Exclusively carL-Made Olympiad

Saturday, June 8, 2019
1:15PM — 5:45PM

Problem 1. Let \( P(x) \) be a polynomial with integer coefficients such that \( P(0) = 1 \), and let \( c > 1 \) be an integer. Define \( x_0 = 0 \) and \( x_{i+1} = P(x_i) \) for all integers \( i \geq 0 \). Show that there are infinitely many positive integers \( n \) such that \( \gcd(x_n, n + c) = 1 \).

Problem 2. Let \( m, n \geq 2 \) be integers. Carl is given \( n \) marked points in the plane and wishes to mark their centroid*. He has no standard compass or straightedge. Instead, he has a device which, given marked points \( A \) and \( B \), marks the \( m - 1 \) points that divides segment \( AB \) into \( m \) congruent parts (but does not draw the segment).

For which pairs \((m, n)\) can Carl necessarily accomplish his task, regardless of which \( n \) points he is given?

Problem 3. Let \( n \geq 3 \) be a fixed integer. A game is played by \( n \) players sitting in a circle. Initially, each player draws three cards from a shuffled deck of \( 3n \) cards numbered \( 1, 2, \ldots, 3n \). Then, on each turn, every player simultaneously passes the smallest-numbered card in their hand one place clockwise and the largest-numbered card in their hand one place counterclockwise, while keeping the middle card.

Let \( T_r \) denote the configuration after \( r \) turns (so \( T_0 \) is the initial configuration). Show that \( T_r \) is eventually periodic with period \( n \), and find the smallest integer \( m \) for which, regardless of the initial configuration, \( T_m = T_{m+n} \).

*Here, the centroid of \( n \) points with coordinates \( (x_1, y_1), \ldots, (x_n, y_n) \) is the point whose coordinates are \( \left( \frac{x_1 + \cdots + x_n}{n}, \frac{y_1 + \cdots + y_n}{n} \right) \).

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.
Problem 4. Carl is given three distinct non-parallel lines $\ell_1$, $\ell_2$, $\ell_3$ and a circle $\omega$ in the plane. In addition to a normal straightedge, Carl has a special straightedge which, given a line $\ell$ and a point $P$, constructs a new line passing through $P$ parallel to $\ell$. (Carl does not have a compass.) Show that Carl can construct a triangle with circumcircle $\omega$ whose sides are parallel to $\ell_1$, $\ell_2$, $\ell_3$ in some order.

Problem 5. Let $S$ be a nonempty set of positive integers so that, for any (not necessarily distinct) integers $a$ and $b$ in $S$, the number $ab + 1$ is also in $S$. Show that the set of primes that do not divide any element of $S$ is finite.

Problem 6. Snorlax chooses a functional expression $E$ which is a finite nonempty string formed from a set $x_1$, $x_2$, $\ldots$, of variables and applications of a function $f$, together with addition, subtraction, multiplication (but not division), and fixed real constants. He then considers the equation $E = 0$, and lets $S$ denote the set of functions $f: \mathbb{R} \to \mathbb{R}$ such that the equation holds for any choices of real numbers $x_1$, $\ldots$, $x_k$. (For example, if Snorlax chooses the functional equation

$$f(2f(x_1) + x_2) - 2f(x_1) - x_2 = 0,$$

then $S$ consists of one function, the identity function.)

(a) Let $X$ denote the set of functions with domain $\mathbb{R}$ and image exactly $\mathbb{Z}$. Show that Snorlax can choose his functional equation such that $S$ is nonempty but $S \subseteq X$.

(b) Can Snorlax choose his functional equation such that $|S| = 1$ and $S \subseteq X$?

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$\dagger$These can be defined formally in the following way: the set of functional expressions is the minimal one (by inclusion) such that (i) any fixed real constant is a functional expression, (ii) for any integer $i$, the variable $x_i$ is a functional expression, and (iii) if $V$ and $W$ are functional expressions, then so are $f(V)$, $V + W$, $V - W$, and $V \cdot W$. 

Time limit: 4 hours 30 minutes. Each problem is worth 7 points.