angle PAD$ and $\angle PQX = \angle DAQ$, lines AD and AX are isogonal in $\angle PAQ$.



Thus if X' is the reflection of X over the perpendicular bisector of \overline{PQ} , then X' lies on (APQ) and satisfies $DA \cdot DX' = DP \cdot DQ$. Thus X' is the inverse of A about a negative inversion at D with radius $\sqrt{DP \cdot DQ}$.

Since A moves along a circle, X' moves along a circle, and so does X.

G5. Let *ABC* be an acute triangle with circumcircle ω . Let *P* be a variable point on the arc *BC* of ω not containing *A*. Squares *BPDE* and *PCFG* are constructed such that *A*, *D*, *E* lie on the same side of line *BP* and *A*, *F*, *G* lie on the same side of line *CP*. Let *H* be the intersection of lines *DE* and *FG*. Show that as *P* varies, *H* lies on a fixed circle.

(Karthik Vedula)

Construct square BCXY such that A, X, Y lie on the same side of \overline{BC} . Let the tangents to (ABC) at B and C intersect at T, and let $U = \overline{XY} \cap \overline{BT}$ and $V = \overline{XY} \cap \overline{CT}$.



Claim. U lies on \overline{DE} .

Proof. Note $\triangle BYE \cong \triangle BCP$ by rotation about B, so

$$\measuredangle YUB = \measuredangle CBT = \measuredangle CPB = \measuredangle YEB,$$

implying UYEB is cyclic. Similarly UXDB is cyclic.

As $\triangle BDE \sim \triangle BXY$, point *B* is the Miquel point of *DEYX*. The concyclicities then imply $U = \overline{DE} \cap \overline{XY}$.

Now since $\overline{BC} \parallel \overline{UV}$, $\overline{PB} \parallel \overline{HU}$, $\overline{PC} \parallel \overline{HV}$, and $T = \overline{BU} \cap \overline{CV}$, we have that $\triangle PBC$ and $\triangle HUV$ are homothetic at T. The homothety Ψ at T sending \overline{BC} to \overline{UV} is fixed as P varies, so $H = \Psi(P)$ lies on $\Psi((ABC))$.

G6. Let *ABCDEF* be a convex cyclic hexagon such that quadrilateral *ABDF* is a square, and the incenter of $\triangle ACE$ lines on \overline{BF} . Diagonal *CE* intersects diagonals *BD* and *DF* at points *P* and *Q*, respectively. Prove that the circumcircle of $\triangle DPQ$ is tangent to \overline{BF} .

(Elliott Liu)

We present several solutions.

First solution (mine) Let *O* be the circumcenter, and let $L \in \overline{AI}$ be the midpoint of arc *CDE* on the circumcircle.



The key claim is this:

Claim. $\angle AIP = \angle AIQ = 135^{\circ}$.

Proof. Construct P' and Q' on \overline{DF} and \overline{DB} so that $\angle AIP' = \angle AIQ'$. We will show lines CE and P'Q' coincide. To do so, it will suffice to check that (i) $\overline{OL} \perp \overline{P'Q'}$ and (ii) the distance from I to $\overline{P'Q'}$ is the inradius of $\triangle ACE$.

(i) Let $X = \overline{AB} \cap \overline{IP'}$ and $Y = \overline{AF} \cap \overline{IQ'}$. Then $\angle XIQ' = 90^\circ$, so BXIQ' is cyclic, and since \overline{BI} bisects $\angle ABD$, we have IX = IQ'. Since $\angle AIX = 45^\circ = \angle IBX$, line AI is tangent to (BXIQ'), so we have $\overline{XQ'} \parallel \overline{AI}$. Similarly IY = IP' and $\overline{YP'} \parallel \overline{AI}$. It follows that P'Q'XY is an isosceles trapezoid with bases parallel to \overline{AI} .

Moreover since $\angle XIY = 90^\circ = \angle XAY$, we have AXIY cyclic. Since $\angle AXY = \angle AIY = 45^\circ$, we have $\overline{XY} \parallel \overline{BF}$. Thus $\measuredangle(\overline{P'Q'}, \overline{AI}) = \measuredangle(\overline{AI}, \overline{XY})$.

$$\begin{split} \measuredangle(\overline{OL}, \overline{P'Q'}) &= \measuredangle(\overline{OL}, \overline{AI}) + \measuredangle(\overline{AI}, \overline{P'Q'}) \\ &= \measuredangle(\overline{AI}, \overline{OA}) + \measuredangle(\overline{XY}, \overline{AI}) \\ &= \measuredangle(\overline{XY}, \overline{OA}) = 90^{\circ}. \end{split}$$

(ii) Since $\overline{XY} \parallel \overline{BF}$, the reflection I' of I over O lies on (AXIY). Let \overline{AO} reintersect (AXIY) at T.

Since \overline{XY} contains the center of (AXIY), we have

$$OI^{2} = OI \cdot OI' = OA \cdot OT = OA \cdot (OA - 2\operatorname{dist}(O, \overline{XY})),$$

so dist (O, \overline{XY}) equals the inradius of $\triangle ABC$. But we already know dist $(I, \overline{P'Q'}) =$ dist (O, \overline{XY}) .

This proves the claim.

Since $\angle PDQ = \angle PIQ = 90^\circ$, we have $I \in (DPQ)$. But $\angle BIQ = 45^\circ = \angle XYI = \angle IPQ$, so \overline{BF} is tangent to (DPIQ), as desired.

Second solution by reflections (Pitchayut Saengrungkongka, unedited) Let Ω be the circumcircle of *ABCDEF*. Extend *AI* to meet Ω again at *M*, and reflect *I* over *MB* and *MF* to obtain points B_1 and F_1 , respectively.

Claim 1. M, D, B_1, F_1 are colinear.

Proof. We have $\angle IMF_1 = 2\angle IMF = 2\angle AMF = 90^\circ$. Similarly, $\angle IMB_1 = 90^\circ$. Moreover, $\angle IMD = \angle AMD = 90^\circ$. Thus, M, D, B_1, F_1 all lie on a line perpendicular to IM.

Claim 2. C, I, E, B_1, F_1 are concyclic on ω .

Proof. By Fact 5, MC = MI = ME. Moreover, since $\triangle IB_1F_1$ is right isosceles with IM being the altitude, we must have $MI = MB_1 = MF_1$, done.

Claim 3. B, B_1, I, D are concyclic.

Proof. Follows from $\angle IB_1D = 45^\circ = \angle DBI$.

Claim 4. I, P, B_1 are colinear. (Similarly, I, Q, F_1 are colinear.)

Proof. By radical center theorem on $\odot(IBB_1D)$, $\Omega \equiv \odot(BMCE)$, and $\omega \equiv \odot(IB_1CE)$, lines IB_1, BM, CE are concurrent. This point must be P.

Thus, $\angle PIQ = \angle B_1IF_1 = 90^\circ$, implying that $I \in \odot(PDQ)$. To finish the problem, just notice that

$$\angle BIP = \angle BB_1I = \angle BDI = \angle PDI.$$

Third solution by inversion (Maxim Li, unedited) Let Ω be the circumcircle, let I be the incenter of ACE, and let AI, CI, EI meet Ω again at M_A, M_C, M_E , respectively. Let DI meet Ω at T, and let T' lie on Ω with $TT' \parallel BF$.

Consider the inversion through I that sends Ω to itself. Then P^* is the intersection of (FIT) and (IM_CM_E) , and Q^* is the intersection of (BIT) and (IM_CM_E) . I claim that P^*, Q^*, T lie on a line parallel to BF, which would finish the problem.

First note that $TM_A \parallel AD$, so $\angle M_A TT' = 90$. Thus, $M_A T'$ is a diameter of Ω , and since I is the orthocenter of $M_A M_C M_E$, this means $(IM_C M_E)$ is the reflection of Ω over the midpoint of IT'.

Now let TT' meet (FIT) again at P_1 . It suffices to show P_1 lies on (IM_CM_E) , since that would mean $P_1 = P^*$. But now note that $\angle P_1IF = \angle TFI = \angle T'BI$ since P_1TIF is an isosceles trapezoid. Thus, $P_1I \parallel T'B$, so $P_1T'BI$ is a parallelogram. In particular, this means P_1 is the reflection of B over the midpoint of IT', and so lies on (IM_CM_E) , so we are done.

G7. Let \mathcal{E} be an ellipse with foci F_1 and F_2 , and let P be a point on \mathcal{E} . Suppose lines PF_1 and PF_2 intersect \mathcal{E} again at distinct points A and B, and the tangents to \mathcal{E} at A and B intersect at point Q. Show that the midpoint of \overline{PQ} lies on the circumcircle of $\triangle PF_1F_2$.

(Karthik Vedula)

We present two solution paths.

Elementary solution (Espen Slettnes) Let $d = PF_1 + PF_2 = AF_1 + AF_2 = BF_1 + BF_2$. Then $\triangle PAF_1$ and $\triangle PBF_2$ have semiperimeter d, so they share a P-excircle Ω .



Let Ω touch \overline{PA} and \overline{PB} at X and Y, so PX = PY = d. Let R and S be the reflections of P over F_1 and F_2 , so PR + PS = 2d.

Then $\triangle QXR \stackrel{+}{\cong} \triangle QYS$, so *PRQS* is cyclic, are we are done by homothety at *P*.

Projective approaches (Eric Shen, Max Lu) Let M be the midpoint of \overline{PQ} , let O be the center of \mathcal{E} , let P' be the antipode of P, and let P^* lie on \mathcal{E} so that $\overline{PP^*} \parallel \overline{F_1F_2}$.



Claim 1. \overline{PM} bisects $\angle F_1 PF_2$.

First proof by Pitot. By converse Pitot, $\{\overline{PF_1}, \overline{PF_2}, \overline{AF_2}, \overline{BF_1}\}$ has an excircle. Since \overline{QA} and \overline{QB} bisect $\angle F_1AF_2$ and $\angle F_1BF_2$, the excircle has center Q, so \overline{PQ} bisects $\angle F_1PF_2$ as needed.

Second proof by harmonic bundle. It is known that the tangent to \mathcal{E} at P bisects $\angle F_1 P F_2$. But since -1 = P(PQ; AB), so does \overline{PQ} .

Third proof by DDIT. By DDIT from P to \mathcal{E} , we have $(\overline{PP}, \overline{PP})$, $(\overline{PA}, \overline{PB})$, $(\overline{PQ}, \overline{PQ})$ are pairs of an involution. Since \overline{PP} bisects $\angle F_1 PF_2$, so does \overline{PQ} .

Claim 2. $MF_1 = MF_2$.

First proof by harmonic bundle. Note

$$-1 = (F_1 F_2; O \infty_{F_1 F_2}) \stackrel{P}{=} (AB; P'P^*)_{\mathcal{E}},$$

so Q, P', P^* are collinear, i.e. $\overline{QP'} \perp \overline{F_1F_2}$. Then $\overline{MO} \perp \overline{F_1F_2}$ by homothety at P.

Second proof by DIT. By DIT from $\overline{F_1F_2}$ to PPAB, we have (F_1, F_2) , $(\mathcal{E} \cap \overline{F_1F_2}, \mathcal{E} \cap \overline{F_1F_2})$, $(\overline{PP} \cap \overline{F_1F_2}, \overline{AB} \cap \overline{F_1F_2})$, are pairs of an involution. This involution must be reflection about the center of \mathcal{E} , so $\overline{P * P*}$, \overline{AB} , $\overline{F_1F_2}$ concur, implying $-1 = (AB; P^*P')$. Thus P^* , P', Q are collinear, and we are done by homothety at P.

These two claims together are sufficient.

Remark. Claim 2 may be proven by taking an affine transformation sending \mathcal{E} to a circle, and proceeding by a method of choice, e.g. complex numbers.

G8. Similar quadrilaterals $ABCD \sim A_1B_1C_1D_1 \sim A_2B_2C_2D_2$ lie in the plane such that points A, A_1 , B_2 , B are collinear, points B, B_1 , C_2 , C are collinear, points C, C_1 , D_2 , D are collinear, and points D, D_1 , A_2 , A are collinear. Prove that the intersections $\overline{AC} \cap \overline{BD}$, $\overline{A_1C_1} \cap \overline{B_1D_1}$, and $\overline{A_2C_2} \cap \overline{B_2D_2}$ are collinear.



(Holden Mui)

Let X be the center of spiral similarity between ABCD and $A_1B_1C_1D_1$, and let Y be the center of spiral similarity between ABCD and $A_2B_2C_2D_2$. Let $\theta := \measuredangle XAB = \measuredangle XBC = \measuredangle XCD = \measuredangle XDA$ and $\theta' := \measuredangle YBA = \measuredangle YCB = \measuredangle YDC = \measuredangle YAD$.

Claim 1. $\theta = \theta'$; i.e. X and Y are isogonal conjugates.

Proof. Assume for contradiction $\theta < \theta'$ (without loss of generality). Then by the law of sines in $\triangle PDA$ and $\triangle QAB$, we have

$$\frac{PD}{PA} = \frac{\sin(\angle A - \theta)}{\sin \theta} > \frac{\sin(\angle A - \theta')}{\sin \theta'} = \frac{QB}{QA}$$

Multiplying cyclically gives

$$1 = \prod_{\text{cyc}} \frac{PD}{PA} > \prod_{\text{cyc}} \frac{QB}{QA} = 1,$$

contradiction.



Claim 2. *ABCD* is cyclic.

Proof. Note

$$\measuredangle AXD = \measuredangle XAD + \measuredangle ADX = \measuredangle XAD + \measuredangle BAX = \measuredangle BAD$$

and similarly $\measuredangle BXC = \measuredangle BCD$.

Since X has an isogonal conjugate,

$$\measuredangle BAD = \measuredangle AXD = \measuredangle BXC = \measuredangle BCD.$$

Claim 3. *ABCD* is harmonic.

Proof. Observe that

$$\triangle AXD \sim \triangle AYB \implies \frac{AY}{AX} = \frac{AB}{AD}$$

$$\triangle BXA \sim \triangle BYC \implies \frac{CY}{AX} = \frac{BC}{AB}$$

$$\triangle CXB \sim \triangle CYD \implies \frac{CY}{CX} = \frac{CD}{BC}$$

$$\triangle DXC \sim \triangle DYA \implies \frac{AY}{CX} = \frac{AD}{BC}.$$

Hence

$$1 = \frac{AY}{AX} \cdot \frac{CX}{AY} \cdot \frac{CY}{CX} \cdot \frac{AX}{CY} = \frac{AB}{AD} \cdot \frac{BC}{AD} \cdot \frac{CD}{BC} \cdot \frac{AB}{BC},$$

implying $AB \cdot CD = AD \cdot BC$.

Now let $R = \overline{AC} \cap \overline{BD}$, $R_1 = \overline{A_1C_1} \cap \overline{B_1D_1}$, $R_2 = \overline{A_2C_2} \cap \overline{B_2D_2}$. Let M be the midpoint of \overline{AC} and N the midpoint of \overline{BD} .

Claim 4. R, X, Y, M, N are concyclic.

Proof. Since $\measuredangle ANB = \measuredangle ADC = \measuredangle AXB$, we have $X \in (ANB)$, and similarly $X \in (BMC)$. Thus

$$\measuredangle XNR = \measuredangle XAB = \measuredangle XBC = \measuredangle XMR$$

implies $X \in (RMN)$, and similarly $Y \in (RMN)$.

Finally

$$\measuredangle XRY = \measuredangle XKY = \measuredangle XKA + \measuredangle AKY = \measuredangle XBA + \measuredangle ADY = -2\theta$$

 \mathbf{SO}

$$\measuredangle R_1 R R_2 = \measuredangle R_1 R X + \measuredangle X R Y + \measuredangle Y R R_2 = \theta + (-2\theta) + \theta = 0^\circ,$$

as desired.

N1. Let *m* be a positive integer. Find all polynomials P(x) with integer coefficients such that for every integer *n*, there exists an integer *k* such that $P(k) = n^m$.

(Raymond Feng)

The answers are $P(x) \equiv (x+a)^d$ and $P(x) \equiv (-x+a)^d$, where $d \mid n$. These work, so we show they are the only solutions.

There exists k_0 so that $P(k_0) = 0^m$. Shift P so that $k_0 \mapsto 0$, so P(0) = 0, implying $x \mid P(x)$ for all x.

For each prime p, there is some k_p with $P(k_p) = p^m$. Since $k_p \mid P(k_p) = p^m$, we find that $k_p \in \{\pm 1, \pm p, \ldots, \pm p^m\}$. By Pigeonhole, there is some $0 \le r \le m$ such that either $k_p = p^r$ for infinitely many p or $k_p = -p^r$ for infinitely many p.

In the former case, we must have $P(x) \equiv x^{m/r}$, and in the latter case we must have $P(x) \equiv (-x)^{m/r}$.

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$$s_1^{s_2} = s_2^{s_3} = \dots = s_{n-1}^{s_n}$$

(Holden Mui)

The answer is n = 5, achieved by

$$256^2 = 2^{16} = 16^4 = 4^8.$$

Now we show n = 5 is maximal.

Evidently there must be a non-perfect power a such that s_1, \ldots, s_{n-1} are all powers of a. Let $s_2 = a^m$ and $s_3 = a^n$, so

$$s_4 = \frac{ma^n}{n}$$

Since s_4 is a power of a, we must have $n = ma^k$ for some nonzero integer k. This gives

$$s_5 = \frac{a^{ma^k}}{a^k - \frac{k}{m}},$$

so $a^k - \frac{k}{m}$ is a power of a.

First case: If k > 0, then

$$a^k - k \le a^k - \frac{k}{m} \le a^{k-1},$$

implying (a, k) = (2, 1) or (a, k) = (2, 2). In both cases, m = 1, and we obtain the two sequences 4, 2, 4, 2, 4, ... (which fails distinctness) and 256, 2, 16, 4, 8, which terminates after five terms.

Second case: Assume k < 0. We may find

$$s_1 = a^{ma^{ma^k} - m} = a^{na^{(-k) + n - na^{-k}}}$$

Since the exponent must be at least 1,

$$n \ge a^{ca^{-k} - (-k) - n} \ge 2^{n2^{-k} - (-k) - n} \ge 2^{n-1},$$

with equality only when a = 2, k = -1, and n = 1 or n = 2. We obtain the two sequences 2, 2, 2, 2, 2, ... and 2, 4, 2, 4, 2, ..., which fail.

N3. Let a and b be positive integers and let $k \leq b$ be a nonnegative integer. A lemonade stand owns $n \geq k$ cups, of which k are initially full and n - k are initially empty. The lemonade stand also has a filling machine and an emptying machine, which operate according to the following rules:

- If at any moment, *a* completely empty cups are available, the filling machine spends the next *a* minutes filling those *a* cups simultaneously and doing nothing else.
- If at any moment, b completely full cups are available, the emptying machine spends the next b minutes emptying those b cups simultaneously and doing nothing else.

Suppose that after a sufficiently long time has passed, both the filling machine and emptying machine work without pausing. In terms of a, b, and k, what is the least possible value of n?

(Raymond Feng)

Let $d = \gcd(a, b)$. The answer is

$$2(a+b-d) + (k \bmod d).$$

We view the problem through two models:

- the *discrete model* where cups are filled instantly at the end of each *a*-minute period, and cups are emptied instantly at the end of each *b*-minute period; and
- the *continuous model*, where cups are filled at a constant rate during each *a*-minute period, and cups are emptied at a constant rate during each *b*-minute period.

We begin by assuming d = 1.

Lower bound for d = 1: Assume that at some time, say t = 0, both the filling machine and the emptying machine are starting their next cycle. Suppose that c cups are fill at t = 0. Using the discrete model, it suffices to consider when t is a multiple of a or b.

- At t = ka, the number of full cups is $c + (ka \mod b)$, whose maximum value is c + b 1. For the machines to continue working without pausing, we must have $n \ge (c + b - 1) + a$.
- At $t = \ell b$, the number of full cups is $c (\ell b \mod a)$, whose minimum value is c a + 1. For the machines to continue workign without pausing, we must have $0 \le (c - a + 1) - b$.

Thus $n \ge 2(a + b - 1)$.

Upper bound for d = 1: Assume n = 2(a + b - 1), and consider the continuous model. Let t be the time and L be the total amount of liquid in the cups.

- When t is an integer and $L \ge a + b 1$, there is at most a 1 total liquid in (at most a) cups being filled and thus at least b totally filled cups. Hence the emptying machine is active and decreases L by 1 per minute.
- When t is an integer and $L \le a + b 1$, there is at most 1 total liquid in (at most b) cups being emptied and thus at least a totally empty cups. Hence the filling machine is active and increases L by 1 per minute.

At any point in time when $L \neq a + b - 1$, one of the machines is active and L gets 1 closer to a + b - 1 every minute. When L = a + b - 1, both machines remain active indefinitely, L remains constant.

Finish for d > 1: Using the discrete model, events only happen when t is a multiple of d, and moreover the amount of total liquid is always k (mod d).

Hence there is a bijection $x \mapsto (x - (k \mod d))/d$ mapping the current situation to the problem where d = 1. This gives the desired bound of

$$2(a+b-d) + (k \mod d).$$

N4. Let d(n) denote the number of positive divisors of n. The sequence a_0, a_1, a_2, \ldots is defined as follows: $a_0 = 1$, and for all integers $n \ge 1$,

$$a_n = d(a_{n-1}) + d(d(a_{n-2})) + \dots + \underbrace{d(d(\dots d(a_0) \dots))}_{n \text{ times}}.$$

Show that for all integers $n \ge 1$, we have $a_n \le 3n$.

(Karthik Vedula)

We first prove:

Lemma

 $d(n) \leq \sqrt{3n}$, with equality iff n = 12.

Proof. We proceed by strong induction. Assume $n \ge 2$ and the lemma holds for integers less than n. The result is clear if n is prime, so assume n composite. Let p be the largest prime dividing n and let $n = p^e \cdot m$, where $p \nmid m$.

If $p \ge 5$, then since $p^e \ge 5^e \ge (e+1)^2$,

$$d(n) = (e+1)d(m) \le (e+1)\sqrt{3m} \le \sqrt{3p^e m}$$

proving the inductive step.

Now our base case is numbers of the form $n = 2^x 3^y$. But $2^x \ge \frac{4}{9}(x+1)^2$ and $3^y \ge \frac{3}{4}(y+1)^2$, implying

$$2^{x}3^{y+1} \ge (x+1)^{2}(y+1)^{2},$$

with equality iff (x, y) = (2, 1).

Claim. Let $f(m) = \sqrt{3m}$ and let $\delta(n)$ be the largest number such that $f^{\delta(n)}(n) \ge 4$. For $n \ge 12$ we have $\delta(n) = 1$

$$\sum_{k=1}^{n-1} f^k(n) \le \sqrt{3} \cdot f(n) - 2.$$

Proof. We proceed by strong induction. If f(n) < 12, then $\delta(n) \le 2$, but $f(n) \le \sqrt{3}f(n) - 2$, proving the claim.

If $f(n) \ge 12$, the inductive hypothesis gives

$$\sum_{k=2}^{\delta(n)-1} f^k(n) \le \sqrt{3} \cdot f^2(n) - 2 \le (\sqrt{3} - 1)f(n) - 2,$$

proving the claim.

Then for $n \ge 26$,

$$a_n = \sum_{k=1}^n d^k(a_{n-k}) \le \left[\sum_{k=1}^{\delta(n)-1} f^k(a_{n-1})\right] + 5 + 3 + 2(n-3)$$
$$\le 2n + \sqrt{3}f(a_{n-1}) \le 2n + 3\sqrt{3(n-1)} \le 3n.$$

The rest is a finite case check.

N5. An ordered pair (k, n) of positive integers is *good* if there exists an ordered quadruple (a, b, c, d) of positive integers such that $a^3 + b^k = c^3 + d^k$ and abcd = n. Prove that there exist infinitely many positive integers n such that (2022, n) is not good but (2023, n) is good.

(Luke Robitaille)

First observe:

Claim 1. If $\nu_7(n) = 1$, then (2022, *n*) is not good.

Proof. Note $a^3, c^3 \in \{0, 1, -1\} \pmod{7}$ and $b^{2022}, d^{2022} \equiv \{0, 1\} \pmod{7}$, with exactly one of these equal to 0 mod 7. This is a clear contradiction to $a^3 + b^{2022} = c^3 + d^{2022}$.

Claim 2. If (2023, n) is good, then $(2023, nt^{4052})$ is good for every t.

Proof. If the quadruple (a, b, c, d) works, then $(at^{2023}, bt^3, ct^{2023}, dt^3)$ works.

Therefore it suffices to find a quadruple (a, b, c, d) with $a^3 + b^{2023} = c^3 + d^{2023}$ and $\nu_7(abcd) = 1$. Select a = 7, c = 6, and b and d with $7 \nmid b, d$ and $b \equiv d \equiv 1 \pmod{127}$. It follows that

$$x = \frac{d^{2023} - b^{2023}}{a^3 - c^3} \in \mathbb{Z}$$

Finally the quadruple $(ax^{674}, bx, cx^{674}, dx)$ works.

Remark. The key difficulty in construction said quadruple (a, b, c, d). The motivation is that for any a, b, c, d, if we define x as we did above, then $(ax^{674}, bx, cx^{674}, dx)$ is good. It will then suffice to force x to be an integer.