Olympians Enjoy Mixed-up Letters

Year: **2021**

Day: **1**

23^{rd} ELMO
127.0.0.1

Thursday, June 17, 2021
2:00PM — 6:30PM EDT

Problem 1. In $\triangle ABC$, points P and Q lie on sides AB and AC, respectively, such that the circumcircle of $\triangle APQ$ is tangent to side BC at D. Let E lie on side BC such that $BD = EC$. Line DP intersects the circumcircle of $\triangle CDQ$ again at X, and line DQ intersects the circumcircle of $\triangle BDP$ again at Y. Prove that D, E, X, and Y are concyclic.

Problem 2. Let $n > 1$ be an integer and let $a_1, a_2, \ldots, a_n$ be integers such that $n \mid a_i - i$ for all integers $1 \leq i \leq n$. Prove there exists an infinite sequence $b_1, b_2, \ldots$ such that

- $b_k \in \{a_1, a_2, \ldots, a_n\}$ for all positive integers k, and
- $\displaystyle\sum_{k=1}^{\infty} \frac{b_k}{n^k}$ is an integer.

Problem 3. Each cell of a 100×100 grid is colored with one of 101 colors. A cell is *diverse* if, among the 199 cells in its row or column, every color appears at least once. Determine the maximum possible number of diverse cells.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.

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Year: **2021**

Day: **2**

Friday, June 18, 2021
2:00PM — 6:30PM EDT

Problem 4. The set of positive integers is partitioned into n disjoint infinite arithmetic progressions S_1, S_2, …, S_n with common differences $d_1, d_2, \ldots, d_n$, respectively. Prove that there exists exactly one index $1 \le i \le n$ such that

$$\frac{1}{d_i}\prod_{j=1}^{n} d_j \in S_i.$$

Problem 5. Let n and k be positive integers. Two infinite sequences $\{s_i\}_{i\ge1}$ and $\{t_i\}_{i\ge1}$ are *equivalent* if, for all positive integers i and j, $s_i = s_j$ if and only if $t_i = t_j$. A sequence $\{r_i\}_{i\ge1}$ has *equi-period* k if $r_1, r_2, \ldots$ and $r_{k+1}, r_{k+2}, \ldots$ are equivalent.

Suppose M infinite sequences with equi-period k whose terms are in the set $\{1, \ldots, n\}$ can be chosen such that no two chosen sequences are equivalent to each other. Determine the largest possible value of M in terms of n and k.

Problem 6. In $\triangle ABC$, points D, E, and F lie on sides BC, CA, and AB, respectively, such that each of the quadrilaterals $AFDE$, $BDEF$, and $CEFD$ has an incircle. Prove that the inradius of $\triangle ABC$ is twice the inradius of $\triangle DEF$.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.