ELMO Literally Moved Online

Year: **2020**

22nd ELMO
127.0.0.1

Day: **1**

Monday, July 20, 2020
2:00PM — 6:30PM EDT

Problem 1. Let $\mathbb{N}$ be the set of all positive integers. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ such that*

$$f^{f^{f(x)}(y)}(z) = x + y + z + 1$$

for all $x, y, z \in \mathbb{N}$.

Problem 2. Define the Fibonacci numbers by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Let k be a positive integer. Suppose that for every positive integer m there exists a positive integer n such that $m \mid F_n - k$. Must k be a Fibonacci number?

Problem 3. Janabel has a device that, when given two distinct points U and V in the plane, draws the perpendicular bisector of UV. Show that if three lines forming a triangle are drawn, Janabel can mark the orthocenter of the triangle using this device, a pencil, and no other tools.

*Here, $f^a(b)$ denotes the result of a repeated applications of f to b. Formally, we define $f^1(b) = f(b)$, and $f^{a+1}(b) = f(f^a(b))$ for all $a > 0$.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.

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Problem 4. Let acute scalene triangle ABC have orthocenter H and altitude AD with D on side BC. Let M be the midpoint of side BC, and let D' be the reflection of D over M. Let P be a point on line $D'H$ such that lines AP and BC are parallel, and let the circumcircles of $\triangle AHP$ and $\triangle BHC$ meet again at $G \neq H$. Prove that $\angle MHG = 90^\circ$.

Problem 5. Let m and n be positive integers. Find the smallest positive integer s for which there exists an $m \times n$ rectangular array of positive integers such that

- each row contains n distinct consecutive integers in some order,
- each column contains m distinct consecutive integers in some order, and
- each entry is less than or equal to s.

Problem 6. For any positive integer n, let

- $\tau(n)$ denote the number of positive integer divisors of n,
- $\sigma(n)$ denote the sum of the positive integer divisors of n, and
- $\varphi(n)$ denote the number of positive integers less than or equal to n that are relatively prime to n.

Let $a, b > 1$ be integers. Brandon has a calculator with three buttons that replace the integer n currently displayed with $\tau(n)$, $\sigma(n)$, or $\varphi(n)$, respectively. Prove that if the calculator currently displays a, then Brandon can make the calculator display b after a finite (possibly empty) sequence of button presses.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.