15^{th} Everyone Lives at Most Once

ELMO 2013

Lincoln, Nebraska

OFFICIAL SOLUTIONS

1. Let $a_1, a_2, \ldots, a_9$ be nine real numbers, not necessarily distinct, with average m. Let A denote the number of triples $1 \le i < j < k \le 9$ for which $a_i + a_j + a_k \ge 3m$. What is the minimum possible value of A?

   Proposed by Ray Li.

   Answer. $A \ge 28$.

   Solution 1. Call a 3-set *good* iff it has average at least m, and let S be the family of good sets.

   The equality case $A = 28$ can be achieved when $a_1 = \cdots = a_8 = 0$ and $a_9 = 1$. Here $m = \frac{1}{9}$, and the good sets are precisely those containing a_9. This gives a total of $\binom{8}{2} = 28$.

   To prove the lower bound, suppose we have exactly N good 3-sets, and let $p = \frac{N}{\binom{9}{3}}$ denote the probability that a randomly chosen 3-set is good. Now, consider a random permutation π of $\{1, 2, \ldots, 9\}$. Then the corresponding partition $\bigcup_{i=0}^{2}\{\pi(3i+1), \pi(3i+2), \pi(3i+3)\}$ has at least 1 good 3-set, so by the linearity of expectation,

$$\begin{aligned} 1 &\le \mathbb{E}\left[\sum_{i=0}^{2}[\{\pi(3i+1), \pi(3i+2), \pi(3i+3)\} \in S]\right] \\ &= \sum_{i=0}^{2}\left[\mathbb{E}[\{\pi(3i+1), \pi(3i+2), \pi(3i+3)\} \in S]\right] \\ &= \sum_{i=0}^{2} 1 \cdot p = 3p. \end{aligned}$$

   Hence $N = p\binom{9}{3} \ge \frac{1}{3}\binom{9}{3} = 28$, establishing the lower bound. ■

   This problem and solution were proposed by Ray Li.

   Remark. One can use double-counting rather than expectation to prove $N \ge 28$. In any case, this method generalizes effortlessly to larger numbers.

   Solution 2. Proceed as above to get an upper bound of 28.

   On the other hand, we will show that we can partition the $\binom{9}{3} = 84$ 3-sets into 28 groups of 3, such that in any group, the elements $a_1, a_2, \cdots, a_9$ all appear. This will imply the conclusion, since if $A < 28$, then there are at least 57 sets with average at most m, but by pigeonhole three of them must be in such a group, which is clearly impossible.

   Consider a 3-set and the following array:

$$\begin{matrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{matrix}$$

   Consider a set $|S| = 3$. We obtain the other two 3-sets in the group as follows:

- If S contains one element in each column, then shift the elements down cyclically mod 3.
- If S contains one element in each row, then shift the elements right cyclically mod 3. Note that the result coincides with the previous case if both conditions are satisfied.
- Otherwise, the elements of S are "constrained" in a 2×2 box, possibly shifted diagonally. In this case, we get an L-tromino. Then shift diagonally in the direction the L-tromino points in.

One can verify that this algorithm creates such a partition, so we conclude that $A \geq 28$. ■

This second solution was suggested by Lewis Chen.

2. Let a, b, c be positive reals satisfying $a + b + c = \sqrt[7]{a} + \sqrt[7]{b} + \sqrt[7]{c}$. Prove that $a^a b^b c^c \geq 1$.

Proposed by Evan Chen.

Solution 1. By weighted AM-GM we have that

$$\begin{aligned} 1 &= \sum_{\text{cyc}} \left(\frac{\sqrt[7]{a}}{a+b+c} \right) \\ &= \sum_{\text{cyc}} \left(\frac{a}{a+b+c} \cdot \frac{1}{\sqrt[7]{a^6}} \right) \\ &\geq \left(\frac{1}{a^a b^b c^c} \right)^{\frac{6/7}{a+b+c}}. \end{aligned}$$

Rearranging yields $a^a b^b c^c \geq 1$. ■

This problem and solution were proposed by Evan Chen.

Remark. The problem generalizes easily to n variables, and exponents other than $\frac{1}{7}$. Specifically, if positive reals $x_1 + \cdots + x_n = x_1^r + \cdots + x_n^r$ for some real number $r \neq 1$, then $\prod_{i \geq 1} x_i^{x_i} \geq 1$ if and only if $r < 1$. When $r \leq 0$, a Jensen solution is possible using only the inequality $a + b + c \geq 3$.

Solution 2. First we claim that $a, b, c < 5$. Assume the contrary, that $a \geq 5$. Let $f(x) = x - \sqrt[7]{x}$. Since $f'(x) > 0$ for $x \geq 5$, we know that $f(a) \geq 5 - \sqrt[7]{5} > 3$. But this means that WLOG $b - \sqrt[7]{b} < -1.5$, which is clearly false since $b - \sqrt[7]{b} \geq 0$ for $b \geq 1$, and $b - \sqrt[7]{b} \geq -\sqrt[7]{b} \geq -1$ for $0 < b < 1$. So indeed $a, b, c < 5$.

Now rewrite the inequality as

$$\sum a \ln a \geq 0 \Leftrightarrow \sum \left(\frac{a^{\frac{1}{7}}}{a^{\frac{1}{7}} + b^{\frac{1}{7}} + c^{\frac{1}{7}}} \right) (a^{\frac{6}{7}} \ln a) \geq 0.$$

Now note that if $g(x) = x^{\frac{6}{7}} \ln x$, then $g''(x) = \frac{35 - 6 \ln x}{49 x^{\frac{8}{7}}} > 0$ for $x \in (0, 5)$. Therefore g is convex and we can use Jensen's Inequality to get

$$\sum \left(\frac{a^{\frac{1}{7}}}{a^{\frac{1}{7}} + b^{\frac{1}{7}} + c^{\frac{1}{7}}} \right) (a^{\frac{6}{7}} \ln a) \geq \left(\sum \frac{a^{\frac{8}{7}}}{a^{\frac{1}{7}} + b^{\frac{1}{7}} + c^{\frac{1}{7}}} \right)^{\frac{6}{7}} \ln \left(\sum \frac{a^{\frac{8}{7}}}{a^{\frac{1}{7}} + b^{\frac{1}{7}} + c^{\frac{1}{7}}} \right).$$

Since $\sum a = \sum a^{\frac{1}{7}}$, it suffices to show that $\sum a^{\frac{8}{7}} \geq \sum a$. But by weighted AM-GM we have

$$6a^{\frac{8}{7}} + a^{\frac{1}{7}} \geq 7a \implies a^{\frac{8}{7}} - a \geq \frac{1}{6}(a - \sqrt[7]{a}).$$

Adding up the analogous inequalities for b, c gives the desired result. ■

This second solution was suggested by David Stoner.

Solution 3. Here we unify the two solutions above.

It's well-known that weighted AM-GM follows from (and in fact, is equivalent to) the convexity of e^x (or equivalently, the concavity of $\ln x$), as $\sum w_i e^{x_i} \ge e^{\sum w_i x_i}$ for reals x_i and nonnegative weights w_i summing to 1. However, it also follows from the convexity of $y \ln y$ (or equivalently, the concavity of ye^y) for $y > 0$. Indeed, letting $y_i = e^{x_i} > 0$, and taking logs, weighted AM-GM becomes

$$\sum w_i y_i \cdot \frac{1}{y_i} \log \frac{1}{y_i} \ge (\sum w_i y_i) \frac{\sum w_i y_i \cdot \frac{1}{y_i}}{\sum w_i y_i} \log \frac{\sum w_i y_i \cdot \frac{1}{y_i}}{\sum w_i y_i},$$

which is clear.

To find Evan's solution, we can use the concavity of $\ln x$ to get $\sum a \ln a^{-s} \le (\sum a) \ln \sum \frac{a \cdot a^{-s}}{\sum a} = 0$. (Here we take $s = 6/7 > 0$.)

For a cleaner version of David's solution, we can use the convexity of $x \ln x$ to get

$$\sum a \ln a^s = \sum a^{1-s} \cdot a^s \ln a^s \ge (\sum a^{1-s}) \frac{\sum a^{1-s} \cdot a^s}{\sum a^{1-s}} \ln \frac{\sum a^{1-s} \cdot a^s}{\sum a^{1-s}} = 0$$

(where we again take $s = 6/7 > 0$).

Both are pretty intuitive (but certainly not obvious) solutions once one realizes direct Jensen goes in the wrong direction. In particular, $s = 1$ doesn't work since we have $a + b + c \le 3$ from the power mean inequality. ■

This third solution was suggested by Victor Wang.

Solution 4. From $e^t \ge 1 + t$ for $t = \ln x^{-\frac{6}{7}}$, we find $\frac{6}{7} \ln x \ge 1 - x^{-\frac{6}{7}}$. Thus

$$\frac{6}{7} \sum a \ln a \ge \sum a - a^{\frac{1}{7}} = 0,$$

as desired. ■

This fourth solution was suggested by chronodecay.

Remark. Polya once dreamed a similar proof of n-variable AM-GM: $x \ge 1 + \ln x$ for positive x, so $\sum x_i \ge n + \ln \prod x_i$. This establishes AM-GM when $\prod x_i = 1$; the rest follows by homogenizing.

3. Let $m_1, m_2, \ldots, m_{2013} > 1$ be 2013 pairwise relatively prime positive integers and $A_1, A_2, \ldots, A_{2013}$ be 2013 (possibly empty) sets with $A_i \subseteq \{1, 2, \ldots, m_i - 1\}$ for $i = 1, 2, \ldots, 2013$. Prove that there is a positive integer N such that

$$N \le (2|A_1| + 1)(2|A_2| + 1) \cdots (2|A_{2013}| + 1)$$

and for each $i = 1, 2, \ldots, 2013$, there does *not* exist $a \in A_i$ such that m_i divides $N - a$.

Proposed by Victor Wang.

Remark. As Solution 3 shows, the bound can in fact be tightened to $\prod_{i=1}^{2013} (|A_i| + 1)$.

Solution 1. We will show that the smallest integer N such that $N \notin A_i \pmod{m_i}$ is less than the bound provided.

The idea is to use pigeonhole and the "Lagrange interpolation"-esque representation of CRT systems. Define integers t_i satisfying $t_i \equiv 1 \pmod{m_i}$ and $t_i \equiv 0 \pmod{m_j}$ for $j \ne i$. If we find nonempty sets B_i of distinct residues mod m_i with $B_i - B_i \pmod{m_i}$ and $A_i \pmod{m_i}$ disjoint, then by pigeonhole, a positive integer solution with $N \le \frac{m_1 m_2 \cdots m_{2013}}{|B_1| \cdot |B_2| \cdots |B_{2013}|}$ must exist (more precisely, since

$$b_1 t_1 + \cdots + b_{2013} t_{2013} \pmod{m_1 m_2 \cdots m_{2013}}$$

is injective over $B_1 \times B_2 \times \cdots \times B_{2013}$, some two consecutively ordered solutions must differ by at most $\frac{m_1 m_2 \cdots m_{2013}}{|B_1| \cdot |B_2| \cdots |B_{2013}|}$).

On the other hand, since $0 \notin A_i$ for every i, we know such nonempty B_i must exist (e.g. take $B_i = \{0\}$). Now suppose $|B_i|$ is maximal; then every $x \pmod{m_i}$ lies in at least one of B_i, $B_i + A_i$, $B_i - A_i$ (note that $x - x = 0$ is not an issue when considering $(B_i \cup \{x\}) - (B_i \cup \{x\})$), or else $B_i \cup \{x\}$ would be a larger working set. Hence $m_i \le |B_i| + |B_i + A_i| + |B_i - A_i| \le |B_i|(1 + 2|A_i|)$, so we get an upper bound of $\prod_{i=1}^{2013} \frac{m_i}{|B_i|} \le \prod_{i=1}^{2013} (2|A_i| + 1)$, as desired. ■

Remark. We can often find $|B_i|$ significantly larger than $\frac{m_i}{2|A_i|+1}$ (the bounds $|B_i + A_i|, |B_i - A_i| \le |B_i| \cdot |A_i|$ seem really weak, and $B_i + A_i, B_i - A_i$ might not be that disjoint either). For instance, if $A_i \equiv -A_i \pmod{m_i}$, then we can get (the ceiling of) $\frac{m_i}{|A_i|+1}$.

Remark. By translation and repeated application of the problem, one can prove the following slightly more general statement: "Let $m_1, m_2, \ldots, m_{2013} > 1$ be 2013 pairwise relatively prime positive integers and $A_1, A_2, \ldots, A_{2013}$ be 2013 (possibly empty) sets with A_i a proper subset of $\{1, 2, \ldots, m_i\}$ for $i = 1, 2, \ldots, 2013$. Then for every integer n, there exists an integer x in the range $(n, n + \prod_{i=1}^{2013}(2|A_i| + 1)]$ such that $x \notin A_i \pmod{m_i}$ for $i = 1, 2, \ldots, 2013$. (We say A is a *proper subset* of B if A is a subset of B but $A \ne B$.)"

Remark. Let f be a non-constant integer-valued polynomial with $\gcd(\ldots, f(-1), f(0), f(1), \ldots) = 1$. Then by the previous remark, we can easily prove that there exist infinitely many positive integers n such that the smallest prime divisor of $f(n)$ is at least $c \log n$, where $c > 0$ is any constant. (We take m_i the ith prime and $A_i \equiv \{n : m_i \mid f(n)\} \pmod{m_i}$—if $f = \frac{a}{b} x^d + \cdots$, then $|A_i| \le d$ for all sufficiently large i.)

Solution 2. We will mimic the proof of 2010 RMM Problem 1.

Suppose $1, 2, \ldots, N$ (for some $N \ge 1$) can be covered by the sets $A_i \pmod{m_i}$.

Observe that for fixed m and $1 \le a \le m$, exactly $1 + \lfloor \frac{N-a}{m} \rfloor$ of $1, 2, \ldots, N$ are $a \pmod{m}$. In particular, we have lower and upper bounds of $\frac{N-m}{m}$ and $\frac{N+m}{m}$, respectively, so PIE yields

$$N \le \sum_i |A_i| \frac{N + m_i}{m_i} - \sum_{i<j} |A_i| \cdot |A_j| \frac{N - m_i m_j}{m_i m_j} \pm \cdots .$$

It follows that

$$N \prod_i \left(1 - \frac{|A_i|}{m_i}\right) \le \prod_i (1 + |A_i|),$$

so $N \le \prod_i \frac{m_i}{m_i - |A_i|}(1 + |A_i|)$.

Note that $\frac{m_i}{m_i - |A_i|} \le \frac{2|A_i|+1}{|A_i|+1}$ iff $m_i \ge 2|A_i| + 1$, so we're done unless $m_i \le 2|A_i|$ for some i.

In this case, there exists (by induction) $1 \le N \le \prod_{j \ne i}(2|A_j| + 1)$ such that $N \notin m_i^{-1} A_j \pmod{m_j}$ for all $j \ne i$. Thus $m_i N \notin A_j \pmod{m_j}$ and we trivially have $m_i N \equiv 0 \notin A_i \pmod{m_i}$, so $m_i N \le \prod_k (2|A_k| + 1)$, as desired. ■

This problem and the above solutions were proposed by Victor Wang.

Solution 3. We can in fact get a bound of $\prod(|A_k| + 1)$ directly.

Let $t = 2013$. Suppose $1, 2, \ldots, N$ are covered by the $A_k \pmod{m_k}$; then

$$z_n = \prod_{1 \le k \le t, a \in A_k} \left(1 - e^{\frac{2\pi i}{m_k}(n-a)}\right)$$

is a linear recurrence in $e^{2\pi i \sum_{k=1}^{t} \frac{j_k}{m_k}}$ (where each j_k ranges from 0 to $|A_k|$). But $z_0 \neq 0 = z_1 = \cdots = z_N$, so N must be strictly less than the degree $\prod(|A_k|+1)$ of the linear recurrence. Thus $1, 2, \ldots, \prod(|A_k|+1)$ cannot all be covered, as desired. ■

This third solution was suggested by Zhi-Wei Sun.

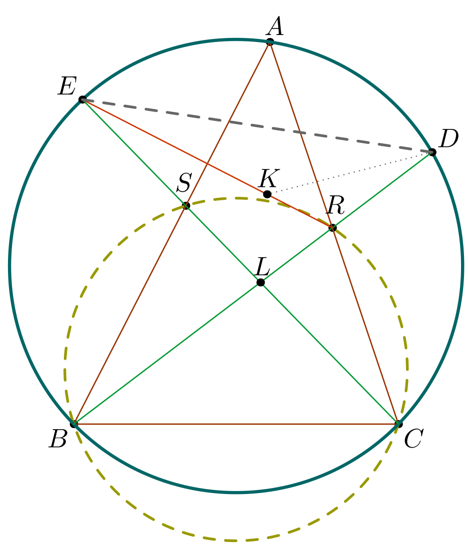
Remark. Solution 3 doesn't require the m_k to be coprime. Note that if $|A_1| = \cdots = |A_t| = b-1$, then a base b construction shows the bound of $\prod(b-1+1) = b^t$ is "tight" (if we remove the restriction that the m_k must be coprime).

However, Solutions 2 and 3 "ignore" the additive structure of CRT solution sets encapsulated in Solution 1's Lagrange interpolation representation.

4. Triangle ABC is inscribed in circle ω. A circle with chord BC intersects segments AB and AC again at S and R, respectively. Segments BR and CS meet at L, and rays LR and LS intersect ω at D and E, respectively. The internal angle bisector of $\angle BDE$ meets line ER at K. Prove that if $BE = BR$, then $\angle ELK = \frac{1}{2}\angle BCD$.

Proposed by Evan Chen.

Solution 1.



First, we claim that $BE = BR = BC$. Indeed, construct a circle with radius $BE = BR$ centered at B, and notice that $\angle ECR = \frac{1}{2}\angle EBR$, implying that it lies on the circle.

Now, CA bisects $\angle ECD$ and DB bisects $\angle EDC$, so R is the incenter of $\triangle CDE$. Then, K is the incenter of $\triangle LED$, so $\angle ELK = \frac{1}{2}\angle ELD = \frac{1}{2}\left(\frac{\widehat{ED}+\widehat{BC}}{2}\right) = \frac{1}{2}\frac{\widehat{BED}}{2} = \frac{1}{2}\angle BCD$. ■

This problem and solution were proposed by Evan Chen.

Solution 2. Note $\angle EBA = \angle ECA = \angle SCR = \angle SBR = \angle ABR$, so AB bisects $\angle EBR$. Then by symmetry $\angle BEA = \angle BRA$, so $\angle BCR = \angle BCA = 180 - \angle BEA = 180 - \angle BRA = \angle BRC$, so $BE = BR = BC$. Proceed as above. ■

This second solution was suggested by Michael Kural.

5. For what polynomials $P(n)$ with integer coefficients can a positive integer be assigned to every lattice point in $\mathbb{R}^3$ so that for every integer $n \geq 1$, the sum of the n^3 integers assigned to any $n \times n \times n$ grid of lattice points is divisible by $P(n)$?

Proposed by Andre Arslan.

Answer. All P of the form $P(x) = cx^k$, where c is a nonzero integer and k is a nonnegative integer.

Solution. Suppose $P(x) = x^k Q(x)$ with $Q(0) \neq 0$ and Q is nonconstant; then there exist infinitely many primes p dividing some $Q(n)$; fix one of them not dividing $Q(0)$, and take a sequence of pairwise coprime integers $m_1, n_1, m_2, n_2, \ldots$ with $p \mid Q(m_i), Q(n_i)$ (we can do this with CRT).

Let $f(x, y, z)$ be the number written at (x, y, z). Note that $P(m)$ divides every $mn \times mn \times m$ grid and $P(n)$ divides every $mn \times mn \times n$ grid, so by Bezout's identity, $(P(m), P(n))$ divides every $mn \times mn \times (m, n)$ grid. It follows that p divides every $m_i n_i \times m_i n_i \times 1$ grid. Similarly, we find that p divides every $m_i n_i m_j n_j \times 1 \times 1$ grid whenever $i \neq j$, and finally, every $1 \times 1 \times 1$ grid. Since p was arbitrarily chosen from an infinite set, f must be identically zero, contradiction.

For the other direction, take a solution g to the one-dimensional case using repeated CRT (the key relation $\gcd(P(m), P(n)) = P(\gcd(m, n))$ prevents "conflicts"): start with a positive multiple of $P(1) \neq 0$ at zero, and then construct $g(1), g(-1), g(2), g(-2)$, etc. in that order using CRT. Now for the three-dimensional version, we can just let $f(x, y, z) = g(x)$. ■

This problem and solution were proposed by Andre Arslan.

Remark. The crux of the problem lies in the 1D case. (We use the same type of reasoning to "project" from d dimension to $d-1$ dimensions.) Note that the condition $P(n) \mid g(i) + \cdots + g(i+n-1)$ (for the 1D case) is "almost" the same as $P(n) \mid g(i) - g(i+n)$, so we immediately find $\gcd(P(m), P(n)) \mid g(i) - g(i + \gcd(m, n))$ by Bezout's identity. In particular, when m, n are coprime, we will intuitively be able to get $\gcd(P(m), P(n))$ as large as we want unless P is of the form cx^k (we formalize this by writing $P = x^k Q$ with $Q(0) \neq 0$).

Conversely, if $P = cx^k$, then $\gcd(P(m), P(n)) = P(\gcd(m, n))$ renders our derived restriction $\gcd(P(m), P(n)) \mid g(i) - g(i + \gcd(m, n))$ superfluous. So it "feels easy" to find *nonconstant* g with $P(n) \mid g(i) - g(i+n)$ for all i, n, just by greedily constructing $g(0), g(1), g(-1), \ldots$ in that order using CRT. Fortunately, $g(i) + \cdots + g(i+m-1) - g(i) - \cdots - g(i+n-1) = g(i+n) + \cdots + g(i+n+(m-n)-1)$ for $m > n$, so the inductive approach still works for the stronger condition $P(n) \mid g(i) + \cdots + g(i+n-1)$.

Remark. Note that polynomial constructions cannot work for $P = cx^{d+1}$ in d dimensions. Suppose otherwise, and take a minimal degree $f(x_1, \ldots, x_d)$; then f isn't constant, so $f'(x_1, \ldots, x_d) = f(x_1+1, \ldots, x_d+1) - f(x_1, \ldots, x_d)$ is a working polynomial of strictly smaller degree.

6. Consider a function $f : \mathbb{Z} \to \mathbb{Z}$ such that for every integer $n \geq 0$, there are at most $0.001n^2$ pairs of integers (x, y) for which $f(x+y) \neq f(x) + f(y)$ and $\max\{|x|, |y|\} \leq n$. Is it possible that for some integer $n \geq 0$, there are more than n integers a such that $f(a) \neq a \cdot f(1)$ and $|a| \leq n$?

Proposed by David Yang.

Answer. No.

Solution. Call an integer conformist if $f(n) = n \cdot f(1)$. Call a pair (x, y) good if $f(x+y) = f(x) + f(y)$ and bad otherwise. Let $h(n)$ denote the number of conformist integers with absolute value at most n.

Let $\epsilon = 0.001$, S be the set of conformist integers, $T = \mathbb{Z} \setminus S$ be the set of non-conformist integers, and $X_n = [-n, n] \cap X$ for sets X and positive integers n (so $|S_n| = h(n)$); clearly $|T_n| = 2n + 1 - h(n)$.

First we can easily get $h(n) = 2n + 1$ ($-n$ to n are all conformist) for $n \leq 10$.

Lemma 1. *Suppose a, b are positive integers such that $h(a) > a$ and $b \leq 2h(a) - 2a - 1$. Then $h(b) \geq 2b(1 - \sqrt{\epsilon}) - 1$.*

Proof. For any integer t, we have

$$\begin{aligned} |S_a \cap (t - S_a)| &= |S_a| + |t - S_a| - |S_a \cup (t - S_a)| \\ &\geq 2h(a) - (\max(S_a \cup (t - S_a)) - \min(S_a \cup (t - S_a)) + 1) \\ &\geq 2h(a) - (\max(a, t + a) - \min(-a, t - a) + 1) \\ &= 2h(a) - (|t| + 2a + 1) \\ &\geq b - |t|. \end{aligned}$$

But (x, y) is bad whenever $x, y \in S$ yet $x + y \in T$, so summing over all $t \in T_b$ (assuming $|T_b| \geq 2$) yields

$$\begin{aligned} \epsilon b^2 \geq g(b) &\geq \sum_{t \in T_b} |S_a \cap (t - S_a)| \\ &\geq \sum_{t \in T_b} (b - |t|) \geq \sum_{k=0}^{\lfloor |T_b|/2 \rfloor - 1} k + \sum_{k=0}^{\lceil |T_b|/2 \rceil - 1} k \geq 2\frac{1}{2}(|T_b|/2)(|T_b|/2 - 1), \end{aligned}$$

where we use $\lfloor r/2 \rfloor + \lceil r/2 \rceil = r$ (for $r \in \mathbb{Z}$) and the convexity of $\frac{1}{2}x(x-1)$. We conclude that $|T_b| \leq 2 + 2b\sqrt{\epsilon}$ (which obviously remains true without the assumption $|T_b| \geq 2$) and $h(b) = 2b + 1 - |T_b| \geq 2b(1 - \sqrt{\epsilon}) - 1$. □

Now we prove by induction on n that $h(n) \geq 2n(1 - \sqrt{\epsilon}) - 1$ for all $n \geq 10$, where the base case is clear. If we assume the result for $n - 1$ ($n > 10$), then in view of the lemma, it suffices to show that $2h(n-1) - 2(n-1) - 1 \geq n$, or equivalently, $2h(n-1) \geq 3n - 1$. But

$$2h(n-1) \geq 4(n-1)(1 - \sqrt{\epsilon}) - 2 \geq 3n - 1,$$

so we're done. (The second inequality is equivalent to $n(1 - 4\sqrt{\epsilon}) \geq 5 - 4\sqrt{\epsilon}$; $n \geq 11$ reduces this to $6 \geq 40\sqrt{\epsilon} = 40\sqrt{0.001} = 4\sqrt{0.1}$, which is obvious.) ■

This problem and solution were proposed by David Yang.