Elmo, Let Me Out

Year: 2022

24^{th} ELMO
PITTSBURGH, PA

Day: 1

Saturday, June 11, 2022
1:15PM–5:45PM EDT

Problem 1. Let $n > 1$ be an integer. The numbers $1, \ldots, n$ are written on a board. Aliceurill and Bobasaur take turns circling an uncircled number on the board, with Aliceurill going first. When the product of the circled numbers becomes a multiple of n, the game ends and the last player to have circled a number loses. For which values of n can Bobasaur guarantee victory?

Problem 2. Find all monic nonconstant polynomials P with integer coefficients for which there exist positive integers a and m such that for all positive integers $n \equiv a \pmod{m}$, $P(n)$ is nonzero, and

$$2022 \cdot \frac{(n+1)^{n+1} - n^n}{P(n)}$$

is an integer.

Problem 3. Let $\mathcal{P}$ be a regular 2022-gon with area 1. Find a real number c such that, if points A and B are chosen independently and uniformly at random on the perimeter of $\mathcal{P}$, then the probability that $AB \geq c$ is $\frac{1}{2}$.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.

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Year: 2022

Day: 2

24th ELMO
PITTSBURGH, PA

Sunday, June 19, 2022
1:15PM–5:45PM EDT

Problem 4. Let $ABCDE$ be a convex pentagon such that $\triangle ABE$, $\triangle BEC$, and $\triangle EDB$ are similar (with vertices in order). Lines BE and CD intersect at point T. Prove that line AT is tangent to the circumcircle of $\triangle ACD$.

Problem 5. Let $n \geq 3$ be a positive integer. There are n^3 users on a social media network called Everyone Likes Meeting Online (ELMO), and some pairs of these users are ELMObuddies. A set of at least three ELMO users forms an ELMOclub if and only if all pairs of members of the set are ELMObuddies. It is known that among every n users, some three form an ELMOclub. Prove that there is an ELMOclub with five members.

Problem 6. Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that, for all integers m and n,

$$f(f(m) - n) + f(f(n) - m) = f(m + n).$$

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.