Exclusively carL-Made Olympiad

Year: 2019

21st ELMO
Pittsburgh, PA

Day: 1

Saturday, June 8, 2019
1:15PM — 5:45PM

Problem 1. Let $P(x)$ be a polynomial with integer coefficients such that $P(0) = 1$, and let $c > 1$ be an integer. Define $x_0 = 0$ and $x_{i+1} = P(x_i)$ for all integers $i \geq 0$. Show that there are infinitely many positive integers n such that $\gcd(x_n, n + c) = 1$.

Problem 2. Let $m, n \geq 2$ be integers. Carl is given n marked points in the plane and wishes to mark their centroid*. He has no standard compass or straightedge. Instead, he has a device which, given marked points A and B, marks the $m - 1$ points that divides segment $\overline{AB}$ into m congruent parts (but does not draw the segment).

For which pairs (m, n) can Carl necessarily accomplish his task, regardless of which n points he is given?

Problem 3. Let $n \geq 3$ be a fixed integer. A game is played by n players sitting in a circle. Initially, each player draws three cards from a shuffled deck of $3n$ cards numbered $1, 2, \ldots, 3n$. Then, on each turn, every player simultaneously passes the smallest-numbered card in their hand one place clockwise and the largest-numbered card in their hand one place counterclockwise, while keeping the middle card.

Let T_r denote the configuration after r turns (so T_0 is the initial configuration). Show that T_r is eventually periodic with period n, and find the smallest integer m for which, regardless of the initial configuration, $T_m = T_{m+n}$.

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*Here, the *centroid* of n points with coordinates $(x_1, y_1), \ldots, (x_n, y_n)$ is the point whose coordinates are $\left(\frac{x_1+\cdots+x_n}{n}, \frac{y_1+\cdots+y_n}{n}\right)$.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.

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Year: 2019

Day: 2

21st ELMO
Pittsburgh, PA

Sunday, June 16, 2019
1:15PM — 5:45PM

Problem 4. Carl is given three distinct non-parallel lines ℓ_1, ℓ_2, ℓ_3 and a circle ω in the plane. In addition to a normal straightedge, Carl has a special straightedge which, given a line ℓ and a point P, constructs a new line passing through P parallel to ℓ. (Carl does not have a compass.) Show that Carl can construct a triangle with circumcircle ω whose sides are parallel to ℓ_1, ℓ_2, ℓ_3 in some order.

Problem 5. Let S be a nonempty set of positive integers so that, for any (not necessarily distinct) integers a and b in S, the number $ab + 1$ is also in S. Show that the set of primes that do not divide any element of S is finite.

Problem 6. Snorlax chooses a *functional expression*[†] E which is a finite nonempty string formed from a set x_1, x_2, …, of variables and applications of a function f, together with addition, subtraction, multiplication (but not division), and fixed real constants. He then considers the equation $E = 0$, and lets S denote the set of functions $f\colon \mathbb{R} \to \mathbb{R}$ such that the equation holds for any choices of real numbers x_1, …, x_k. (For example, if Snorlax chooses the functional equation

$$f(2f(x_1) + x_2) - 2f(x_1) - x_2 = 0,$$

then S consists of one function, the identity function.)

(a) Let X denote the set of functions with domain $\mathbb{R}$ and image exactly $\mathbb{Z}$. Show that Snorlax can choose his functional equation such that S is nonempty but $S \subseteq X$.

(b) Can Snorlax choose his functional equation such that $|S| = 1$ and $S \subseteq X$?

[†]These can be defined formally in the following way: the set of functional expressions is the minimal one (by inclusion) such that (i) any fixed real constant is a functional expression, (ii) for any integer i, the variable x_i is a functional expression, and (iii) if V and W are functional expressions, then so are $f(V)$, $V + W$, $V - W$, and $V \cdot W$.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.