Problem 1. Let $a_1, a_2, \ldots, a_n$ be positive integers with product $P$, where $n$ is an odd positive integer. Prove that
\[
gcd(a_1^n + P, a_2^n + P, \ldots, a_n^n + P) \leq 2 \gcd(a_1, \ldots, a_n)^n.
\]

Problem 2. Let $ABC$ be a triangle with orthocenter $H$, and let $M$ be the midpoint of $BC$. Suppose that $P$ and $Q$ are distinct points on the circle with diameter $AH$, different from $A$, such that $M$ lies on line $PQ$. Prove that the orthocenter of $\triangle APQ$ lies on the circumcircle of $\triangle ABC$.

Problem 3. nicky is drawing kappas in the cells of a square grid. However, he does not want to draw kappas in three consecutive cells (horizontally, vertically, or diagonally). Find all real numbers $d > 0$ such that for every positive integer $n$, nicky can label at least $dn^2$ cells of an $n \times n$ square.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.
Problem 4. An integer \( n > 2 \) is called tasty if for every ordered pair of positive integers \((a, b)\) with \( a + b = n \), at least one of \( \frac{a}{b} \) and \( \frac{b}{a} \) is a terminating decimal. Do there exist infinitely many tasty integers?

Problem 5. The edges of \( K_{2017} \) are each labelled with 1, 2, or 3 such that any triangle has sum of labels at least 5. Determine the minimum possible average of all \( \binom{2017}{2} \) labels.

(Here \( K_{2017} \) is defined as the complete graph on 2017 vertices, with an edge between every pair of vertices.)

Problem 6. Find all functions \( f : \mathbb{R} \to \mathbb{R} \) such that for all real numbers \( a, b, \) and \( c \):

(i) If \( a + b + c \geq 0 \) then \( f(a^3) + f(b^3) + f(c^3) \geq 3f(abc) \).

(ii) If \( a + b + c \leq 0 \) then \( f(a^3) + f(b^3) + f(c^3) \leq 3f(abc) \).