Shortlisted Problems

19^{th} ELMO

Pittsburgh, PA, 2017

Note of Confidentiality

The shortlisted problems should be kept strictly confidential until disclosed publicly by the committee on the ELMO.

Contributing Students

The Problem Selection Committee for ELMO 2017 thanks the following proposers for contributing **45 problems** to this year's Competition:

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Problem Selection Committee

The Problem Selection Committee for ELMO 2017 was led by **Evan Chen** and consisted of:

- Ashwin Sah
- James Lin
- Kevin Ren
- Mihir Singhal
- Michael Ma
- Michael Ren
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Problems

A1. Let $0 < k < \frac{1}{2}$ be a real number and let a_0 and b_0 be arbitrary real numbers in $(0,1)$. The sequences $(a_n)_{n\ge 0}$ and $(b_n)_{n\ge 0}$ are then defined recursively by

$$a_{n+1} = \frac{a_n+1}{2} \qquad \text{and} \qquad b_{n+1} = b_n^k$$

for $n \ge 0$. Prove that $a_n < b_n$ for all sufficiently large n.

(Michael Ma)

A2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers a, b, and c:

(i) If $a+b+c \ge 0$ then $f(a^3)+f(b^3)+f(c^3) \ge 3f(abc)$.

(ii) If $a+b+c \le 0$ then $f(a^3)+f(b^3)+f(c^3) \le 3f(abc)$.

(Ashwin Sah)

C1. Let m and n be fixed distinct positive integers. A wren is on an infinite chessboard indexed by $\mathbb{Z}^2$, and from a square (x,y) may move to any of the eight squares $(x\pm m, y\pm n)$ or $(x\pm n, y\pm m)$. For each $\{m,n\}$, determine the smallest number k of moves required for the wren to travel from $(0,0)$ to $(1,0)$, or prove that no such k exists.

(Michael Ren)

C2. The edges of K_{2017} are each labelled with 1, 2, or 3 such that any triangle has sum of labels at least 5. Determine the minimum possible average of all $\binom{2017}{2}$ labels.

(Michael Ma)

C3. Consider a finite binary string b with at least 2017 ones. Show that one can insert some plus signs in between pairs of digits such that the resulting sum, when performed in base 2, is equal to a power of two.

(David Stoner)

C4. nicκy is drawing kappas in the cells of a square grid. However, he does not want to draw kappas in three consecutive cells (horizontally, vertically, or diagonally). Find all real numbers $d > 0$ such that for every positive integer n, nicκy can label at least dn^2 cells of an $n \times n$ square.

(Mihir Singhal and Michael Kural)

C5. There are n MOPpers $p_1, \ldots, p_n$ designing a carpool system to attend their morning class. Each p_i's car fits $\chi(p_i)$ people ($\chi\colon \{p_1,\ldots,p_n\} \to \{1,2,\ldots,n\}$). A c-fair carpool system is an assignment of one or more drivers on each of several days, such that each MOPper drives c times, and all cars are full on each day. (More precisely, it is a sequence of sets $(S_1,\ldots,S_m)$ such that $|\{k : p_i \in S_k\}| = c$ and $\sum_{x\in S_j} \chi(x) = n$ for all i, j.)

Suppose it turns out that a 2-fair carpool system is possible but not a 1-fair carpool system. Must n be even?

(Nathan Ramesh and Palmer Mebane)

G1. Let ABC be a triangle with orthocenter H, and let M be the midpoint of $\overline{BC}$. Suppose that P and Q are distinct points on the circle with diameter $\overline{AH}$, different from A, such that M lies on line PQ. Prove that the orthocenter of $\triangle APQ$ lies on the circumcircle of $\triangle ABC$.

(Michael Ren)

G2. Let ABC be a scalene triangle with $\angle A = 60^\circ$. Let E and F be the feet of the angle bisectors of $\angle ABC$ and $\angle ACB$ respectively, and let I be the incenter of $\triangle ABC$. Let P, Q be distinct points such that $\triangle PEF$ and $\triangle QEF$ are equilateral. If O is the circumcenter of $\triangle APQ$, show that $\overline{OI} \perp \overline{BC}$.

(Vincent Huang)

G3. Call the ordered pair of distinct circles (ω, γ) scribable if there exists a triangle with circumcircle ω and incircle γ. Prove that among n distinct circles there are at most $(n/2)^2$ scribable pairs.

(Daniel Liu)

G4. Let ABC be an acute triangle with incenter I and circumcircle ω. Suppose a circle ω_B is tangent to BA, BC, and internally tangent to ω at B_1, while a circle ω_C is tangent to CA, CB, and internally tangent to ω at C_1. If B_2, C_2 are the points on ω opposite to B, C, respectively, and X denotes the intersection of B_1C_2, B_2C_1, prove that $XA = XI$.

(Vincent Huang and Nathan Weckwerth)

N1. Let $a_1, a_2, \dots, a_n$ be positive integers with product P, where n is an odd positive integer. Prove that

$$\gcd(a_1^n + P, a_2^n + P, \dots, a_n^n + P) \le 2\gcd(a_1, \dots, a_n)^n.$$

(Daniel Liu)

N2. An integer $n > 2$ is called *tasty* if for every ordered pair of positive integers (a, b) with $a + b = n$, at least one of $\frac{a}{b}$ and $\frac{b}{a}$ is a terminating decimal. Do there exist infinitely many tasty integers?

(Vincent Huang)

N3. For each integer $C > 1$, decide whether there exists pairwise distinct positive integers a_1, a_2, a_3, $\dots$ such that for every $k \ge 1$,

$$a_{k+1}^k \quad \text{divides} \quad C^k a_1 a_2 \dots a_k.$$

(Daniel Liu)

Solutions

A1. Let $0 < k < \frac{1}{2}$ be a real number and let a_0 and b_0 be arbitrary real numbers in $(0, 1)$. The sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are then defined recursively by

$$a_{n+1} = \frac{a_n + 1}{2} \qquad \text{and} \qquad b_{n+1} = b_n^k$$

for $n \geq 0$. Prove that $a_n < b_n$ for all sufficiently large n.

(Michael Ma)

It should be clear that both sequences converge to 1. In the first sequence, the distance from 1 is halved every time and converges to 0. In the second sequence $b_n = b_0^{k^n}$ and since k^n converges to 0, b_i converges to 1.

The key lemma to solve the problem is the following:

Lemma. *If $k < \frac{1}{2}$ then there exists $0 < x_0 < 1$ such that whenever $x_0 < x < 1$,*

$$x^k > \frac{2k+1}{4}x + \frac{3-2k}{4}.$$

Proof. First notice that if we take the tangent to $y = x^k$ at $(1, 1)$ we get the equation $y = kx + (1 - k)$. We can see by taking the first derivative of

$$kx + (1 - k) - x^k$$

to get

$$k - kx^{k-1}$$

which is negative as $kx + (1 - k) - x^k$ is decreasing from 0 to 1. Furthermore x^k is concave and increasing from 0 to 1. Now it if we take a line of higher slope than k passing through $(1, 1)$ for large enough x the line will fall under x^k. $\square$

Now let x_0 be as above, and let $a = \frac{2k+1}{4} < \frac{1}{2}$ for convenience. Now we can see that

$$b_{n+1} > ab_n + (1 - a).$$

Take the smallest M such that a_M and b_M are both larger than x_0. By iterating both recurrences we can see that for $\ell = 0, 1, \ldots$ we have

$$a_{M+\ell} = 1 - \left(\frac{1}{2}\right)^{\ell}(1 - a_M) \quad \text{and} \quad b_{M+\ell} > 1 - a^{\ell}(1 - b_M).$$

Since $\frac{1}{2a} > 1$ we can take a sufficiently large positive integer ℓ_0 such that $\left(\frac{1}{2a}\right)^{\ell_0} > \frac{1-b_M}{1-a_M}$. Then taking $N = M + \ell_0$ we are done since $b_N > a_N$ and

$$x^k > ax + (1 - a) > \frac{x+1}{2}$$

for $x > x_0$.

A2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers a, b, and c:

(i) If $a+b+c \ge 0$ then $f(a^3)+f(b^3)+f(c^3) \ge 3f(abc)$.

(ii) If $a+b+c \le 0$ then $f(a^3)+f(b^3)+f(c^3) \le 3f(abc)$.

(Ashwin Sah)

The answer is $f(x) = kx + \ell$ where k and ℓ are any real numbers with $k \ge 0$.
We begin with some weird optimizations:

- Since f can be shifted by a constant, we get $f(0) = 0$.
- Put $c = 0$ and $b = -a$ to get $f(a^3) + f(-a^3) = 0$, so that f is odd.
- Put $c = 0$ now to get $f(a^3) + f(b^3) \ge 0$ whenever $a + b \ge 0$. Combined with f odd, this implies f is weakly increasing.

Now, we let $c = -a - b$ to get:

$$f(a^3) + f(b^3) + f(-(a+b)^3) = 3f(-ab(a+b))$$

Using oddness and rearranging:

$$f(a^3) + f(b^3) + 3f(ab(a+b)) = f((a+b)^3)$$

Call this property $P(a,b)$.

Lemma. *$f(2^k m) = 2^k f(m)$ for all integer k and real $m > 0$.*

Proof. $P(d^{1/3}, d^{1/3})$ gives $2f(d) + 3f(2d) = f(8d)$. Consider the sequence $\alpha_k = f(2^k m)$. We have a linear recurrence: $\alpha_{k+3} = 3\alpha_{k+1} + 3\alpha_k$. Its characteristic equation has roots $2, -1, -1$, so we have $f(2^k m) = \alpha_k = c_1 2^k + c_2(-1)^k + c_3(-1)^k k$ for some c_1, c_2, c_3 that may depend on m but not on k. This can be extended to negative k as well. Note that since $f(x)$ is increasing and $f(0) = 0$, $\alpha_k \ge 0$ for all k. Now, if either c_2 or c_3 is nonzero, you can take $k \to -\infty$ with the right parity, and you will get $\alpha_k < 0$, a contradiction. Thus $c_2 = c_3 = 0$, so $f(2^k m) = c_1 2^k$. Plugging in $k = 0$, we get $c_1 = f(m)$, so $f(2^k m) = 2^k f(m)$ as desired. $\square$

Lemma. *$f(\phi^{3k} m) = \phi^{3k} f(m)$ for all integer k and real $m > 0$.*

Proof. $P(d^{1/3}, \phi d^{1/3})$ gives $f(d) + 4f(\phi^3 d) = f(\phi^6 d)$. Again, this gives a linear recurrence for the sequence $\beta_k = \phi^{3k} m$, $\beta_{k+2} = 4\beta_{k+1} + \beta_k$. Its characteristic equation has roots $\phi^3, -\phi^{-3}$, so we have $f(\phi^{3k} m) = \beta_k = c_4 \phi^{3k} + c_5(-\phi^{-3})^k$ for some c_4, c_5 that may depend on m but not on k. As before, c_5 must be zero, so $f(\phi^{3k} m) = c_4 \phi^{3k}$. Plugging in $k = 0$, $c_4 = f(m)$, so $f(\phi^{3k} m) = \phi^{3k} f(m)$ as desired. $\square$

Now I claim that $f(x) = f(1)x$ for all x. Since f is odd, we only need to prove this for positive x. If $f(1) = 0$, we are done by Lemma 1. Otherwise, for a contradiction, let $f(n) \neq f(1)n$ for some $n > 0$. (note that $f(n) \ge 0$). Let $f(n) > f(1)n$; the case where $f(n) < f(1)n$ is similar. By Dirichlet's approximation theorem, we can find r, s such that:

$$n < \frac{2^s}{\phi^{3r}} < \frac{f(n)}{f(1)}$$

or, expanding,

$$\phi^{3r} n < 2^s \implies \phi^{3r} f(n) > 2^s f(1)$$

But, by Lemmas 1 and 2:

$$f(\phi^{3r} n) = \phi^{3r} f(n) \qquad \text{and} \qquad f(2^s) = 2^s f(1)$$

a contradiction to the fact that f is increasing. Thus, $f(x) = f(1)x$ for all x. Re-adjusting for the assumption that $f(0) = 0$, $f(x)$ is linear. Plugging back in to the condition, $f(x)$ can be any linear function with a nonnegative coefficient of x.

C1. Let m and n be fixed distinct positive integers. A wren is on an infinite chessboard indexed by $\mathbb{Z}^2$, and from a square (x, y) may move to any of the eight squares $(x \pm m, y \pm n)$ or $(x \pm n, y \pm m)$. For each $\{m, n\}$, determine the smallest number k of moves required for the wren to travel from $(0, 0)$ to $(1, 0)$, or prove that no such k exists.

(Michael Ren)

Sorry, the answer we had originally was wrong. The user `talkon` gives an answer of:

- If $\gcd(m, n) > 1$ then no such sequence exists.
- If $m \equiv n \equiv 1 \pmod 2$ then no such sequence exists.
- Otherwise, suppose m is even. Then the answer is

$$\max\{2p, m\} + \max\{q, n\}$$

where $p \geq 0$ is minimal such that $2mp \equiv \pm 1 \pmod n$, and q is $\frac{2pm \pm 1}{n}$, whichever is the smallest integer.

(The obvious guess $k = m + n$ is not correct.) See https://artofproblemsolving.com/community/c6h1472063.

This problem is actually known already. The question was raised by Alasdair Iain Houston in the 1970s, with members of the Fairy Chess Correspondence Circle. It appeared in print in George Jelliss's paper *Theory of Leapers* in Chessics 24, 1985. (Chessics was a fairy chess and recreational mathematics journal published and edited by Jelliss; issue 24 is available https://www.mayhematics.com/p/p.htm and the discussion of Houston's problem begins page 96.)

C2. The edges of K_{2017} are each labelled with 1, 2, or 3 such that any triangle has sum of labels at least 5. Determine the minimum possible average of all $\binom{2017}{2}$ labels.

(Michael Ma)

In general, the answer for $2m+1$ is $2 - \frac{1}{2m+1}$.

We prove the lower bound by induction on m: assume some edge vw is labeled 1. Then we delete it, noting that edges touching v and w contribute a sum of at least $4 \cdot (2m-1) = 8m-4$. Thus by induction hypothesis the total is at least

$$\binom{2m-1}{2}\left(2 - \frac{1}{2m-1}\right) + (8m-4) + 1 = \binom{2m+1}{2}\left(2 - \frac{1}{2m+1}\right)$$

as desired.

Interestingly, there are (at least) two equality cases. One is to have all edges be 2 except for m disjoint edges, which have weight 1. Another is to split the vertex set into two sets $A \cup B$ with $|A| = m$ and $|B| = m+1$, then weight all edges in $A \times B$ with 1 and the remaining edges with 3.

Remark. In fact, given any equality case on c vertices, one can generate one on $c+2$ vertices by two vertices u and v, connected to the previous c vertices with weight 2, and then equipping uv with weight 1.

C3. Consider a finite binary string b with at least 2017 ones. Show that one can insert some plus signs in between pairs of digits such that the resulting sum, when performed in base 2, is equal to a power of two.

(David Stoner)

Solution by Mihir Singhal:

We first note that, given any binary string with n ones, we can achieve any integer value in the range $[n, \frac{3n}{2}]$ as follows: first, put pluses between every digit. Then, remove the plus directly after every other 1. Doing this one at a time gives everything from n to $\frac{3n}{2}$.

Now we prove the result for $n \geq 17$. Let n be the number of ones. If any power of 2 is in the range $[n, \frac{3n}{2}]$, then we are done already. Otherwise, we must have $2^\alpha + 1 \leq n < \frac{2^{\alpha+2}}{3}$ for some integer α. We claim that $2^{\alpha+1}$ is achievable via the following algorithm:

0. Put pluses in between every digit, so that we have a current sum n.
1. Cut off the part of the string from the fourth to right 1 onwards; call this the tail, and the rest the head.
2. Starting at the leftmost ungrouped 1, group that one with the two digits immediately following it.
3. Repeat step 2 until the sum is $\geq 2^{\alpha+1}$.
4. Adjust the result until the sum is exactly $2^{\alpha+1}$.

We first show that the condition in 3 occurs before step 2 becomes impossible. Note that since there are at least 13 ones in the head, at least four full groups can be attained before step 2 becomes problematic. Note that the group transformations take $1+1+1 \to 7, 1+0+1 \to 5, 1+1+0 \to 6, 1+0+0 \to 4$. In particular, the sum value v becomes $\geq 2v+1$. Suppose that ℓ is the number of leftover ones in the tail after all possible groups have been formed in the manner described, and g is the number of groups formed. The sum at this point is at least:

$$2(n-\ell-4)+g+\ell+4 = 2n+g-\ell-4$$

Since $g \geq 4$ and $\ell \leq 2$, this is at least $2n-2 \geq 2^{\alpha+1}$. So, the condition in step 3 will indeed arise before step 2 becomes impossible.

Now we clarify step 4. Suppose that on the formation of group $1+b_0+b_1 \to 4+2b_0+b_1$ the sum first becomes $\geq 2^{\alpha+1}$. If it equals $2^{\alpha+1}$, we are done. Otherwise, since every grouping increases the sum by at most 4, the beforehand sum is in the set $\{2^{\alpha+1}-3, 2^{\alpha+1}-2, 2^{\alpha+1}-1\}$.

- If the sum is $2^{\alpha+1}-3$, then change $1+b_0$ to $1b_0$ and the tail sum from 4 to 6 (possibly by the lemma).
- If the sum is $2^{\alpha+1}-2$, then change the tail sum from 4 to 6.
- If the sum is $2^{\alpha+1}-1$, then change the tail sum from 4 to 5.

In any case, a final sum of $2^{\alpha+1}$ is attained, as desired.

C4. nicκy is drawing kappas in the cells of a square grid. However, he does not want to draw kappas in three consecutive cells (horizontally, vertically, or diagonally). Find all real numbers $d > 0$ such that for every positive integer n, nicκy can label at least dn^2 cells of an $n \times n$ square.

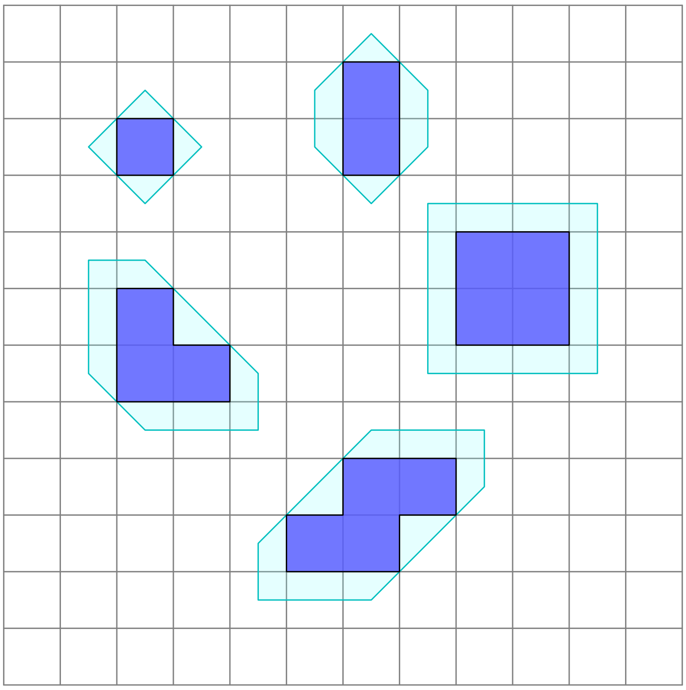
(Mihir Singhal and Michael Kural)

Solution by Yevhenii Diomidov, Kada Williams and Mihir Singhal:

The answer is $d \le \frac{1}{2}$. The construction consists of placing kappas in all squares of the forms $(2k, 4\ell)$, $(2k, 4\ell+1)$, $(2k+1, 4\ell+2)$, and $(2k+1, 4\ell+3)$.

To prove that this is minimal, consider all connected components consisting of squares that contain kappas that are connected via edges. It is easy to see that there are only five different kinds of connected components.

Extend each connected component into a larger figure as shown below:



Due to the fact that there are no three kappas in a line and due to the nature of the extensions, one can see that after extension, the interiors of the figures remain disjoint. However, note that the extended area of each figure is at least twice its original area (it is exactly twice except for the 2 by 2 square, for which it is $\frac{9}{4}$ times the original area). Some of the extended regions may fall outside the square, but this is fine since the error is at most $O(n)$.

Thus, Nicky can cover at most $\frac{n^2}{2} + O(n)$ of the squares with kappas, which is what we wanted to show.

C5. There are n MOPpers $p_1, \ldots, p_n$ designing a carpool system to attend their morning class. Each p_i's car fits $\chi(p_i)$ people ($\chi\colon \{p_1, \ldots, p_n\} \to \{1, 2, \ldots, n\}$). A c-fair carpool system is an assignment of one or more drivers on each of several days, such that each MOPper drives c times, and all cars are full on each day. (More precisely, it is a sequence of sets $(S_1, \ldots, S_m)$ such that $|\{k : p_i \in S_k\}| = c$ and $\sum_{x \in S_j} \chi(x) = n$ for all i, j.)

Suppose it turns out that a 2-fair carpool system is possible but not a 1-fair carpool system. Must n be even?

(Nathan Ramesh and Palmer Mebane)

First solution (Palmer Mebane) Let $n = 5 \cdot 2^{20} + 2^{15} - 1$ which is odd. For all but 15 people, set $\chi(x) = n$. Biject the 15 people to two element subsets of $\{1, 2, 3, 4, 5, 6\}$, and construct a complete graph K_6 where 1 to 6 are the vertices and each person $\{i, j\}$ is an edge from i to j. There are 15 perfect matchings (so 3 edges) on K_6. Number these matchings from 0 to 14, and assign each edge the matching numbers it's a part of, so each person/edge has 3 matching numbers assigned to them. If the three numbers for person p_i are x, y, z, set $\chi(p_i) = 2^{20} + 2^x + 2^y + 2^z$. We claim this is 2-fair but not 1-fair.

It is 2-fair because we can take 6 sets S_i such that S_i contains all people whose subsets are of the form $\{i, j\}$ for some $j \neq i$. This is because the 15 matching numbers assigned to 5 people all incident to the same vertex are distinct; that's how matchings work.

However it is not 1-fair, because we constructed χ so that those sets S_i are the only ways to choose a subset of people whose χ values sum to n. The $5 \cdot 2^{20}$ term in n forces us to choose exactly 5 people. Then each of these 5 people comes with three matching numbers, and the only way to get the $2^{15} - 1$ term by summing 15 powers of 2 is to sum $2^0 + 2^1 + \cdots + 2^{14}$. So our 5 people have to be assigned each matching number from 0 to 14 exactly once between them. But if the edges we choose don't all come from the same vertex, then two of the edges will be in the same matching, so that matching number is repeated and we can't get 15 powers of 2 to sum to $2^{15} - 1$.

Second solution (Krit Boonsiriseth) Here is a counterexample with $n = 23$: the capacities are 2^4, 7^3, 3^2, 8^3, 17, 18, 23^9. It is not 1-fair since the 17 needs either all the 2's or all the 3's while the 18 needs a 2 and a 3. However, a 2-fair carpool system is:

- $2 + 2 + 2 + 17$
- $2 + 7 + 7 + 7$
- $7 + 8 + 8$
- $7 + 8 + 8$
- $7 + 8 + 8$
- $7 + 8 + 8$
- $2 + 3 + 18$
- $3 + 3 + 17$
- eighteen 23's.

G1. Let ABC be a triangle with orthocenter H, and let M be the midpoint of $\overline{BC}$. Suppose that P and Q are distinct points on the circle with diameter $\overline{AH}$, different from A, such that M lies on line PQ. Prove that the orthocenter of $\triangle APQ$ lies on the circumcircle of $\triangle ABC$.

(Michael Ren)

We present seven different solutions.

First solution (Michael Ren) Let R be the intersection of (AH) and (ABC), and let D, E, and F respectively be the orthocenter of APQ, the foot of the altitude from A to PQ, and the reflection of D across E. Note that F lies on (AH) and E lies on (AM). Let S and H' be the intersection of AH with BC and (ABC) respectively. Note that R is the center of spiral similarity taking DEF to $H'SH$, so D lies on (ABC), as desired.

Second solution (Vincent Huang, Evan Chen) Let DEF be the orthic triangle of ABC. Let N and S be the midpoints of PQ and AH. Then $\overline{MS}$ is the diameter of the nine-point circle, so since $\overline{SN}$ is the perpendicular bisector of $\overline{PQ}$ the point N lies on the nine-point circle too. Now the orthocenter of $\triangle APQ$ is the reflection of H across N, hence lies on the circumcircle (homothety of ratio 2 takes the nine-point circle to (ABC)).

Third solution (Zack Chroman) Let R be the midpoint of PQ, and X the point such that $(M, X; P, Q) = -1$. Take E and F to be the feet of the B, C altitudes. Recall that ME, MF are tangents to the circle (AH), so EF is the polar of M.

Then note that $MP \cdot MQ = MX \cdot MR = ME^2$. Then, since X is on the polar of M, R lies on the nine-point circle — the inverse of that polar at M with power ME^2. Then by dilation the orthocenter $2\vec{R} - \vec{H}$ lies on the circumcircle of ABC.

Fourth solution (Zack Chroman) We will prove the following more general claim which implies the problem:

Claim. For a circle γ with a given point A and variable point B, consider a fixed point X not on γ. Let C be the second intersection of XB and γ, then the locus of the orthocenter of ABC is a circle

Proof. Complex numbers is straightforward, but suppose we want a more synthetic solution. Let D be the midpoint of BC. If O is the center of the circle, $\angle OMX = 90$, so M lies on the circle (OX). Then

$$H = 4O - A - B - C = 4O - A - 2D.$$

So H lies on another circle. (Here we can use complex numbers, vectors, coordinates, whatever; alternatively we can use the same trick as above and say that H is the reflection of a fixed point over D). $\square$

Fifth solution (Kevin Ren) Let O be the midpoint of AH and N be the midpoint of PQ. Let K be the orthocenter of APQ.

Because $AP \perp KQ$ and $KP \perp HP$, we have $KQ \parallel PH$. Similarly, $KP \parallel QH$. Thus, $KPHQ$ is a parallelogram, which means KH and PQ share the same midpoint N.

Since N is the midpoint of chord PQ, we have $\angle ONM = 90^\circ$. Hence N lies on the 9-point circle. Take a homothety from H mapping N to K. This homothety maps the 9-point circle to the circumcircle, so K lies on the circumcircle.

Sixth solution (Evan Chen, complex numbers) We use complex numbers with $(AHEF)$ the unit circle, centered at N. Let a, e, f denote the coordinates of A, E, F, and hence $h = -a$. Since M is the pole of $\overline{EF}$, we have $m = \frac{2ef}{e+f}$. Now, the circumcenter O of $\triangle ABC$ is given by $o = \frac{2ef}{e+f} + a$, due to the fact that $ANMO$ is a parallelogram.

The unit complex numbers p and q are now known to satisfy

$$p + q = \frac{2ef}{e+f} + \frac{2pq}{e+f}$$

so

$$(a+p+q) - o = \frac{2pq}{e+f} \qquad \text{and} \qquad a - o = \frac{2ef}{e+f}$$

which clearly have the same magnitude. Hence the orthocenter of $\triangle APQ$ and A are equidistant from O.

Seventh solution (Evan Chen, complex numbers) Here is another complex solution using (APQ) as the unit circle. We let the fourth point M satisfy $m + pq\overline{m} = p + q$. Moreover, let D be the reflection of H across M; we wish to show $a + p + q$ lies on the circle with diameter $\overline{AD}$. This is:

$$\begin{aligned} \frac{(a+p+q)-a}{(a+p+q)-(2m-h)} &= \frac{p+q}{p+q-2m} \\ \overline{\left(\frac{p+q}{p+q-2m}\right)} &= \frac{\frac{1}{p}+\frac{1}{q}}{\frac{1}{p}+\frac{1}{q}-2\overline{m}} = \frac{p+q}{p+q-2pq\overline{m}} \\ &= \frac{p+q}{p+q-2(p+q-m)} = \frac{p+q}{2m-p-q}. \end{aligned}$$

G2. Let ABC be a scalene triangle with $\angle A = 60^\circ$. Let E and F be the feet of the angle bisectors of $\angle ABC$ and $\angle ACB$ respectively, and let I be the incenter of $\triangle ABC$. Let P, Q be distinct points such that $\triangle PEF$ and $\triangle QEF$ are equilateral. If O is the circumcenter of $\triangle APQ$, show that $\overline{OI} \perp \overline{BC}$.

(Vincent Huang)

WLOG assume $AB < AC$. Also suppose P is on the same side of EF as A, so that A, P, E, F are concyclic. Basic angle-chasing tells us $\angle EIF = 120^\circ$, hence I lies on the same circle as A, E, F, P.

Let the circumcircle of $\triangle BFI$ meet BC again at point Q'. By Miquel's Theorem on $\triangle ABC$ and points Q', EF we have that Q', I, C, E are concyclic. Hence $\angle EQ'F = \angle EQ'I + \angle FQ'I = \angle ECI + \angle FBI = \frac{1}{2}(\angle B + \angle C) = 60^\circ$, implying that E, F, Q, Q' are concyclic.

Since $\angle FEI = \angle FAI = 30^\circ = \frac{1}{2}\angle FEQ$ and $FE = EQ$, we know that F, Q are reflections about BI, so since $F \in AB$ we have $Q \in BC$. Now since I must lie on the perpendicular bisector of QQ', we deduce that if X is the midpoint of QQ', then $IX \perp BC$.

Since AP is the exterior angle bisector of $\angle BAC$ it's well-known that AP, EF, BC concur at a point R, hence $RA \cdot RP = RE \cdot RF = RQ \cdot RQ'$, implying A, P, Q, Q' are concyclic, hence $OX \perp BC \implies OI \perp BC$ as desired.

G3. Call the ordered pair of distinct circles (ω, γ) scribable if there exists a triangle with circumcircle ω and incircle γ. Prove that among n distinct circles there are at most $(n/2)^2$ scribable pairs.

(Daniel Liu)

The main point is to show that there are no triangles in the graph of scribable pairs, after which Turan's theorem finishes the proof. This is essentially Poncelet porism but we give a direct proof.

Suppose there exist three circles A, B, C with radii a, b, c respectively (with $a > b > c > 0$) such that $(A, B), (B, C), (A, C)$ are scribable. Then by triangle inequality and Euler's formula, we have

$$\sqrt{a(a-2b)} + \sqrt{b(b-2c)} \geq \sqrt{a(a-2c)}.$$

However note that

$$\sqrt{a(a-2c)} - \sqrt{a(a-2b)} = \frac{\sqrt{a}(2b-2c)}{\sqrt{a-2c}+\sqrt{a-2b}} > \frac{\sqrt{a}(2b-2c)}{\sqrt{a}+\sqrt{a}} = b - c$$

and

$$\sqrt{b(b-2c)} \leq \sqrt{b^2 - 2bc + c^2} = b - c$$

establishing a contradiction.

G4. Let ABC be an acute triangle with incenter I and circumcircle ω. Suppose a circle ω_B is tangent to BA, BC, and internally tangent to ω at B_1, while a circle ω_C is tangent to CA, CB, and internally tangent to ω at C_1. If B_2, C_2 are the points on ω opposite to B, C, respectively, and X denotes the intersection of B_1C_2, B_2C_1, prove that $XA = XI$.

(Vincent Huang and Nathan Weckwerth)

Solution by Ankan:

Let M_B and N_B be the midpoints of the minor and major arcs AC, and define M_C and N_C similarly. It's well known that $I = \overline{N_B B_1} \cap \overline{N_C C_1}$.

The case where $O = I$ is left to the reader as an exercise. If $O \neq I$, Pascal on $M_B B B_2 C_1 N_C M_C$ and $M_C C C_2 B_1 N_B M_B$ give $\overline{M_B M_C} \cap \overline{C_1 B_2} \in \overline{OI}$ and $\overline{M_B M_C} \cap \overline{B_1 C_2} \in \overline{OI}$, so $X = \overline{B_1 C_2} \cap \overline{C_1 B_2} \in \overline{M_B M_C}$.

But this is equivalent to $XA = XI$, so done. (One way to see this is to let I_A, I_B, and I_C be the A-, B-, and C-excenters of $\triangle ABC$, and consider the homothety with ratio $\frac{1}{2}$ centered at I; it takes $\overline{I_B I_C}$ to $\overline{M_B M_C}$.)

N1. Let $a_1, a_2, \dots, a_n$ be positive integers with product P, where n is an odd positive integer. Prove that

$$\gcd(a_1^n + P, a_2^n + P, \dots, a_n^n + P) \le 2 \gcd(a_1, \dots, a_n)^n.$$

(Daniel Liu)

The inequality is homogenous, so we may assume $\gcd(a_1, \dots, a_n) = 1$. Then we want to show

$$\gcd(a_1^n + P, \dots, a_n^n + P) \le 2.$$

So it suffices to show that neither 4 nor any odd prime divides the gcd.

First, let p be an odd prime. Suppose that $p \mid a_i^n + P$ for all i. Then $a_i^n \equiv -P \pmod{p}$, so multiplying this for all i, we get $P^n \equiv -P^n \pmod{p}$. Then we see that $p \mid P$, so p divides a_i^n for each i, contradiction.

If $p = 4$, similarly $2 \mid P$ and $2 \mid a_i^n$ for each i, contradiction.

N2. An integer $n > 2$ is called *tasty* if for every ordered pair of positive integers (a, b) with $a + b = n$, at least one of $\frac{a}{b}$ and $\frac{b}{a}$ is a terminating decimal. Do there exist infinitely many tasty integers?

(Vincent Huang)

The answer is no. (In fact, a computation implies that $n = 21$ is the largest one.)

First, we recall the well-known fact that the fraction $\frac{a}{b}$, with $\gcd(a, b) = 1$, is terminating if and only if the prime factorization of b consists only of 2s and 5s.

Consider some tasty number n and all pairs (a, b) with $a+b = n, \gcd(a, n) = 1, a \le 0.5n$. It's clear that there are $0.5\phi(n)$ of these pairs, and since $\gcd(a, b) = 1$ we must have that at least one of a and b has a prime factorization of only 2s and 5s.

But considering all numbers $2^x5^y \le n$, we know $x \le \log_2 n + 1, y \le \log_5 n + 1$, hence there are at most $(\log_2 n + 1)(\log_5 n + 1)$ such numbers, so we deduce that $(\log_2 n + 1)(\log_5 n + 1) \ge 0.5\phi(n)$.

Lemma. *For every $n > 2$, $\phi(n) \ge 0.5\sqrt{n}$.*

Proof. Decompose n into prime powers $p_i^{e_i}$. For each $p_i > 2$, it's easy to show that $p_i^{e_i-1}(p - 1) \ge \sqrt{p_i^{e_i}}$. For $p_i = 2$, we can show that $p_i^{e_i-1}(p - 1) \ge 0.5\sqrt{p_i^{e_i}}$, hence multiplying these bounds gives the desired. $\square$

Therefore, for n to be tasty, we need $(\log_2 n + 1)(\log_5 n + 1) \ge 0.25\sqrt{n}$, which only holds for finitely many n as desired.

N3. For each integer $C > 1$, decide whether there exists pairwise distinct positive integers a_1, a_2, a_3, ... such that for every $k \geq 1$,

$$a_{k+1}^k \quad \text{divides} \quad C^k a_1 a_2 \dots a_k.$$

(Daniel Liu)

No sequence exists for any C. Note that the divisibility is homogenous with respect to the a_i so we can shift the sequence and WLOG assume that $a_1 = 1$.

Note that any prime divisor of the a_i must also be a prime divisor of C. Let

$$C = p_1^{c_1} p_2^{c_2} \dots p_k^{c_k}$$

be the prime factorization of C.

Claim. Fix $p = p_j$ and $c = c_j$. Then for any index k we have

$$\nu_p(a_i) \leq cH_k + \nu_p(a_1).$$

Proof. Let $b_i = \nu_p(a_i)$. We apply strong induction: Base case of $k = 1$ is trivial. Now assume $b_i \leq H_{i-1}c + b_1$ for $i \leq k$; then

$$\begin{aligned}
b_{k+1} &\leq c + \frac{\sum_{i=1}^k b_i}{k} \\
&\leq c + \frac{\sum_{i=1}^k H_{i-1}c + b_1}{k} \\
&= c\left(1 + \frac{\sum_{i=1}^k H_{i-1}}{k}\right) + b_1 \\
&= c\left(1 + \frac{\frac{k-1}{1} + \frac{k-2}{2} + \dots + \frac{1}{k-1}}{k}\right) + b_1 \\
&= c\left(1 + \frac{k-1}{k \cdot 1} + \frac{k-2}{k \cdot 2} + \dots + \frac{1}{(k-1) \cdot k}\right) + b_1 \\
&= c\left(1 + \frac{1}{1} - \frac{1}{k} + \frac{1}{2} - \frac{1}{k} + \dots + \frac{1}{k-1} - \frac{1}{k}\right) + b_1 \\
&= cH_k + b_1
\end{aligned}$$

and the induction is complete. $\square$

Now, let N be a positive integer, and let $m = 1 + \max_j \nu_{p_j}(a_1)$. We have that

$$\nu_{p_j} a_i \leq c_j H_i + \nu_{p_j}(a_1) \leq c_j(m + \log N)$$

if $i \leq N$. Hence, there are at most

$$\prod_{j=1}^k [1 + c_j(m + \log N)] = O\left((\log N)^k\right)$$

possible k-triples that $(\nu_{p_1} a_i, \nu_{p_2} a_i, \dots, \nu_{p_k} a_i)$ can be. But this also needs to be at least $N + 1$, which is impossible for large N.