

# Shortlisted Problems

20<sup>th</sup> ELMO

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## Note of Confidentiality

The shortlisted problems should be kept strictly confidential until disclosed publicly by the committee on the ELMO.

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# Problems

**A1.** Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a bijective function. Does there always exist an infinite number of functions  $g: \mathbb{R} \rightarrow \mathbb{R}$  such that  $f(g(x)) = g(f(x))$  for all  $x \in \mathbb{R}$ ?

(Daniel Liu)

**A2.** Let  $a_1, a_2, \dots, a_m$  be a finite sequence of positive integers. Prove that there exist nonnegative integers  $b, c$ , and  $N$  such that

$$\left\lfloor \sum_{i=1}^m \sqrt{n + a_i} \right\rfloor = \left\lfloor \sqrt{bn + c} \right\rfloor$$

holds for all integers  $n > N$ .

(Carl Schildkraut)

**A3.** Let  $a, b, c, x, y, z$  be positive reals such that  $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$ . Prove that

$$a^x + b^y + c^z \geq \frac{4abcxyz}{(x + y + z - 3)^2}.$$

(Daniel Liu)

**A4.** Elmo calls a monic polynomial with real coefficients *tasty* if all of its coefficients are in  $[-1, 1]$ . A monic polynomial  $P$  with real coefficients and complex roots  $\chi_1, \dots, \chi_m$  (counted with multiplicity) is given to Elmo, and he discovers that there does not exist a monic polynomial  $Q$  with real coefficients such that  $P \cdot Q$  is tasty. Find all possible values of  $\max(|\chi_1|, \dots, |\chi_m|)$ .

(Carl Schildkraut)

**C1.** Let  $n$  be a positive integer. There are  $2018n + 1$  cities in the Kingdom of Sellke Arabia. King Mark wants to build two-way roads that connect certain pairs of cities such that for each city  $C$  and integer  $1 \leq i \leq 2018$ , there are exactly  $n$  cities that are a distance  $i$  away from  $C$ . (The *distance* between two cities is the least number of roads on any path between the two cities.)

For which  $n$  is it possible for Mark to achieve this?

(Michael Ren)

**C2.** We say that a positive integer  $n$  is *m-expressible* if one can write an expression evaluating to  $n$  in base 10, where the expression consists only of

- exactly  $m$  numbers from the set  $\{0, 1, \dots, 9\}$
- the six operations  $+$ ,  $-$ ,  $\times$ ,  $\div$ , exponentiation  $^$ , concatenation  $\oplus$ , and
- some number (possibly zero) of left and right parentheses.

For example, 5625 is 3-expressible (in two ways), as  $5625 = 5 \oplus (5^4) = (7 \oplus 5)^2$ , say. Does there exist a positive integer  $A$  such that all positive integers with  $A$  digits are  $(A - 1)$ -expressible?

(Krit Boonsiriseth)

**C3.** A *windmill* in the plane consists of a line segment of unit length with a distinguished endpoint, the *pivot*. Geoff has a finite set of windmills, such that no two windmills intersect, and any two pivots are distance more than  $\sqrt{2}$  apart. In an operation, Geoff can choose a windmill and rotate it about its pivot, either clockwise or counterclockwise and by any amount, as long as no two windmills intersect during or after the rotation. Show that Geoff can, in finitely many operations, rotate the windmills so that they all point in the same direction.

(Michael Ren)

**G1.** Let  $ABC$  be an acute triangle with orthocenter  $H$ , and let  $P$  be a point on the nine-point circle of  $ABC$ . Lines  $BH$ ,  $CH$  meet the opposite sides  $AC$ ,  $AB$  at  $E$ ,  $F$ , respectively. Suppose that the circumcircles of  $\triangle EHP$  and  $\triangle FHP$  intersect lines  $CH$ ,  $BH$  a second time at  $Q$ ,  $R$ , respectively. Show that as  $P$  varies along the nine-point circle of  $ABC$ , the line  $QR$  passes through a fixed point.

(Brandon Wang)

**G2.** Let  $ABC$  be a scalene triangle with orthocenter  $H$  and circumcenter  $O$ . Let  $P$  be the midpoint of  $\overline{AH}$  and let  $T$  be on line  $BC$  with  $\angle TAO = 90^\circ$ . Let  $X$  be the foot of the altitude from  $O$  onto line  $PT$ . Prove that the midpoint of  $\overline{PX}$  lies on the nine-point circle of  $\triangle ABC$ .

(Zack Chroman)

**G3.** Let  $A$  be a point in the plane, and  $\ell$  a line not passing through  $A$ . Evan doesn't have a straightedge, but instead has a special compass which has the ability to draw a circle through three distinct noncollinear points. (The center of the circle is *not* marked in this process.) Additionally, Evan can mark the intersections between two objects drawn, and can mark an arbitrary point on a given object or on the plane.

- (i) Can Evan construct the reflection of  $A$  over  $\ell$ ?
- (ii) Can Evan construct the foot of the altitude from  $A$  to  $\ell$ ?

(Zack Chroman)

**G4.** Let  $ABCDEF$  be a convex hexagon inscribed in a circle  $\Omega$  such that triangles  $ACE$  and  $BDF$  have the same orthocenter. Suppose that  $\overline{BD}$  and  $\overline{DF}$  intersect  $\overline{CE}$  at  $X$  and  $Y$ , respectively. Show that there is a point common to  $\Omega$ , the circumcircle of  $DXY$ , and the line through  $A$  perpendicular to  $\overline{CE}$ .

(Michael Ren and Vincent Huang)

**G5.** Let scalene triangle  $ABC$  have altitudes  $\overline{AD}$ ,  $\overline{BE}$ ,  $\overline{CF}$  and circumcenter  $O$ . The circumcircles of  $\triangle ABC$  and  $\triangle ADO$  meet at  $P \neq A$ . The circumcircle of  $\triangle ABC$  meets lines  $\overline{PE}$  at  $X \neq P$  and  $\overline{PF}$  at  $Y \neq P$ . Prove that  $\overline{XY} \parallel \overline{BC}$ .

(Daniel Hu)

**N1.** Determine all nonempty finite sets  $S = \{a_1, \dots, a_n\}$  of  $n$  distinct positive integers such that  $a_1 \cdots a_n$  divides  $(x + a_1) \cdots (x + a_n)$  for every positive integer  $x$ .

(Ankan Bhattacharya)

**N2.** Call a number  $n$  *good* if it can be expressed in the form  $2^x + y^2$  where  $x$  and  $y$  are nonnegative integers.

- Prove that there exist infinitely many sets of 4 consecutive good numbers.
- Find all sets of 5 consecutive good numbers.

(Michael Ma)

**N3.** Let  $a_1, a_2, \dots$  be an infinite sequence of positive integers satisfying  $a_1 = 1$  and

$$a_n \mid a_k + a_{k+1} + \cdots + a_{k+n-1}$$

for all positive integers  $k$  and  $n$ . Find the maximum possible value of  $a_{2018}$ .

(Krit Boonsiriseth)

**N4.** Fix a positive integer  $n > 1$ . We say a nonempty subset  $S$  of  $\{0, 1, \dots, n-1\}$  is  $d$ -coverable if there exists a polynomial  $P$  with integer coefficients and degree at most  $d$ , such that  $S$  is exactly the set of residues modulo  $n$  that  $P$  attains as it ranges over the integers.

For each  $n$ , determine the smallest  $d$  such that any nonempty subset of  $\{0, \dots, n-1\}$  is  $d$ -coverable, or prove that no such  $d$  exists.

(Carl Schildkraut)

## Solutions

**A1.** Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a bijective function. Does there always exist an infinite number of functions  $g: \mathbb{R} \rightarrow \mathbb{R}$  such that  $f(g(x)) = g(f(x))$  for all  $x \in \mathbb{R}$ ?

*(Daniel Liu)*

Yes. It's clear  $f^0, f^1, f^2, \dots$  all commute with  $f$ . If  $f$  doesn't have finite order this collection is infinite and valid.

Else, suppose that  $f^n = \text{id}$ , where  $n$  is minimal. If  $n = 1$  the problem is clear, so suppose  $n > 1$ . Then  $f$  is composed of some cycles; some cycle length  $d \mid n$  appears infinitely many times. Let a countable number of these cycles be  $x_{r,1} \rightarrow x_{r,2} \rightarrow \dots \rightarrow x_{r,d} \rightarrow x_{r,1}$  for  $r \in \mathbb{Z}$ .

Then for every integer  $s$ , create a new function  $h_s$  fixing everything except the  $x_{k,\ell}$ , and send every  $x_{r,a} \rightarrow x_{r+s,a}$ . It is clear that every  $h_s$  commutes with  $f$ .

This gives infinitely many  $g$ , unless all but finitely many of the cycles have length 1. In that case, we can find more  $g$  by swapping any two fixed points of  $f$  and leaving everything else intact.

**A2.** Let  $a_1, a_2, \dots, a_m$  be a finite sequence of positive integers. Prove that there exist nonnegative integers  $b, c$ , and  $N$  such that

$$\left\lfloor \sum_{i=1}^m \sqrt{n + a_i} \right\rfloor = \left\lfloor \sqrt{bn + c} \right\rfloor$$

holds for all integers  $n > N$ .

(Carl Schildkraut)

If all the  $a_i$  are equal, then  $\sum_{i=1}^m \sqrt{n + a_i} = \sqrt{m^2 n + m^2 a_1}$  and so  $(b, c) = (m^2, m^2 a_1)$  works fine.

Let us assume this is not the case. Instead, will take  $b = m^2$  and  $c = m(a_1 + \dots + a_m) - 1$  and claim it works for  $N$  large enough.

On the one hand,

$$\begin{aligned} \sum_{i=1}^m \sqrt{n + a_i} &< m \cdot \sqrt{n + \frac{a_1 + \dots + a_m}{m}} \\ &= \sqrt{m^2 \cdot n + c + 1} \leq \left\lceil \sqrt{m^2 \cdot n + c + 1} \right\rceil \leq \left\lfloor \sqrt{m^2 \cdot n + c} \right\rfloor + 1. \end{aligned}$$

On the other hand, let  $\lambda = \frac{c}{2(c+1)} < \frac{1}{2}$ . We use the following estimate.

**Claim.** If  $n$  is large enough in terms of  $(a_1, \dots, a_m)$  then  $\sqrt{n + a_i} \geq \sqrt{n} + \frac{\lambda a_i}{\sqrt{n}}$ .

*Proof.* Squaring both sides, it's equivalent to  $a_i \geq 2\lambda \cdot a_i + \frac{\lambda^2 a_i^2}{n}$ , which holds for  $n$  big enough as  $2\lambda < 1$ .  $\square$

Now,

$$\begin{aligned} \sum_{i=1}^m \sqrt{n + a_i} &\geq \sum_{i=1}^m \left( \sqrt{n} + \frac{\lambda a_i}{\sqrt{n}} \right) \\ &\geq m\sqrt{n} + \frac{\lambda \cdot (a_1 + \dots + a_m)}{\sqrt{n}} \\ &= m\sqrt{n} + \frac{\lambda \cdot (c + 1)}{m\sqrt{n}} \\ &= m\sqrt{n} + \frac{c}{2m\sqrt{n}} > \sqrt{m^2 \cdot n + c} \geq \left\lfloor \sqrt{m^2 n + c} \right\rfloor. \end{aligned}$$

This finishes the problem.

**Remark.** Obviously,  $b = m^2$  for asymptotic reasons (by taking  $n$  large). As for possible values of  $c$ :

- If  $a_1 = \dots = a_m$ , then one can show  $c = m(a_1 + \dots + a_m)$  is the only valid choice. Indeed, taking  $n$  of the form  $n = k^2 - a$  and  $n = \frac{k^2 - 1}{m^2} - a$  is enough to see this.
- But if not all  $a_i$  are equal, the natural guess of taking  $c = m(a_1 + \dots + a_m)$  is not valid in general. For example, we have that

$$\left\lfloor \sqrt{n} + \sqrt{n+2} \right\rfloor \neq \left\lfloor \sqrt{4n+4} \right\rfloor \quad n \in \{t^2 - 1 \mid t = 2, 3, \dots\}.$$

I think one can actually figure out exactly which  $c$  are valid, though the answer will depend on some quadratic residues, and we do not pursue this line of thought here.

So any correct solutions must distinguish these two cases.



**A3.** Let  $a, b, c, x, y, z$  be positive reals such that  $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$ . Prove that

$$a^x + b^y + c^z \geq \frac{4abcxyz}{(x + y + z - 3)^2}.$$

(Daniel Liu)

We present three solutions.

**First solution, proof without words (by proposer)**

$$\begin{aligned} a^x + b^y + c^z &= yz \cdot \frac{a^x}{yz} + zx \cdot \frac{a^y}{zx} + xy \cdot \frac{a^z}{xy} \\ &\geq (xy + yz + zx) \left( \left( \frac{a^x}{yz} \right)^{yz} \left( \frac{b^y}{zx} \right)^{zx} \left( \frac{c^z}{xy} \right)^{xy} \right)^{\frac{1}{xy+yz+zx}} \\ &= (xy + yz + zx) \cdot \frac{(abc)^{\frac{xyz}{xy+yz+zx}}}{x^{\frac{xy+zx}{xy+yz+zx}} y^{\frac{yz+xy}{xy+yz+zx}} z^{\frac{zx+yz}{xy+yz+zx}}} \\ &\geq (xy + yz + zx) \cdot \frac{(abc)^{\frac{xyz}{xy+yz+zx}}}{\left( \frac{x \cdot \frac{xy+zx}{xy+yz+zx} + y \cdot \frac{yz+xy}{xy+yz+zx} + z \cdot \frac{zx+yz}{xy+yz+zx}}{2} \right)^2} \\ &= (xy + yz + zx) \cdot \frac{4(abc)^{\frac{xyz}{xy+yz+zx}}}{\left( \sum_{\text{cyc}} x \cdot \left( 1 - \frac{yz}{xy+yz+zx} \right) \right)^2} \\ &= \frac{4abc(xy + yz + zx)}{\left( x + y + z - 3 \frac{xyz}{xy+yz+zx} \right)^2} \\ &= \frac{4abcxyz}{(x + y + z - 3)^2}. \end{aligned}$$

**Second solution, by weighted AM-GM (Andrew Gu)** By weighted AM-GM,

$$\frac{1}{x} \cdot xa^x + \frac{1}{y} \cdot yb^y + \frac{1}{z} \cdot zc^z \geq x^{\frac{1}{x}} y^{\frac{1}{y}} z^{\frac{1}{z}} abc.$$

Hence it suffices to show

$$x^{\frac{1}{x}} y^{\frac{1}{y}} z^{\frac{1}{z}} \geq \frac{4xyz}{(x + y + z - 3)^2}.$$

By weighted AM-GM,

$$2x^{\frac{1}{2}(1-\frac{1}{x})} y^{\frac{1}{2}(1-\frac{1}{y})} z^{\frac{1}{2}(1-\frac{1}{z})} \leq 2 \cdot \frac{1}{2} (x - 1 + y - 1 + z - 1) = x + y + z - 3.$$

Squaring both sides and rearranging proves the required inequality.

**Third solution, by Hölder and Schur/Muirhead (Evan Chen)** By Hölder and weighted AM-GM we have

$$\sqrt{\left( \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \right)} (a^x + b^y + c^z) \geq \frac{1}{x} \cdot a^{x/2} + \frac{1}{y} \cdot b^{y/2} + \frac{1}{z} \cdot c^{z/2} \geq (abc)^{1/2}.$$

Hence, it suffices to prove that

$$(x + y + z - 3)^2 \geq 4xyz(1/x^2 + 1/y^2 + 1/z^2) \quad \forall \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$$

which is a 3-variable symmetric inequality. It also happens to be is MOP 2011, K4.1, done in my SOS handout. We give a proof below (with  $a = 1/x$ , etc).

**Claim** (Black MOP 2011, Test 4, Problem 1). If  $a, b, c > 0$  then

$$\left( (a + b + c) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) - 3 \right)^2 \geq 4 \left( \frac{(a + b + c)(a^2 + b^2 + c^2)}{abc} \right)$$

*Proof.* Expanding and clearing denominators it's just

$$\sum_{\text{sym}} a^4 b^2 + \sum_{\text{cyc}} a^3 b^3 + 6a^2 b^2 c^2 \geq 2 \sum_{\text{cyc}} a^4 bc + 2 \sum_{\text{sym}} a^3 b^2 c$$

which can also be written as

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & 0 & 0 & \\ & & & 1 & -2 & 1 & \\ & & 2 & -2 & -2 & 2 & \\ & 1 & -2 & 6 & -2 & 1 & \\ 0 & -2 & -2 & -2 & -2 & 0 & \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 \end{array}$$

in Chinese dumbass notation. This rewrites as

$$\sum_{\text{cyc}} a^4 (b - c)^2 + 2 \sum_{\text{cyc}} ab(ab - bc)(ab - ac) \geq 0$$

which is evident (the latter sum is “upsidedown triangle Schur”). □

**A4.** Elmo calls a monic polynomial with real coefficients *tasty* if all of its coefficients are in  $[-1, 1]$ . A monic polynomial  $P$  with real coefficients and complex roots  $\chi_1, \dots, \chi_m$  (counted with multiplicity) is given to Elmo, and he discovers that there does not exist a monic polynomial  $Q$  with real coefficients such that  $P \cdot Q$  is tasty. Find all possible values of  $\max(|\chi_1|, \dots, |\chi_m|)$ .

(Carl Schildkraut)

We claim the answer is  $r > 1$ .

**Part 1: Any value of  $r > 1$  can be achieved.** To prove this, we will show that the polynomial

$$P(x) = x^n - r^n$$

has no tasty multiples if  $r^n \geq 2$  (such an  $n$  exists because  $r > 1$ ). Set  $M = r^n$ . Assume we have a polynomial

$$R(x) = \sum_{i=0}^N a_i x^i$$

so that  $-1 \leq a_i \leq 1$  for all  $i$  ( $a_N = 1$ ) and  $P|R$ . Taking  $R$  modulo  $P$ , we get that, with  $N = bn + c$  and  $0 \leq c < n$  (setting  $a_k = 0$  if  $k > N$ ),

$$R(x) = \sum_{j=0}^{n-1} \sum_{k=0}^b a_{kn+j} x^{kn+j} \equiv \sum_{j=0}^{n-1} x^j \left[ \sum_{k=0}^b a_{kn+j} R^k \right].$$

We have this must be the zero polynomial (since  $P|R$ ); specifically, taking  $j = c$ ,

$$\begin{aligned} \sum_{k=0}^b a_{nk+c} R^k &= 0 \\ \sum_{k=0}^{b-1} (-a_{nk+c}) R^k &= a_{bn+c} R^b \\ \sum_{k=0}^{b-1} |a_{nk+c}| R^k &\geq R^b \end{aligned}$$

(since  $a_{bn+c} = a_N = 1$ ). However, since  $|a_{nk+c}| \leq 1$ , we then have

$$\begin{aligned} \sum_{k=0}^{b-1} R^k &\geq R^b \\ \frac{R^b - 1}{R - 1} &\geq R^b \\ R^b - 1 &\geq R^{b+1} - R^b \\ R^b(2 - R) &\geq 1. \end{aligned}$$

However, as  $R \geq 2$ , this is false.

**Part 2: Any polynomial with  $r \leq 1$  has a tasty multiple.** Define the *sparsity* of a polynomial to be the greatest common divisor of the exponents  $m$  for which the coefficient of  $x^m$  in  $P$  is not zero. Equivalently, it is the largest integer  $d$  so that  $P(x) = Q(x^d)$  for some polynomial  $Q$ .

We prove the following theorem:

**Theorem.** *Given any complex number  $z$  for which  $|z| \leq 1$ , there exist tasty polynomials with  $z$  as a root that have arbitrarily large sparsities.*

*Proof.* Let  $z = re^{i\theta}$ . If  $\theta$  is a rational multiple of  $\pi$  (say,  $\theta = a\pi/b$ ), then we take the polynomial  $x^{bn} - r^{bn}$  for any integer  $n$ ; this has sparsity  $bn$  and is tasty (as  $r \leq 1$ ,  $r^{bn} \leq 1$ ). So, it suffices to prove this in the case where  $\theta$  is not a rational multiple of  $\pi$ , and we henceforth assume this.

We claim that, for infinitely many  $n$ , the polynomial

$$x^{2n} - 2 \cos(n\theta) r^n x^n + r^{2n}$$

is tasty (note that this polynomial has sparsity  $n$  and as such the theorem is implied by this claim). First note that this polynomial reduces to

$$x^n = r^n e^{\pm ni\theta} = \left( r e^{\pm i\theta} \right)^n,$$

which is true at  $x = r e^{i\theta} = z$ , so  $z$  is in fact a root.

We recall the following lemma:

**Lemma.** *For any real number  $\phi$  which is not a rational multiple of  $\pi$ , the sequence  $a_n = \cos(n\phi)$  has infinitely many terms in the range  $[-1/2, 1/2]$ .*

Indeed, let  $\{x\}$  be the fractional part of  $x$ , and consider the sequence

$$\alpha_n = \left\{ \frac{n\phi}{2\pi} \right\}.$$

We see that  $-1/2 \leq a_n \leq 1/2$  iff  $1/6 \leq \alpha_n \leq 1/3$  or  $2/3 \leq \alpha_n \leq 5/6$ . It is well known that the sequence  $x_n = \{nx\}$  is dense in  $[0, 1]$  for any irrational  $x$ , so this is true. Thus, for infinitely many  $n$ , as  $\theta$  has been assumed not to be a rational multiple of  $\pi$ , the coefficients of  $P$  are bounded above in absolute value by  $r^n$  and  $r^{2n}$  for infinitely many  $n$ , both of which are  $\leq 1$  as  $r \leq 1$ .  $\square$

We now provide a second lemma.

**Lemma.** *If  $P(x)$  and  $Q(x)$  are both tasty polynomials and the sparsity  $D$  of  $P$  is greater than the degree  $d$  of  $Q$ , then the product  $R(x) = P(x)Q(x)$  is also tasty.*

*Proof.* Write

$$P(x) = \sum_{j=0}^s a_j x^{Dj}, \quad Q(x) = \sum_{k=0}^d b_k x^k.$$

Then,

$$P(x)Q(x) = \sum_{j=0}^s \sum_{k=0}^d a_j b_k x^{Dj+k}.$$

As  $D > d$ , none of these terms interfere with one another (for each integer  $n$ , there is at most one choice of  $0 \leq j \leq s$ ,  $0 \leq k \leq d$  so that  $Dj + k = n$ ), so the coefficients of  $R(x)$  are just the values of  $a_j b_k$  as  $j$  and  $k$  range over the desired range; as each  $a_j$  and  $b_k$  are of magnitude  $\leq 1$ , each pairwise product is as well, finishing the proof.  $\square$

Given a polynomial  $P$  with roots  $\chi_1, \dots, \chi_m$  in  $\mathbb{C}$  (possibly with duplicates), we will inductively construct the polynomial  $R(x)$  that is tasty and that  $P$  divides. We define a sequence of polynomials  $R_0, \dots, R_m$  so that  $R_0(x) = 1$ , and for each  $0 < k \leq m$ , we take a tasty polynomial  $P_k(x)$  with root  $\chi_k$  and sparsity greater than the degree of  $R_{k-1}$ , and take  $R_k(x) = R_{k-1}(x)P_k(x)$ . Such a  $P_k(x)$  is guaranteed to exist by our theorem, and the product  $R_{k-1}(x)P_k(x)$  is guaranteed to be tasty by our lemma. Thus, we may take  $R = R_m$ , finishing the proof.

**Remark.** A polynomial  $P$  that has a tasty multiple exists for all  $r < 2$ : We have upon fixing  $r < 2$  that for large enough  $n$ , we know  $r^n - r^{n-1} - \dots - r - 1 \leq 0$ . If  $n$  is minimal,  $r^n - r^{n-1} - \dots - r > 0$ , and we can thus take some value  $0 \leq c \leq 1$  for the constant term by the intermediate value theorem so that  $P(x) = x^n - x^{n-1} - \dots - x - c$  has a root at  $r$ . If  $r \geq 2$ , then  $n = 1$  can be taken in Part 1 and thus no tasty multiples exist.

**C1.** Let  $n$  be a positive integer. There are  $2018n + 1$  cities in the Kingdom of Sellke Arabia. King Mark wants to build two-way roads that connect certain pairs of cities such that for each city  $C$  and integer  $1 \leq i \leq 2018$ , there are exactly  $n$  cities that are a distance  $i$  away from  $C$ . (The *distance* between two cities is the least number of roads on any path between the two cities.)

For which  $n$  is it possible for Mark to achieve this?

*(Michael Ren)*

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The answer is  $n$  even.

To see that  $n$  odd fails, note that by taking  $i = 1$  we see the graph is  $n$ -regular; since it has an odd number of vertices we need  $n$  to be even.

On the other hand, if  $n$  is even, then consider the graph formed by taking the vertices of a regular  $(2018n + 1)$ -gon and drawing edges between vertices which are at most  $n/2$  apart. Then this works.

**C2.** We say that a positive integer  $n$  is  $m$ -expressible if one can write an expression evaluating to  $n$  in base 10, where the expression consists only of

- exactly  $m$  numbers from the set  $\{0, 1, \dots, 9\}$
- the six operations  $+$ ,  $-$ ,  $\times$ ,  $\div$ , exponentiation  $^$ , concatenation  $\oplus$ , and
- some number (possibly zero) of left and right parentheses.

For example, 5625 is 3-expressible (in two ways), as  $5625 = 5 \oplus (5^4) = (7 \oplus 5)^2$ , say. Does there exist a positive integer  $A$  such that all positive integers with  $A$  digits are  $(A - 1)$ -expressible?

(Krit Boonsiriseth)

Here is a solution by Evan Chen achieving  $A = 6 \cdot 10^6$ , and reprising the joke “six consecutive zeros”.

We will replace “exactly  $m$  numbers” with “at most  $m$  numbers”, since this is the same. Suppose we group the digits of  $N$  into base 1000000, so that we have

$$N = s_1 s_2 s_3 \dots s_m$$

where each  $s_m$  is a group of six digits ( $s_1$  padded with leading zeros, if needed, but  $s_1 \neq 000000$ ). We consider two cases.

- Suppose some group is zero; then we find that  $N$  has six consecutive zeros in its decimal representations. Thus  $N$  has the form

$$N = X \oplus (b \cdot (1 \oplus 0)^6) \oplus Y$$

for some strings  $X$  and  $Y$  (possibly empty), which are formed by repeated concatenation.

- Otherwise, note that  $m \geq 10^6$ . By a classical pigeonhole argument there exist indices  $i < j$  such that  $s_i + \dots + s_j \equiv 0 \pmod{999999}$ . Let  $n = \frac{1}{999999} s_i \dots s_j$ . Then we can write

$$N = X \oplus [(1 \oplus 0)^6 - 1] \cdot n \oplus Y$$

for strings  $X = s_1 \dots s_i$  and  $Y = s_{j+1} \dots s_n$ .

**Remark** (Possible motivational remarks). Ankan Bhattacharya says: I knew that the answer had to be yes — the obvious counting argument to show answer no doesn’t work, and the given elements are unrelated enough that proving a no answer would be very difficult.

Evan says: I think you really do have to use exponentiation, since otherwise the numbers aren’t big enough; but exponentiation is really painful to deal with, so I tried to find a way to use it only once. This is less daunting than it seems because you can concatenate digits “for free” from a size perspective; thus you just need a substring that you can “save space” on. After a bit of guesswork I came upon the idea of taking modulo  $10^6 - 1 = 999999$  (which saves about two digits) and from there I had it.

**Remark** (Possible motivational remarks). Ankan Bhattacharya points out that if we fix all  $N - 2$  operations, then there are only  $10^{N-1}$  choices, compared to  $9 \cdot 10^{N-1}$  numbers we need to obtain. Thus we need to use different operations to reach different numbers. This suggests that all solutions are likely to use some amount of casework.

Unlike Ankan, I did not find the case split to be a substantial part of the problem. It came up naturally because I had an edge case where six consecutive zeros might appear in my argument, and the first case was patch-only in that situation.



**C3.** A *windmill* in the plane consists of a line segment of unit length with a distinguished endpoint, the *pivot*. Geoff has a finite set of windmills, such that no two windmills intersect, and any two pivots are distance more than  $\sqrt{2}$  apart. In an operation, Geoff can choose a windmill and rotate it about its pivot, either clockwise or counterclockwise and by any amount, as long as no two windmills intersect during or after the rotation. Show that Geoff can, in finitely many operations, rotate the windmills so that they all point in the same direction.

(Michael Ren)

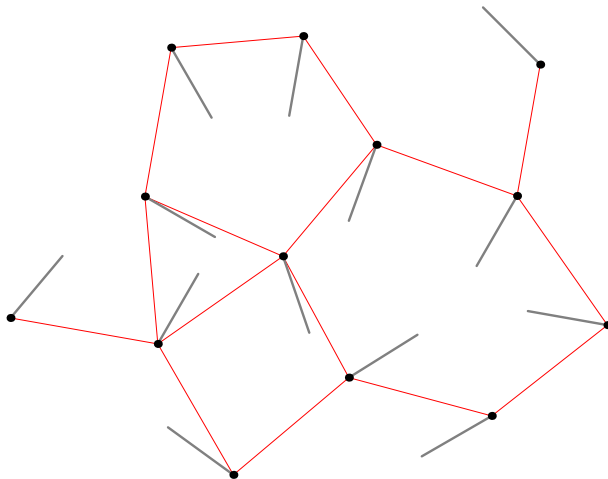
Throughout the solution we will general denote pivots by  $P, Q, R, \dots$  and non-pivots by  $A, B, C, \dots$

We say that a configuration of windmills around  $S$  is *admissible* if no two windmills intersect. The problem is equivalent to showing one can reach any admissible configuration from any other (and the final position with the windmills pointing the same direction is just one example of a clearly admissible configuration).

Draw a red line segment between any two pivots which have distance at most 2 (thus these windmills could intersect). This naturally gives us a graph  $\mathcal{G}$ .

**Lemma.** For  $c \geq \sqrt{2}$ , the graph  $\mathcal{G}$  is planar.

*Proof.* Indeed, if  $\overline{PA}$  and  $\overline{QB}$  intersect, we can consider convex quadrilateral  $PQAB$ , one of whose angles is at least  $90^\circ$ . WLOG it is  $\angle PQA$ , in which case  $PA^2 \geq PQ^2 + QA^2 > 2 + 2 = 4$ , so  $\overline{PA}$  should not be red.  $\square$



Clearly, we can ignore any isolated vertices. We can also ignore any leaves in  $\mathcal{G}$ ; indeed suppose  $P$  is a pivot with  $\overline{PQ}$  the only red edge. Then we can rotate the windmill at  $P$  to point away from  $Q$  and it will never obstruct other windmills since  $c \geq 1$ , so we can delete the pivot  $P$  from consideration (and use induction on the number of pivots, say).

Thus, we may assume  $\mathcal{G}$  is a finite planar graph with no leaves. Thus it makes sense to speak of the faces of planar graph  $\mathcal{G}$ , consisting of several polygons.

**Lemma.** A windmill with pivot  $P$  can never intersect a red edge other than those touching  $P$ .

*Proof.* Suppose windmill  $\overline{PA}$  intersects red edge  $\overline{QR}$ . Then the altitude from  $\overline{PH}$  to  $\overline{QR}$  has length at most 1. WLOG that  $QH < RH$ , so  $QH < \frac{1}{2}QR = 1$ . Then  $PQ^2 < QH^2 + HP^2 < 1 + 1 = 2$ , contradiction.  $\square$

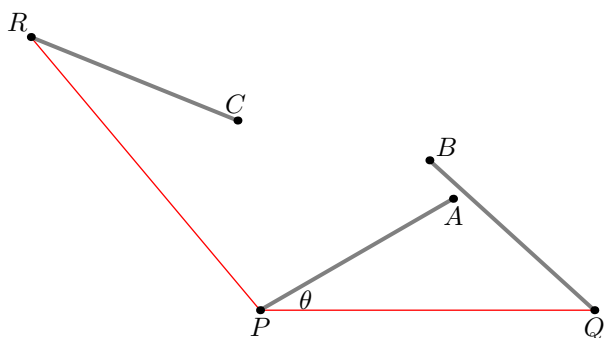
From now on, a windmill  $\overline{PA}$  is said to *hug* a red edge  $\overline{PQ}$  if the angle  $\angle QPA < \varepsilon$  for some sufficiently small  $\varepsilon$  in terms of  $\mathcal{G}$ ; each red edge  $\overline{PQ}$  has at most two windmills hugging it (namely the windmills with pivots  $P$  and  $Q$ ; if this happens, the windmills are on opposite sides of  $\overline{PQ}$ ). Call a windmill configuration *cuddly* if every windmill is hugging an edge.

**Claim.** We can reach some cuddly configuration from any admissible one.

*Proof.* Indeed, consider a windmill  $\overline{PA}$  not hugging any edge, and an edge  $\overline{PQ}$ , and such that  $\angle APQ = \theta$  is minimal among all such pairs. Let  $\angle RPQ$  be the corresponding angle of the face containing  $\overline{PA}$ , and let  $\overline{QB}, \overline{RC}$  be windmills.

If  $\overline{QB}$  is hugging  $\overline{PQ}$ , we perturb it slightly so that  $A$  and  $B$  are on opposite sides of  $\overline{PQ}$ ; thus  $\overline{QB}$  is no longer in the way.

We rotate  $\overline{PA}$  towards  $\overline{PQ}$  now. Because we assumed  $\theta = \angle APQ$  was minimal, it is impossible for the body of the windmill to collide with the points  $B$  or  $C$ . So the only way it can be obstructed is if the point  $A$  collides with the interior of  $\overline{QB}$  or  $\overline{RC}$ .

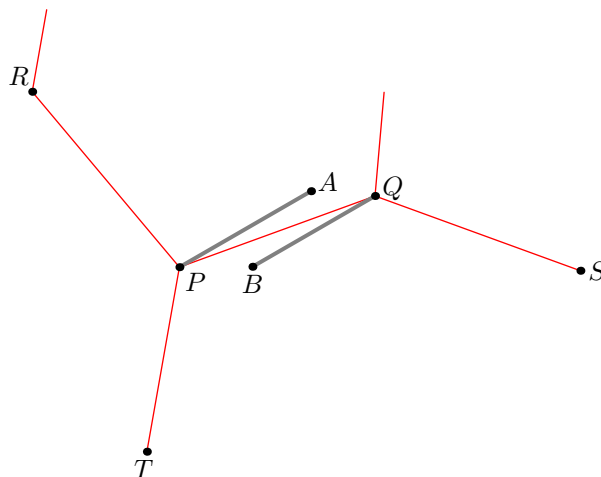


Suppose that  $A$  collided with  $\overline{QB}$ . At the moment of collision, we would have to have  $\angle PAQ \leq 90^\circ$ . (This is because just before the collision  $\overline{PA}$  was still disjoint from  $\overline{QB}$ , and if  $\angle PAQ \geq 90^\circ$  just before then it would remain disjoint as  $\overline{PA}$  rotated.) But then  $PQ^2 \leq PA^2 + AQ^2 \leq 2$ , contradiction. A similar proof works for  $\overline{RC}$ .

Thus we can rotate the windmills one by one so they hug the edges, as desired.  $\square$

It remains to show any two cuddly configurations can be reached from each other. For this, we make two observations.

- Suppose  $\overline{PA}$  and  $\overline{QB}$  both hug  $\overline{PQ}$ . We show we can interchange the two. Assume  $\angle RPQ$  is the angle of a face containing  $A$ , and  $\angle TPQ, \angle PQS$  are the angles of the face containing  $B$ .



Rotate  $\overline{PA}$  so it hugs  $\overline{PR}$  (possibly perturbing the windmill at  $R$ ), and then rotate  $\overline{QB}$  so it hugs  $\overline{QS}$  (possibly perturbing the windmill at  $S$ ). Then rotate  $\overline{PA}$  so it hugs  $\overline{PT}$ , then move  $\overline{QB}$  back so it hugs  $\overline{PQ}$  from the other side, and rotate  $\overline{PA}$  back.

- Now suppose  $\overline{PA}$  hugs  $\overline{PQ}$ , and  $\angle RPQ$  is the angle of a face containing  $A$ . Then we can rotate it so that  $\overline{PA}$  hugs  $\overline{PR}$  (here  $\overline{PA}$  could be blocked by  $\overline{QB}$  initially, but then we perform the switching operation above).

Together these two observations finish the problem.

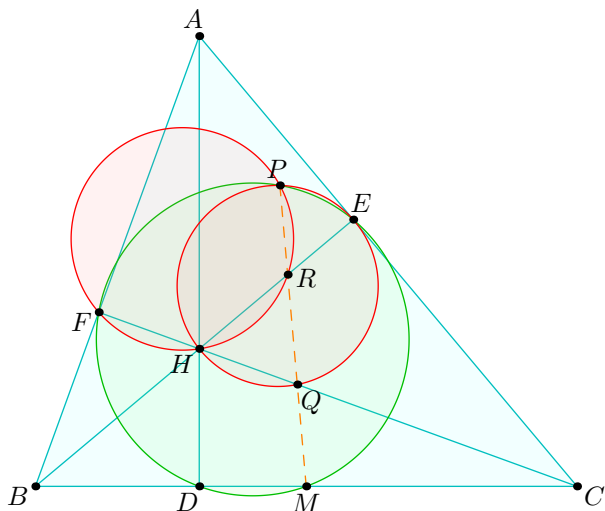
**Remark** (Michael Ren). Here is a solution achieving just  $c = \sqrt{3}$ .

Draw a disk of radius  $1 + \epsilon$  around every point in  $S$  such that the distance between any two points in  $S$  is more than  $\sqrt{3}(1 + \epsilon)$  for some  $\epsilon > 0$  that clearly exists. Note that no three disks can intersect. Indeed, if disks centered at  $A$ ,  $B$ , and  $C$  intersected, then the circumradius of  $ABC$  is at most  $1 + \epsilon$ , which means that some two of  $A, B, C$  are at most a distance of  $\sqrt{3}(1 + \epsilon)$  apart. In light of this, for any two points  $A$  and  $B$  in  $S$  that are a distance of at most 2 apart, draw a rhombus  $APBQ$  of length  $1 + \epsilon$ . By our work before, all such rhombi are distinct. Furthermore, windmill collisions only happen inside these rhombi by definition. Now, have Geoff move each of his windmills one by one to Sasha's windmills. If a windmill collision happens, have Geoff move the other windmill out of the way inside the rhombus before moving the windmill by and then restore the position of the other windmill. Hence, he can always get his windmills to coincide, as desired.

**G1.** Let  $ABC$  be an acute triangle with orthocenter  $H$ , and let  $P$  be a point on the nine-point circle of  $ABC$ . Lines  $BH$ ,  $CH$  meet the opposite sides  $AC$ ,  $AB$  at  $E$ ,  $F$ , respectively. Suppose that the circumcircles of  $\triangle EHP$  and  $\triangle FHP$  intersect lines  $CH$ ,  $BH$  a second time at  $Q$ ,  $R$ , respectively. Show that as  $P$  varies along the nine-point circle of  $ABC$ , the line  $QR$  passes through a fixed point.

(Brandon Wang)

Let  $D$  denote the foot of the  $A$ -altitude, and  $M$  the midpoint of  $\overline{BC}$ . We claim that  $R$  and  $Q$  both lie on line  $\overline{PM}$ . That will solve the problem ( $M$  is the fixed point).



By angle chasing, it is not hard to show that

$$\angle FHE = \angle FEM.$$

Now,

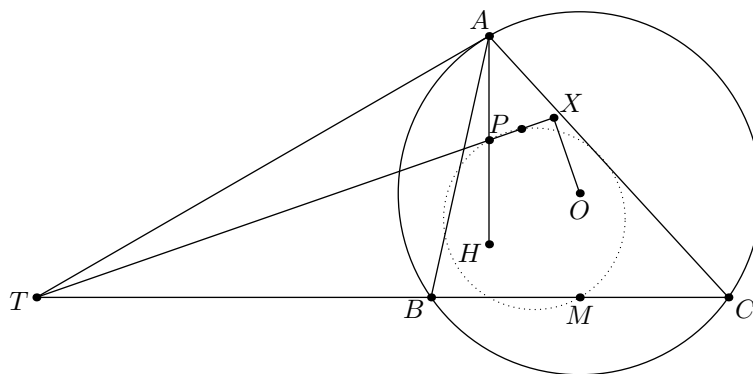
$$\angle FPR = \angle FHR = \angle FHE = \angle FEM = \angle FPM$$

as desired so  $P$ ,  $R$ ,  $M$  are collinear. Similarly,  $P$ ,  $Q$ ,  $M$  are collinear, as desired.

**G2.** Let  $ABC$  be a scalene triangle with orthocenter  $H$  and circumcenter  $O$ . Let  $P$  be the midpoint of  $\overline{AH}$  and let  $T$  be on line  $BC$  with  $\angle TAO = 90^\circ$ . Let  $X$  be the foot of the altitude from  $O$  onto line  $PT$ . Prove that the midpoint of  $\overline{PX}$  lies on the nine-point circle of  $\triangle ABC$ .

(Zack Chroman)

We present two solutions, one synthetic and by complex numbers.



**First solution (Zack Chroman)** Let  $M$  be the midpoint of  $\overline{BC}$ . Note that since  $\overline{AP} \perp \overline{BC}$  and  $\overline{AT} \perp \overline{AO} \parallel \overline{PM}$ , we find that  $P$  is the orthocenter of  $\triangle ATM$ . Thus  $Y = \overline{TP} \cap \overline{AM}$  satisfies  $\angle PYM = 90^\circ$ , so it lies on the 9-point circle.

It then suffices to note that the reflection  $X'$  of  $P$  over  $Y$  lies on the circumcircle of  $(AMT) = (TO)$ , so  $\angle TX'O = 90^\circ \implies X = X'$ .

**Second solution (complex numbers, Evan Chen)** Let  $Q$  denote the reflection of  $P$  over  $M$ , the midpoint of  $\overline{BC}$ .

**Claim.** We have  $\overline{QO} \perp \overline{PT}$ .

*Proof.* By complex numbers. We have

$$\begin{aligned} t &= \frac{aa(b+c) - bc(a+a)}{aa - bc} = \frac{a^2(b+c) - 2abc}{a^2 - bc} \\ t - p &= \frac{a^2(b+c) - 2abc}{a^2 - bc} - \left( a + \frac{b+c}{2} \right) \\ &= \frac{a^2(\frac{1}{2}b + \frac{1}{2}c - a) + (-a + \frac{1}{2}b + \frac{1}{2}c)bc}{a^2 - bc} \\ &= \frac{b+c-2a}{2} \cdot \frac{a^2+bc}{a^2-bc} \\ q &= 2 \cdot \frac{b+c}{2} - p = \frac{b+c-2a}{2} \end{aligned}$$

Since  $\frac{a^2+bc}{a^2-bc} \in i\mathbb{R}$ , the claim is proven.  $\square$

Thus,  $\overline{QOX}$  are collinear. By considering right triangle  $\triangle PQX$  with midpoint  $M$ , we conclude that  $MX = MP$ . Since the nine-point circle is the circle with diameter  $\overline{PM}$ , it passes through the midpoint of  $\overline{PX}$ .

**G3.** Let  $A$  be a point in the plane, and  $\ell$  a line not passing through  $A$ . Evan doesn't have a straightedge, but instead has a special compass which has the ability to draw a circle through three distinct noncollinear points. (The center of the circle is *not* marked in this process.) Additionally, Evan can mark the intersections between two objects drawn, and can mark an arbitrary point on a given object or on the plane.

- (i) Can Evan construct the reflection of  $A$  over  $\ell$ ?
- (ii) Can Evan construct the foot of the altitude from  $A$  to  $\ell$ ?

*(Zack Chroman)*

The trick is to invert the figure around a circle centered at  $A$  of arbitrary radius. We let  $\omega = \ell^*$  denote the image of  $\ell$  under this inversion. Then, under the inversion, Evan's compass has the following behavior:

- Evan can draw a line through two points other than  $A$ ; or
- Evan can draw a circle through three points other than  $A$ .

In other words, the point  $A$  is “invisible” to Evan, but Evan otherwise has a straightedge and the same compass.

It is clear then that the answer to (ii) is no.

Part (i) is equivalent to showing that Evan can construct the center of  $\omega$ ; we give one construction here anyways. Take any cyclic quadrilateral  $WXYZ$  inscribed in  $\omega$ , and let  $P = \overline{WZ} \cap \overline{XY}$ . Then the circumcircles of  $\triangle PWX$  and  $\triangle PYZ$  meet again at the Miquel point  $M$ , and the second intersection of  $(MXZ)$  and  $(MWY)$  is the center of  $\omega$ .

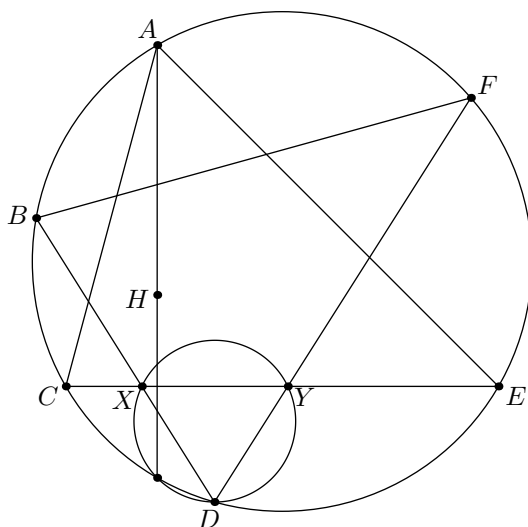
**Remark.** The proof of (ii) implies that it's actually more or less impossible in this context to construct any point other than the reflection of  $A$ , as a function of  $A$  and  $\ell$ .

An alternative proof of (ii) is possible by inverting around a generic point  $P$  on  $\ell$  with radius  $PA$ ; this necessarily preserves the entire construction, but the foot from  $A$  to  $\ell$  is not fixed by this inversion.

**G4.** Let  $ABCDEF$  be a convex hexagon inscribed in a circle  $\Omega$  such that triangles  $ACE$  and  $BDF$  have the same orthocenter. Suppose that  $\overline{BD}$  and  $\overline{DF}$  intersect  $\overline{CE}$  at  $X$  and  $Y$ , respectively. Show that there is a point common to  $\Omega$ , the circumcircle of  $DXY$ , and the line through  $A$  perpendicular to  $\overline{CE}$ .

(Michael Ren and Vincent Huang)

We present many, many solutions. In all of them, we let  $H$  denote the common orthocenter.



**First solution by Simson lines (Vincent Huang)** Let  $AH$  meet  $CE$  and  $\Omega$  again at  $M$  and  $A_1$ , respectively, and  $P$  and  $Q$  be the projections of  $A_1$  onto  $BD$  and  $DF$ , respectively. Note that  $PQ$  is the Simson line of  $A_1$  with respect to  $BDF$ . It is well known that this Simson line bisects the segment between  $A_1$  and  $H$ . Hence,  $M$  lies on  $PQ$ . But  $P$ ,  $M$ , and  $Q$  are respectively the projections of  $A_1$  onto  $DX$ ,  $XY$ , and  $YD$ , so  $A_1$  must lie on the circumcircle of  $DXY$ , as desired.

**Second solution by dual Desargues involution (Michael Ren)** Let  $O$  and  $r$  be the center and radius of  $\Omega$ , respectively. Let  $\mathcal{E}$  be the ellipse with foci  $O$  and  $H$  consisting of the set of points  $P$  such that  $OP + HP = r$ . Note that as the reflections of  $H$  over  $AC, CE, EA, BD, DF, FB$  lie on  $\Omega$ ,  $\mathcal{E}$  is tangent to the sides of  $ACE$  and  $BDF$ . Let  $\mathcal{E}$  and  $AD$  meet  $CE$  at  $P$  and  $Q$ , respectively. By the dual of Desargues involution theorem on quadrilateral  $ACPE$  with inscribed conic  $\mathcal{E}$ ,  $D(CE; XY; PQ)$  is an involution. Hence, the circumcircles of  $DCE$ ,  $DXY$ , and  $DPQ$  are coaxial, so it suffices to show that  $A_1DPQ$  is cyclic, where  $A_1$  is the second intersection of  $AH$  and  $\Omega$ . But note that  $A_1$  lies on  $OP$ , so  $\angle QDA_1 = \angle ADA_1 = \frac{\pi}{2} - \angle OA_1A = \frac{\pi}{2} - \angle PA_1A$ , which is the angle between  $PA_1$  and  $PQ$  by the perpendicularity of  $AA_1$  and  $CE$ , as desired.

**Third solution by angle chasing (Mihir Singhal)** Let  $A_1$  be the reflection of  $H$  over  $CE$ . Note  $A_1$  is on  $\Omega$  so it suffices to show that  $DA_1XY$  is cyclic. Let  $M$  be the foot of the altitude from  $A$  to  $\overline{CE}$ . Note that  $M$  is the midpoint of  $\overline{HA_1}$  so since  $A_1$  is on  $\Omega$ ,  $M$  must be on the nine-point circle of  $DBF$ . Let  $R$  and  $S$  be the feet of the altitudes from  $F$  and  $B$  in  $DBF$ .

Note  $MXRH$  and  $MYSH$  are cyclic. Moreover,  $M$  lies on the nine-point circle of  $\triangle BDF$ , and hence  $\angle SMR = 2\angle SDR$ . Then

$$\begin{aligned}\angle XHY &= \angle XHM + \angle MHY \\ &= \angle XRM + \angle MSY = \angle DRM + \angle MSD \\ &= -(\angle RMS + \angle SDR) = \angle SMR + \angle RDS \\ &= 2\angle SDR + \angle RDS = \angle SDR = \angle YDX.\end{aligned}$$

Thus  $\angle XA_1Y = -\angle XHY = \angle RDS = \angle XDY$ , as needed.

**Fourth solution by inversion (James Lin)** Let  $K$  be the second intersection of  $\Omega$  and the perpendicular from  $A$  to  $CE$ . We want to show  $DKXY$  is cyclic. We invert about  $H$ . It's clear that now,  $A'C'E'$  and  $B'D'F'$  share the same circumcircle  $\Omega'$  and incenter  $H$ . Note that  $K$  maps to the midpoint  $M_{A'}$  of the arc  $C'E'$  on  $\Omega'$  not containing  $A'$ . Also note that  $X'$  is the intersection of circles  $(HB'D')$  and  $(HC'E')$ , which are centered at midpoint  $M_{F'}$  of the arc  $B'D'$  on  $\Omega'$  not containing  $F'$  and the midpoint  $M_{D'}$  of the arc  $B'F'$  on  $\Omega'$  not containing  $D'$ , respectively. Thus,  $X'$  is the reflection of  $H$  over  $M_{A'}M_{F'}$ . Similarly,  $Y'$  is the reflection of  $H$  over  $M_{A'}M_{B'}$ . Then, note that  $M_{A'}X = M_{A'}H = M_{A'}Y$ . Now we reformulate the problem by erasing  $A', C'$  and  $E'$ , as the rest of the problem can be defined without them. The reformulated statement is that if we fix  $B, D, F, H$  and vary  $M_{A'}$  along  $\Omega'$ , then  $D'M_{A'}X'Y'$  is always cyclic.

We proceed with directed angles. Note that  $\angle X'D'M_{A'} = \angle X'D'H + \angle HD'M_{A'} = \angle M_{A'}M_{F'}F + \angle M_{D'}M_{F'}M_{A'} = \angle M_{D'}M_{F'}F$ . Similarly,  $\angle Y'D'M_{A'} = \angle M_{D'}M_{B'}B = -\angle M_{D'}M_{F'}F = -\angle X'DM_{A'}$ , so it follows that  $M_{A'}$  lies on an angle bisector of  $\angle X'DY'$ . Assume that  $D'M_{A'}$  and  $X'Y'$  are not perpendicular. Then from  $M_{A'}X' = M_{A'}Y'$ , it follows that  $D'M_{A'}X'$  and  $D'M_{A'}Y'$  have the same circumradius, and if they don't have the same circumcircle, then  $D'M_{A'}$  and  $X'Y'$  must be perpendicular, a contradiction. So  $D'X'M_{A'}Y'$  is cyclic. If  $D'M_{A'}$  and  $X'Y'$  are perpendicular, then use the new problem formulation (without  $A, C$  and  $E$  and just varying  $M_{A'}$ ) to move  $M_{A'}$  by a miniscule amount. Then  $D'M_{A'}$  and  $X'Y'$  will not be perpendicular, so  $D'X'M_{A'}Y'$  is cyclic both after and before moving  $M_{A'}$  by continuity. We are done.

**Fifth solution, by complex numbers (Carl Schildkraut)** Let  $\Omega$  be the unit circle, and let  $A = a$ , etc. We have that

$$c + e = h - a \implies \frac{c + e}{ce} = \bar{h} - \frac{1}{a} \implies ce = \frac{a(h - a)}{a\bar{h} - 1}.$$

Let  $T$  be the second intersection of the line through  $A$  perpendicular to  $CE$  and  $\Omega$ . We see that

$$t = -\frac{ce}{a} = -\frac{h - a}{a\bar{h} - 1}.$$



We endeavor to show that  $DTXY$  is a cyclic quadrilateral. We have that

$$\begin{aligned}
 x &= \frac{ce(b+d) - bd(c+e)}{ce - bd} \\
 &= \frac{\frac{a(b+d)(h-a)}{ah-1} - bd(h-a)}{\frac{a(h-a)}{ah-1} - bd} \\
 &= (h-a) \left( \frac{a(b+d) - bd(a\bar{h} - 1)}{a(h-a) - bd(a\bar{h} - 1)} \right) \\
 &= (h-a) \left( \frac{ab + ad - ab - ad - \frac{abd}{f} + bd}{ab + ad + af - a^2 - ab - ad - \frac{abd}{f} + bd} \right) \\
 &= (h-a) \left( \frac{bd(f-a)}{(af+bd)(f-a)} \right) \\
 &= \frac{bd(h-a)}{af+bd}.
 \end{aligned}$$

Similarly

$$y = \frac{bf(h-a)}{ab+df}.$$

So, we want to show that

$$d, -\frac{h-a}{a\bar{h}-1}, \frac{bd(h-a)}{af+bd}, \frac{bf(h-a)}{ab+df}$$

are concyclic. This is equivalent to, dividing each by  $h-a$  and reciprocating,

$$\frac{h-a}{d}, 1 - a\bar{h}, 1 + \frac{af}{bd}, 1 + \frac{ab}{df}$$

being concyclic. This is equivalent to, subtracting 1 and multiplying by  $ddf$ ,

$$bf(b+f-a), -a(bd+bf+df), ab^2, af^2$$

being concyclic. This is equivalent to, adding  $abf$  and dividing by  $b+f$ ,

$$bf, -ad, ab, af$$

being concyclic. However, all of these points lie on the unit circle, finishing the proof.

**Sixth solution by complex numbers (Evan Chen)** As usual let  $\Omega$  denote the unit circle. We immediately have

$$\begin{aligned}
 c+e &= b+d+f-a \\
 \text{and thus } \frac{1}{c} + \frac{1}{e} &= \frac{c+e}{ce} = \frac{1}{b} + \frac{1}{d} + \frac{1}{f} - \frac{1}{a} \\
 \implies ce &= \frac{b+f+d-a}{\frac{1}{b} + \frac{1}{f} + \frac{1}{d} - \frac{1}{a}}.
 \end{aligned}$$

These two equations let us eliminate  $c$  and  $e$ , leaving only  $a, b, d, f$ .

Now consider the point  $p = -\frac{ce}{a}$  on the circumcircle. We compute

$$\begin{aligned}
\frac{x-p}{b-p} &= \frac{x + \frac{ce}{a}}{b + \frac{ce}{a}} \\
&= \frac{\frac{bd(c+e) - ce(b+d)}{bd-ce} + \frac{ce}{a}}{b + \frac{ce}{a}} \\
&= \frac{abcd + abde - abce - adce + bdce - (ce)^2}{(ab+ce)(bd-ce)} \\
&= \frac{abcde(1/a + 1/e + 1/c - 1/d - 1/b) - (ce)^2}{(ab+ce)(bd-ce)} \\
&= \frac{abcde(1/f) - (ce)^2}{(ab+ce)(bd-ce)} = \frac{(ce)(abd - cef)}{f(ab+ce)(bd-ce)}
\end{aligned}$$

Now, we write

$$\begin{aligned}
ab+ce &= \frac{ab(\frac{1}{b} + \frac{1}{f} + \frac{1}{d} - \frac{1}{a}) + (b+f+d-a)}{\frac{1}{b} + \frac{1}{f} + \frac{1}{d} - \frac{1}{a}} \\
&= \frac{ab(\frac{1}{d} + \frac{1}{f}) + d+f}{\frac{1}{b} + \frac{1}{f} + \frac{1}{d} - \frac{1}{a}} = \frac{\frac{1}{df}(d+f)(ab+df)}{\frac{1}{b} + \frac{1}{f} + \frac{1}{d} - \frac{1}{a}} \\
bd-ce &= \frac{bd(\frac{1}{b} + \frac{1}{d} + \frac{1}{f} - \frac{1}{a}) - (b+f+d-a)}{\frac{1}{b} + \frac{1}{f} + \frac{1}{d} - \frac{1}{a}} \\
&= \frac{bd(\frac{1}{f} - \frac{1}{a}) + (a-f)}{\frac{1}{b} + \frac{1}{f} + \frac{1}{d} - \frac{1}{a}} = \frac{\frac{1}{af}(a-f)(bd+af)}{\frac{1}{b} + \frac{1}{f} + \frac{1}{d} - \frac{1}{a}} \\
abd - cef &= abd - \frac{f(b+f+d-a)}{\frac{1}{b} + \frac{1}{f} + \frac{1}{d} - \frac{1}{a}} \\
&= \frac{abd(\frac{1}{b} + \frac{1}{f} + \frac{1}{d} - \frac{1}{a}) - f(b+f+d-a)}{\frac{1}{b} + \frac{1}{f} + \frac{1}{d} - \frac{1}{a}} \\
&= \frac{(b+f)(\frac{abd}{bf} - f) + b(a-d) + f(a-d)}{\frac{1}{b} + \frac{1}{f} + \frac{1}{d} - \frac{1}{a}} \\
&= \frac{(b+f)(\frac{ad}{f} - f + (a-d))}{\frac{1}{b} + \frac{1}{f} + \frac{1}{d} - \frac{1}{a}} \\
&= \frac{\frac{1}{f}(b+f)(a-f)(f+d)}{\frac{1}{b} + \frac{1}{f} + \frac{1}{d} - \frac{1}{a}}.
\end{aligned}$$

Putting that all together gives

$$\frac{x-p}{b-p} = \frac{ce \cdot adf(b+f)(\frac{1}{b} + \frac{1}{f} + \frac{1}{d} - \frac{1}{a})}{(ab+df)(bd+af)}$$

which is symmetric in  $d$  and  $f$ , so the analogous calculation with  $\frac{y-p}{f-p}$  yields the same result. Consequently,  $P$  is the center of the spiral similarity sending  $\overline{YF}$  to  $\overline{BX}$ , as desired.

**Remark.** Philosophical point: it's necessary to use both  $a + c + e = b + d + f$  and its conjugate, to capture two degrees of freedom.

**Sixth solution, by inversion and moving points (Anant Mudgal, unedited)** Let  $H$  be the common orthocenter. Pick any two vertices  $X, Y$  of either  $\triangle ACE$  or  $\triangle BDF$  and notice that  $\triangle XYH$  has circumradius equal to the radius of  $\Omega$ . Now invert at  $H$ . We obtain the following equivalent problem.

Let  $ABCDEF$  be a cyclic hexagon with  $\triangle ACE$  and  $\triangle BDF$  sharing a common incircle  $\omega$  centered at point  $H$ . Let  $\odot(HBD), \odot(HFD)$  meet  $\odot(CHE)$  again at points  $X$  and  $Y$  respectively. Let  $M$  be the midpoint of arc  $CE$  not containing  $A$ . Then  $\odot(DXY)$  passes through point  $M$ .

Let  $\omega$  touch  $\overline{CE}$  at point  $N$  and  $L = \overline{AD} \cap \overline{CE}$ . Let  $P = \overline{DB} \cap \overline{CE}$  and  $Q = \overline{DF} \cap \overline{CE}$ . By Dual of Desargues Involution Theorem on circumscribed  $ACEN$  and point  $D$ ; we conclude  $(\overline{DN}, \overline{DL}), (\overline{DC}, \overline{DE}), (\overline{DP}, \overline{DQ})$  are pairs of an involution. Notice that  $P$  has equal powers in  $\odot(HBD), \odot(CHE)$  hence  $P$  lies on  $\overline{XH}$ . Similarly,  $Q$  lies on  $\overline{YH}$ .

Let  $\overline{HN}, \overline{HL}$  meet  $\odot(CHE)$  again at  $S, T$ . Project through  $H$  to conclude that  $(C, E), (X, Y), (S, T)$  are pairs of an involution on the circle  $\odot(CHE)$ . Thus, we conclude that lines  $\overline{CE}, \overline{XY}, \overline{ST}$  concur.

**Claim.**  $\overline{CE}, \overline{ST}, \overline{DM}$  concur.

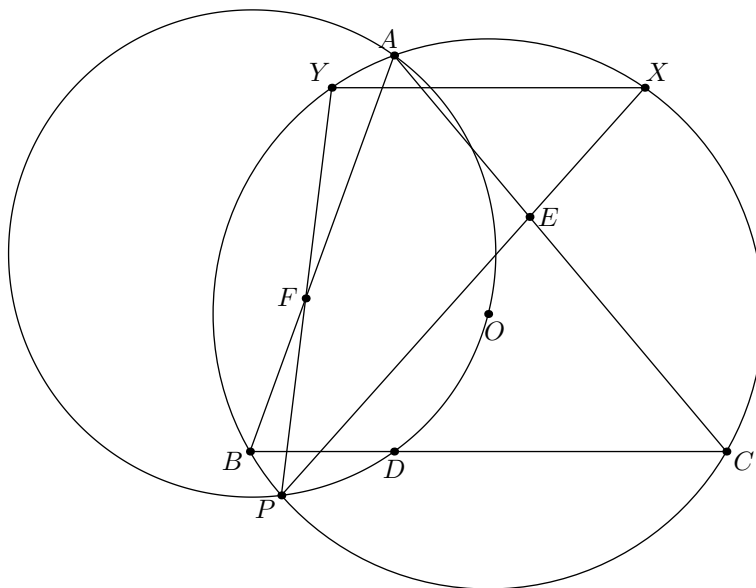
*Proof.* Animate  $D$  on  $\odot(ACE)$ ; then  $D \mapsto L \mapsto T$  is projective. Let  $U = \overline{DM} \cap \overline{CE}$  and  $V = \overline{ST} \cap \overline{CE}$  then  $D \mapsto U$  and  $D \mapsto V$  are also projective. Thus to show  $W \stackrel{\text{def}}{=} U \equiv V$  we need to verify for three choices of point  $D$ ; namely we pick  $\{C, E, M\}$ . These are all clearly true and the lemma is proved.  $\square$

Finally, notice  $WX \cdot WY = WC \cdot WE = WD \cdot WM$  proving  $DXYM$  is cyclic.

**G5.** Let scalene triangle  $ABC$  have altitudes  $\overline{AD}$ ,  $\overline{BE}$ ,  $\overline{CF}$  and circumcenter  $O$ . The circumcircles of  $\triangle ABC$  and  $\triangle ADO$  meet at  $P \neq A$ . The circumcircle of  $\triangle ABC$  meets lines  $\overline{PE}$  at  $X \neq P$  and  $\overline{PF}$  at  $Y \neq P$ . Prove that  $\overline{XY} \parallel \overline{BC}$ .

(Daniel Hu)

Denote by  $\Omega$  and  $H$  the circumcircle and orthocenter of  $\triangle ABC$ . Let  $T$  lie on  $\Omega$  such that  $\overline{AT} \parallel \overline{BC}$ . Let  $\triangle ABC$  have orthocenter  $H$ .



**First solution, synthetic** First we prove a lemma.

**Claim.** The points  $H, P, T$  are collinear.

*Proof.* Let  $\overline{HT}$  meet  $\Omega$  at  $P^* \neq T$ . Let  $\overline{AD}$  meet  $\Omega$  at  $K \neq A$ . By homothety at  $K$ ,  $\overline{HT} \parallel \overline{DO}$ . By angle chasing,  $\angle P^*AD = \angle P^*AK = \angle P^*TK = \angle P^*TO = \angle OP^*T = \angle P^*OD$ , so  $P^*$  lies on the circumcircle of  $\triangle AOD$ . Therefore,  $P \equiv P^*$  as desired.  $\square$

We now provide two finishes.

- First finish: By DDIT on  $AEHF$ , the pairs of lines  $(\overline{PA}, \overline{PH}), (\overline{PB}, \overline{PC}), (\overline{PE}, \overline{PF})$  are part of a single involution, so  $\overline{AT}, \overline{BC}, \overline{XY}$  are concurrent. Since  $\overline{AT} \parallel \overline{BC}$ , this implies that  $\overline{XY} \parallel \overline{BC}$  as desired.
- Second finish: Let  $Q = \overline{AP} \cap \overline{EF}$ . By inversion at  $A$ ,  $BFPQ, CEPQ, DHPQ$  are all cyclic. By the lemma, this implies that  $\angle ABC + \angle ACB = \angle APT = \angle APH = \angle QPH = \angle QDH = \angle QAH$ , so  $\overline{DQ} \perp \overline{EF}$ .

Let  $G = \overline{EF} \cap \overline{BC}$ ; since  $(G, D; B, C) = -1$ ,  $\angle BQD = \angle DQC$ . Thus  $\angle BAY = \angle BPY = \angle BPF = \angle BQF = \angle CQE = \angle CPE = \angle CPX = \angle CAX$ , so  $\overline{XY} \parallel \overline{BC}$  as desired.

**Second solution by complex numbers (Adam Ardeishar)** Let  $ABC$  be the complex unit circle. Then  $D = \frac{1}{2}(a + b + c - \frac{bc}{a})$ , and we know

$$\begin{aligned} \frac{p-a}{p-o} \cdot \frac{d-o}{d-a} &\in \mathbb{R} \\ \frac{p-a}{p} \cdot \frac{a+b+c-\frac{bc}{a}}{b+c-a-\frac{bc}{a}} &= \frac{\frac{1}{p}-\frac{1}{a}}{\frac{1}{p}} \cdot \frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{a}{bc}}{-\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{a}{bc}} \\ \frac{1}{p} \cdot \frac{a+b+c-\frac{bc}{a}}{b+c-a-\frac{bc}{a}} &= \frac{-1}{a} \cdot \frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{a}{bc}}{\frac{1}{b}+\frac{1}{c}-\frac{1}{a}-\frac{a}{bc}} \\ -\frac{a}{p} \cdot \frac{a^2+ab+ac-bc}{ab+ac-a^2-bc} &= \frac{bc+ab+ab-a^2}{ab+ac-bc-a^2} \\ p &= a \cdot \frac{a^2+ab+ac-bc}{a^2-ab-ac-bc} \end{aligned}$$

Now note that  $p+x=e+px\bar{e}$ , so  $x = \frac{p-e}{p\bar{e}-1}$ . But we compute that

$$\begin{aligned} p-e &= a \cdot \frac{a^2+ab+ac-bc}{a^2-ab-ac-bc} - \frac{1}{2}\left(a+b+c-\frac{ac}{b}\right) \\ &= \frac{a^3b+a^3+2a^2b^2+a^2bc+ab^3+ab^2c+b^3c+b^2c^2-a^2c^2}{2b(a^2-ab-ac-bc)} \\ &= \frac{(a+b)(b+c)(a^2+ab-ac+bc)}{2b(a^2-ab-ac-bc)} \end{aligned}$$

And also compute

$$\begin{aligned} p\bar{e}-1 &= a \cdot \frac{a^2+ab+ac-bc}{a^2-ab-ac-bc} \cdot \frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{b}{ac}\right) - 1 \\ &= \frac{a^3b+a^3c+a^2bc+a^2c^2+ab^2c+2abc^2+b^3c+b^2c^2-ab^3}{2bc(a^2-ab-ac-bc)} \\ &= \frac{(a+b)(b+c)(a^2+ac+bc-ab)}{2bc(a^2-ab-ac-bc)} \end{aligned}$$

So

$$x = \frac{\frac{(a+b)(b+c)(a^2+ab-ac+bc)}{2b(a^2-ab-ac-bc)}}{\frac{(a+b)(b+c)(a^2+ac+bc-ab)}{2bc(a^2-ab-ac-bc)}} = c \cdot \frac{a^2+ab+bc-ac}{a^2+ac+bc-ab}$$

By symmetry,

$$y = b \cdot \frac{a^2+ac+bc-ab}{a^2+ab+bc-ac}$$

Now note that  $xy = bc$  to finish.

**N1.** Determine all nonempty finite sets  $S = \{a_1, \dots, a_n\}$  of  $n$  distinct positive integers such that  $a_1 \cdots a_n$  divides  $(x + a_1) \cdots (x + a_n)$  for every positive integer  $x$ .

(Ankan Bhattacharya)

Answer:  $\{a_1, \dots, a_n\} = \{1, \dots, n\}$ . This works since

$$\frac{(x+n) \cdots (x+1)}{n!} = \binom{x+n}{n} \in \mathbb{Z}$$

so we now show that it is the only possibility. There are two approaches.

**First solution** Let  $P(x) = (x + a_1) \cdots (x + a_n)$ . Then,  $a_1 \cdots a_n$  should divide the  $n$ th finite difference of  $P$ , which is  $n!$ . But

$$a_1 \cdots a_n \mid n! \implies \{a_1, \dots, a_n\} = \{1, \dots, n\}$$

for size reasons.

**Second solution (Kevin Sun)** Let  $s + 1$  be the smallest positive integer not in our set  $A$  and denote  $B = A \setminus \{1, \dots, s\}$ .

It's clear that the divisibility holds for negative  $x$  as well. Set  $x = -s - 1$  to obtain

$$\begin{aligned} \mathbb{Z} &\ni \frac{1}{a_1 \cdots a_n} \prod_{a \in A} (x + a) \\ &= \prod_{a \in A} \left(1 + \frac{x}{a}\right) \\ &= \prod_{a \in \{1, \dots, s\}} \left(1 - \frac{s+1}{a}\right) \cdot \prod_{b \in B} \left(1 - \frac{s+1}{b}\right) \\ &= \prod_{a \in \{1, \dots, s\}} \left(\frac{a - (s+1)}{a}\right) \cdot \prod_{b \in B} \left(1 - \frac{s+1}{b}\right) \\ &= \frac{(-s)(-(s-1)) \cdots (-1)}{1 \cdot 2 \cdots s} \cdot \prod_{b \in B} \left(1 - \frac{s+1}{b}\right) \\ &= (-1)^{|A|} \prod_{b \in B} \left(1 - \frac{s+1}{b}\right). \end{aligned}$$

If  $B$  is nonempty this has magnitude strictly between 0 and 1, (since  $\min B > s + 1$  and thus each term is in  $(0, 1)$ ). Thus  $B$  is empty and  $A = \{1, \dots, s\}$ .

**N2.** Call a number  $n$  *good* if it can be expressed in the form  $2^x + y^2$  where  $x$  and  $y$  are nonnegative integers.

- (a) Prove that there exist infinitely many sets of 4 consecutive good numbers.  
 (b) Find all sets of 5 consecutive good numbers.

(Michael Ma)

For (a), note that for any  $t$ , the numbers  $t^2 + 1$ ,  $t^2 + 2$ ,  $t^2 + 4$  are good. So it suffices to show  $t^2 + 3$  is good infinitely often, that is,  $t^2 + 3 = 2^x + y^2$  has infinitely many nonnegative integer solutions (since for fixed  $t$  there are finitely many  $(x, y)$ ). But this rearranges  $t^2 - y^2 = 2^x - 3$  which has a solution for every  $x$ .

We now turn to the laborious task of (b), determining all sets of five consecutive good numbers. The answers are the six tuples  $\{1, 2, 3, 4, 5\}$ ,  $\{2, 3, 4, 5, 6\}$ ,  $\{8, 9, 10, 11, 12\}$ ,  $\{9, 10, 11, 12, 13\}$ ,  $\{288, 289, 290, 291, 292\}$ ,  $\{289, 290, 291, 292, 293\}$ . These all work since

$$\begin{aligned} 1 &= 2^0 + 0^2, & 2 &= 2^0 + 1^2, & 3 &= 2^1 + 1^2, \\ 4 &= 2^2 + 0^2, & 5 &= 2^2 + 1^2, & 6 &= 2^1 + 2^2, \\ 8 &= 2^3 + 0^2, & 9 &= 2^3 + 1^2, & 10 &= 2^0 + 3^2, \\ 11 &= 2^1 + 3^2, & 12 &= 2^3 + 2^2, & 13 &= 2^2 + 3^2, \\ 288 &= 2^5 + 16^2, & 289 &= 2^6 + 15^2, & 290 &= 2^0 + 17^2, \\ 291 &= 2^1 + 17^2, & 292 &= 2^8 + 6^2, & 293 &= 2^2 + 17^2. \end{aligned}$$

We now show they are the only ones. First, consider the following table which shows  $2^x + y^2 \pmod{8}$ :

	$x = 0$	$x = 1$	$x = 2$	$x \geq 3$
$y \equiv 1 \pmod{2}$	2	3	5	1
$y \equiv 0 \pmod{4}$	1	2	4	0
$y \equiv 2 \pmod{4}$	5	6	0	4

Note that from this table, no good number is  $7 \pmod{8}$ . Thus any five good numbers must have a  $3 \pmod{8}$  number. By table can only occur if that good number is of the form  $t^2 + 2^1 = t^2 + 2$  for an odd integer  $t$ .

We now have several cases.

**Case 1:** Suppose the five good numbers are  $\{t^2 + 1, t^2 + 2, t^2 + 3, t^2 + 4, t^2 + 5\}$ .

Note that  $t^2 + 5 \equiv 6 \pmod{8}$ , and by table, this can only occur if  $t^2 + 5 = s^2 + 2^2 = s^2 + 4$  for some integer  $s$ ; hence  $t^2 - s^2 = 1$ , so  $t = 1$  and  $s = 0$ . This gives the solution set  $\{2, 3, 4, 5, 6\}$ .

**Case 2:** Suppose the five good numbers are  $\{t^2, t^2 + 1, t^2 + 2, t^2 + 3, t^2 + 4\}$ .

Since  $t^2$  is good, we have  $t^2 = 2^w + z^2$  for some  $w$  and  $z$ , which we write as  $(t - z)(t + z) = 2^w$ .

We now split into cases.

- **Subcase 2.1:** We handle the situation where  $w < 4$ .
  - If  $w = 0$ , then we get  $t = 1$ , which gives the solution  $\{1, 2, 3, 4, 5\}$ .
  - If  $w = 1$ , then there are no solutions by taking mod 4.

- If  $w = 2$ , then  $t^2 = 4 + z^2$  which implies  $t = 2$ , but  $t$  was odd.
- If  $w = 3$ , we get  $t^2 = 8 + z^2$  which implies  $t = 3$ , which gives  $\{9, 10, 11, 12, 13\}$ .
- If  $w = 4$ , we get  $t^2 = 16 + z^2$  which together with  $t$  odd implies  $t = 5$ , which gives  $\{25, 26, 27, 28, 29\}$ . However, the number 28 is not good, so this is not a solution.
- **Subcase 2.2:** Suppose  $w \geq 5$ . As  $\gcd(t - z, t + z) \mid 2t$  we must have  $t - z = 2$ ,  $t + z = 2^{w-1}$ , and thus  $t = \frac{1}{2}(2 + 2^{w-1}) = 2^{w-2} + 1$ . Since  $t$  was odd, we actually have  $w \geq 3$ .

But  $t^2 + 3$  is also good, so write

$$t^2 + 3 = 2^x + y^2.$$

So we split into cases again.

- **Subcase 2.2.1:** We handle the case  $x < 3$ .
  - \* If  $x = 0$ , we get  $t^2 + 2 = y^2$  which has no solutions.
  - \* If  $x = 1$ , we get  $t^2 + 1 = y^2$  which implies  $t = 0$ , but  $t$  is supposed to be odd.
  - \* If  $x = 2$ , then we get  $t^2 = y^2 + 1$  which implies  $t = 1$ , which was an earlier solution.
- **Subcase 2.2.2:** Otherwise, assume  $x \geq 3$ .

$$\begin{aligned} 2^x + y^2 &= t^2 + 3 \\ \implies 2^x + y^2 &= (2^{w-2} + 1)^2 + 3 \\ &= 2^{2w-4} + 2^{w-1} + 4 \\ \implies 2^{2w-6} + 2^{w-3} + 1 &= 2^{x-2} + (y/2)^2 \end{aligned}$$

since  $y$  is clearly even; the last line implies  $y/2$  is odd, since  $2w - 6 > 0$ ,  $w - 3 > 0$ ,  $x - 2 > 0$ .

Let  $c = w - 3 \geq 2$ ,  $a = x - 2 \geq 1$ ,  $b = y/2 \geq 1$  for brevity; then the equation rewrites as

$$2^{2c} + 2^c + 1 = 2^a + b^2.$$

We rewrite this as

$$(2^c + 1 - b)(2^c + 1 + b) = (2^c + 1)^2 - b^2 = 2^a + 2^c \geq 0.$$

In light of this, we have  $2^a + 2^c \geq (2^c + 1)^2 - 2^{2c} > 2^{c+1}$ , so  $2^a > 2^c$ , ergo  $a > c$ . Thus we may further write

$$(2^c + 1 - b)(2^c + 1 + b) = 2^c(2^{a-c} + 1).$$

The factors on the left-hand side are nonnegative and have gcd dividing  $2b$ , hence one of them has at most one factor of 2. So one of the factors must be divisible by  $2^{c-1}$ . Thus,  $b \equiv \pm 1 \pmod{2^{c-1}}$ .

But,  $b < 2^c + 1$ . So we have four possibilities:

- \* **Subcase 2.2.2.1:** suppose  $b = 1$ . Then we get  $2^{2c} + 2^c = 2^a$ , which is impossible.



- \* **Subcase 2.2.2.2:** suppose  $b = 2^{c-1} - 1$ . Then we get  $(2^{c-1} + 2)(2^c + 2^{c-1}) = 2^c(2^{a-c} + 1)$  and hence  $3 \cdot 2^{c-2} = 2^{a-c} - 2$ . This implies  $a - c = 3$  and  $c - 2 = 1$ , so  $c = 3$ , or  $w = 6$ , hence  $t = 2^{w-2} + 1 = 17$ .

This gives  $\{289, 290, 291, 292, 293\}$  which indeed works.

- \* **Subcase 2.2.2.3:** suppose  $b = 2^{c-1} + 1$ . Then we get  $2^{c-1}(2^c + 2^{c-1} + 2) = 2^c(2^{a-c} + 1)$ , or  $2^{c-1} + 2^{c-2} + 1 = 2^{a-c} + 1$ , which is impossible.
- \* **Subcase 2.2.2.4:** suppose  $b = 2^c - 1$ . This gives  $2 \cdot 2^{c+1} = 2^c(2^{a-c} + 1)$ , which is impossible.

**Case 3:** Suppose the five good numbers are  $\{t^2 - 1, t^2, t^2 + 1, t^2 + 2, t^2 + 3\}$ .

In that case,  $\{t^2, t^2 + 1, t^2 + 2, t^2 + 3, t^2 + 4\}$  is also a set of five consecutive good numbers. Using case 2, the new candidate this now gives are  $\{8, 9, 10, 11, 12\}$  and  $\{288, 289, 290, 291, 292\}$ , which work.

**N3.** Let  $a_1, a_2, \dots$  be an infinite sequence of positive integers satisfying  $a_1 = 1$  and

$$a_n \mid a_k + a_{k+1} + \dots + a_{k+n-1}$$

for all positive integers  $k$  and  $n$ . Find the maximum possible value of  $a_{2018}$ .

(Krit Boonsiriseth)

The answer is  $a_{2018} \leq 2^{1009} - 1$ . To see this is attainable, consider the sequence

$$a_n = \begin{cases} 1 & n \text{ odd} \\ 2^{n/2} - 1 & n \text{ even.} \end{cases}$$

This can be checked to work, so we prove it's optimal.

We have  $a_2 \mid a_1 + a_2 = 1 + a_2 \implies a_2 = 1$ .

Now consider an integer  $n$ , and let  $s = s_n = a_1 + \dots + a_n$ . Then

$$\begin{aligned} a_{n+1} &\mid s \\ a_{n+2} &\mid s + a_{n+1} \\ a_{n+2} &\equiv 1 \pmod{a_{n+1}}. \end{aligned}$$

Thus,  $\gcd(a_{n+2}, a_{n+1}) = 1$ . So  $a_{n+2} \leq \frac{s+a_{n+1}}{a_{n+1}}$ , and thus

$$a_{n+1} + a_{n+2} \leq 1 + a_{n+1} + \frac{s}{a_{n+2}} \leq s + 2.$$

So, we have

$$\begin{aligned} a_1 + a_2 &= 2 \\ a_3 + a_4 &\leq 2 + 2 = 4 \\ a_5 + a_6 &\leq (2 + 4) + 2 = 8 \\ a_7 + a_8 &\leq (2 + 4 + 8) + 2 = 16 \\ &\vdots \\ a_{2017} + a_{2018} &\leq 2^{1009}. \end{aligned}$$

Thus  $a_{2018} \leq 2^{1009} - a_{2017} \leq 2^{1009} - 1$ .

**Remark** (Motivational notes). It's very quick to notice  $a_{n+1} \mid a_1 + \dots + a_n$ , which already means that given the first  $n$  terms of the sequence there are finitely many possibilities for the next one. Thus it's possible to play with "small cases" by drawing a large tree.

When doing so, one might hope that somehow  $a_n = a_1 + \dots + a_{n-1}$  is achievable, but quickly notices in such a tree that if  $a_n$  is the sum of all previous terms, then  $a_{n+1} = 1$  is forced. This gives the idea to try to look at the terms in pairs, rather than one at a time, and this gives the correct bound.

As for extracting the equality case from this argument, there are actually two natural curves to try. We have  $a_3 \mid 1 + 1 = 2$ . If we have  $a_3 = 2$  we get  $a_4 = 1$ ,  $a_5 \leq 5$ , but then  $a_6$  actually gets stuck. But if we have  $a_3 = 1$  instead, we get  $a_4 = 3$ ,  $a_5 = 1$ ,  $a_6 = 7$ , and so on; pushing this gives the equality case above, seen to work. I think it's quite unnatural to guess the correct construction before having the corresponding  $s + 2$  estimate.

**N4.** Fix a positive integer  $n > 1$ . We say a nonempty subset  $S$  of  $\{0, 1, \dots, n-1\}$  is  $d$ -coverable if there exists a polynomial  $P$  with integer coefficients and degree at most  $d$ , such that  $S$  is exactly the set of residues modulo  $n$  that  $P$  attains as it ranges over the integers.

For each  $n$ , determine the smallest  $d$  such that any nonempty subset of  $\{0, \dots, n-1\}$  is  $d$ -coverable, or prove that no such  $d$  exists.

(Carl Schildkraut)

This is possible for  $n = 4$  or  $n$  prime, in which case  $d = n - 1$  is best possible. Let  $P(\mathbb{Z}/n)$  denote the range of a polynomial modulo  $n$ .

- We first note that if  $n = q_1 \dots q_k$  is the product of  $k \geq 2$  distinct prime powers, then

$$|P(\mathbb{Z}/n)| = \prod_{i=1}^k |P(\mathbb{Z}/q_i)|.$$

Hence any subset  $S$  with size  $n - 1$  is not coverable.

- If  $n = p^e$  is a prime power with other than 4 with  $e \geq 2$ , consider the set  $S = \{0, 1, \dots, p-1, p\}$ . We claim it is not coverable.

Indeed, if  $P$  covers it, WLOG  $P(0) = 0$ . Now,  $P$  is surjective modulo  $p$ , hence bijective, and thus  $P(x) \equiv 0 \pmod{p} \iff x \equiv 0 \pmod{p}$ . Now we can write

$$P(x) = a_1x + a_2x^2 + \dots$$

- If  $a_1 \equiv 0 \pmod{p}$ , then  $x \equiv 0 \pmod{p} \implies P(x) \equiv 0 \pmod{p^2}$ , so  $p$  does not appear in the image.
- If  $a_1 \not\equiv 0 \pmod{p}$ , then  $p, 2p, \dots$  all appear in the image, which is wrong for  $n > 4$ .
- Let  $n = 4$ , and consider  $S \pmod{4}$ .
  - If  $S = \{k\}$  take  $P(x) = k$ .
  - If  $S = \{k, k+1\}$  take  $P(x) = x^2 + k$ .
  - If  $S = \{k, k+2\}$  take  $P(x) = 2x^2 + k$ .
  - If  $S = \{k-1, k, k+1\}$  take  $P(x) = x^3 + k$ .

We claim also the example  $S = \{-1, 0, 1\}$  is not 2-coverable. Indeed, WLOG  $P(0) = 0$  so  $P(x) = x(x+c)$ . Then  $P(2) \equiv 0 \pmod{4}$ , meaning  $c$  is even. But then  $P(1) \equiv c+1 \pmod{4}$  and  $P(-1) \equiv 1-c \pmod{4}$ , so  $P(1) \equiv P(-1)$ .

- If  $S = \{0, 1, 2, 3\}$  take  $P(x) = x$ .
- Let  $n = 2$ .
  - If  $S = \{k\}$  take  $P(x) = k$ .
  - If  $S = \{0, 1\}$  take  $P(x) = x$ . This is obviously not 0-coverable.

- If  $n = p$  is an odd prime, we claim  $S = \{1, \dots, p-1\}$  is not  $(p-2)$ -coverable. Indeed, suppose  $P(x) = a_{p-2}x^{p-2} + \dots + a_0$  covered it. Then

$$\sum_x P(x) \equiv \sum_k a_k \sum_x x^k \equiv 0 \pmod{p}.$$

However, if  $P(\mathbb{Z}/p) = \{1, \dots, p-1\}$  then some element appears twice and the others appear once. If  $k$  is the repeated element though, then  $\sum_x P(x) = (1 + \dots + (p-1)) + k \equiv k \not\equiv 0 \pmod{p}$ .