Everyone Lives at Most Once

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Problem Shortlist

Created and Managed by Evan Chen

ELMO regulation:
The shortlist problems should be kept strictly confidential until after the exam.

The Everyone Lives at Most Once committee gratefully acknowledges the receipt of 41 problem proposals from the following 13 contributors:

Andre Arslan	1 problem
Matthew Babbitt	3 problems
Evan Chen	7 problems
Eric Chen	1 problem
Calvin Deng	2 problems
Owen Goff	1 problem
Michael Kural	2 problems
Ray Li	6 problems
Allen Liu	2 problems
Bobby Shen	1 problem
David Stoner	8 problems
Victor Wang	4 problems
David Yang	3 problems

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Part I

Problems

Algebra

A1*

A1*

Find all triples (f, g, h) of injective functions from $\mathbb{R}$ to $\mathbb{R}$ satisfying

$$\begin{aligned} f(x+f(y)) &= g(x)+h(y) \\ g(x+g(y)) &= h(x)+f(y) \\ h(x+h(y)) &= f(x)+g(y) \end{aligned}$$

for all real numbers x and y. (We say a function F is *injective* if $F(x) \neq F(y)$ whenever $x \neq y$.)

Evan Chen

A2

A2

Prove that for all positive reals a, b, c,

$$\frac{1}{a+\frac{1}{b}+1}+\frac{1}{b+\frac{1}{c}+1}+\frac{1}{c+\frac{1}{a}+1} \geq \frac{3}{\sqrt[3]{abc}+\frac{1}{\sqrt[3]{abc}}+1}.$$

David Stoner

A3

A3

Find all $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$, $f(x)+f(y) = f(x+y)$ and $f(x^{2013}) = f(x)^{2013}$.

Calvin Deng

A4

A4

Positive reals a, b, and c obey $\frac{a^2+b^2+c^2}{ab+bc+ca} = \frac{ab+bc+ca+1}{2}$. Prove that

$$\sqrt{a^2+b^2+c^2} \leq 1+\frac{|a-b|+|b-c|+|c-a|}{2}.$$

Evan Chen

A5*

A5*

Let a, b, c be positive reals satisfying $a+b+c = \sqrt[7]{a}+\sqrt[7]{b}+\sqrt[7]{c}$. Prove that $a^a b^b c^c \geq 1$.

Evan Chen

A6

A6

Let a, b, c be positive reals such that $a+b+c=3$. Prove that

$$18\sum_{\text{cyc}}\frac{1}{(3-c)(4-c)}+2(ab+bc+ca)\geq 15.$$

David Stoner

A7*

A7*

Consider a function $f:\mathbb{Z}\to\mathbb{Z}$ such that for every integer $n\geq 0$, there are at most $0.001n^2$ pairs of integers (x,y) for which $f(x+y)\neq f(x)+f(y)$ and $\max\{|x|,|y|\}\leq n$. Is it possible that for some integer $n\geq 0$, there are more than n integers a such that $f(a)\neq a\cdot f(1)$ and $|a|\leq n$?

David Yang

A8*

A8*

Let a, b, c be positive reals with $a^{2013}+b^{2013}+c^{2013}+abc=4$. Prove that

$$\left(\sum a(a^2+bc)\right)\left(\sum\left(\frac{a}{b}+\frac{b}{a}\right)\right)\geq\left(\sum\sqrt{(a+1)(a^3+bc)}\right)\left(\sum\sqrt{a(a+1)(a+bc)}\right).$$

David Stoner

A9

A9

Let a, b, c be positive reals, and let $\sqrt[2013]{\frac{3}{a^{2013}+b^{2013}+c^{2013}}}=P$. Prove that

$$\prod_{\text{cyc}}\left(\frac{(2P+\frac{1}{2a+b})(2P+\frac{1}{a+2b})}{(2P+\frac{1}{a+b+c})^2}\right)\geq\prod_{\text{cyc}}\left(\frac{(P+\frac{1}{4a+b+c})(P+\frac{1}{3b+3c})}{(P+\frac{1}{3a+2b+c})(P+\frac{1}{3a+b+2c})}\right).$$

David Stoner

Combinatorics

C1

C1

Let $n \geq 2$ be a positive integer. The numbers $1, 2, \ldots, n^2$ are consecutively placed into squares of an $n \times n$, so the first row contains $1, 2, \ldots, n$ from left to right, the second row contains $n+1, n+2, \ldots, 2n$ from left to right, and so on. The *magic square value* of a grid is defined to be the number of rows, columns, and main diagonals whose elements have an average value of $\frac{n^2+1}{2}$. Show that the magic-square value of the grid stays constant under the following two operations: (1) a permutation of the rows; and (2) a permutation of the columns. (The operations can be used multiple times, and in any order.)

Ray Li

C2

C2

Let n be a fixed positive integer. Initially, n 1's are written on a blackboard. Every minute, David picks two numbers x and y written on the blackboard, erases them, and writes the number $(x+y)^4$ on the blackboard. Show that after $n-1$ minutes, the number written on the blackboard is at least $2^{\frac{4n^2-4}{3}}$.

Calvin Deng

C3*

C3*

Let $a_1, a_2, \ldots, a_9$ be nine real numbers, not necessarily distinct, with average m. Let A denote the number of triples $1 \leq i < j < k \leq 9$ for which $a_i + a_j + a_k \geq 3m$. What is the minimum possible value of A?

Ray Li

C4

C4

Let n be a positive integer. The numbers $\{1, 2, \ldots, n^2\}$ are placed in an $n \times n$ grid, each exactly once. The grid is said to be *Muirhead-able* if the sum of the entries in each column is the same, but for every $1 \leq i, k \leq n-1$, the sum of the first k entries in column i is at least the sum of the first k entries in column $i+1$. For which n can one construct a Muirhead-able array?

Evan Chen

C5

C5

There is a 2012×2012 grid with rows numbered $1, 2, \ldots 2012$ and columns numbered $1, 2, \ldots, 2012$, and we place some rectangular napkins on it such that the sides of the napkins all lie on grid lines. Each napkin has a positive integer thickness. (in micrometers!)

(a) Show that there exist 2012^2 unique integers $a_{i,j}$ where $i, j \in [1, 2012]$ such that for all $x, y \in [1, 2012]$, the sum

$$\sum_{i=1}^{x} \sum_{j=1}^{y} a_{i,j}$$

is equal to the sum of the thicknesses of all the napkins that cover the grid square in row x and column y.

(b) Show that if we use at most $500,000$ napkins, at least half of the $a_{i,j}$ will be 0.

Ray Li

C6

C6

A 4×4 grid has its 16 cells colored arbitrarily in three colors. A *swap* is an exchange between the colors of two cells. Prove or disprove that it always takes at most three swaps to produce a line of symmetry, regardless of the grid's initial coloring.

Matthew Babbitt

C7*

C7*

A $2^{2013} + 1$ by $2^{2013} + 1$ grid has some black squares filled. The filled black squares form one or more snakes on the plane, each of whose heads splits at some points but never comes back together. In other words, for every positive integer $n > 1$, there do not exist pairwise distinct black squares $s_1, s_2, \ldots, s_n$ such that s_i, s_{i+1} share an edge for $i = 1, 2, \ldots, n$ (here $s_{n+1} = s_1$). What is the maximum possible number of filled black squares?

David Yang

C8

C8

There are 20 people at a party. Each person holds some number of coins. Every minute, each person who has at least 19 coins simultaneously gives one coin to every other person at the party. (So, it is possible that A gives B a coin and B gives A a coin at the same time.) Suppose that this process continues indefinitely. That is, for any positive integer n, there exists a person who will give away coins during the nth minute. What is the smallest number of coins that could be at the party?

Ray Li

C9*

C9*

f_0 is the function from $\mathbb{Z}^2$ to $\{0,1\}$ such that $f_0(0,0) = 1$ and $f_0(x,y) = 0$ otherwise. For each $i > 1$, let $f_i(x,y)$ be the remainder when

$$f_{i-1}(x,y) + \sum_{j=-1}^{1} \sum_{k=-1}^{1} f_{i-1}(x+j, y+k)$$

is divided by 2.

For each $i \geq 0$, let $a_i = \sum_{(x,y)\in\mathbb{Z}^2} f_i(x,y)$. Find a closed form for a_n (in terms of n).

Bobby Shen

C10*

C10*

Let $N \geq 2$ be a fixed positive integer. There are $2N$ people, numbered $1, 2, \ldots, 2N$, participating in a tennis tournament. For any two positive integers i, j with $1 \leq i < j \leq 2N$, player i has a higher skill level than player j. Prior to the first round, the players are paired arbitrarily and each pair is assigned a unique court among N courts, numbered $1, 2, \ldots, N$.

During a round, each player plays against the other person assigned to his court (so that exactly one match takes place per court), and the player with higher skill wins the match (in other words, there are no upsets). Afterwards, for $i = 2, 3, \ldots, N$, the winner of court i moves to court $i-1$ and the loser of court i stays on court i; however, the winner of court 1 stays on court 1 and the loser of court 1 moves to court N.

Find all positive integers M such that, regardless of the initial pairing, the players $2, 3, \ldots, N+1$ all change courts immediately after the Mth round.

Ray Li

Geometry

G1

G1

Let ABC be a triangle with incenter I. Let U, V and W be the intersections of the angle bisectors of angles A, B, and C with the incircle, so that V lies between B and I, and similarly with U and W. Let X, Y, and Z be the points of tangency of the incircle of triangle ABC with BC, AC, and AB, respectively. Let triangle UVW be the *David Yang triangle* of ABC and let XYZ be the *Scott Wu triangle* of ABC. Prove that the David Yang and Scott Wu triangles of a triangle are congruent if and only if ABC is equilateral.

Owen Goff

G2

G2

Let ABC be a scalene triangle with circumcircle Γ, and let D,E,F be the points where its incircle meets BC, AC, AB respectively. Let the circumcircles of $\triangle AEF$, $\triangle BFD$, and $\triangle CDE$ meet Γ a second time at X, Y, Z respectively. Prove that the perpendiculars from A, B, C to AX, BY, CZ respectively are concurrent.

Michael Kural

G3

G3

In $\triangle ABC$, a point D lies on line BC. The circumcircle of ABD meets AC at F (other than A), and the circumcircle of ADC meets AB at E (other than A). Prove that as D varies, the circumcircle of AEF always passes through a fixed point other than A, and that this point lies on the median from A to BC.

Allen Liu

G4*

G4*

Triangle ABC is inscribed in circle ω. A circle with chord BC intersects segments AB and AC again at S and R, respectively. Segments BR and CS meet at L, and rays LR and LS intersect ω at D and E, respectively. The internal angle bisector of $\angle BDE$ meets line ER at K. Prove that if $BE = BR$, then $\angle ELK = \frac{1}{2}\angle BCD$.

Evan Chen

G5

G5

Let ω_1 and ω_2 be two orthogonal circles, and let the center of ω_1 be O. Diameter AB of ω_1 is selected so that B lies strictly inside ω_2. The two circles tangent to ω_2, passing through O and A, touch ω_2 at F and G. Prove that $FGOB$ is cyclic.

Eric Chen

G6

G6

Let $ABCDEF$ be a non-degenerate cyclic hexagon with no two opposite sides parallel, and define $X = AB \cap DE$, $Y = BC \cap EF$, and $Z = CD \cap FA$. Prove that

$$\frac{XY}{XZ} = \frac{BE}{AD} \frac{\sin |\angle B - \angle E|}{\sin |\angle A - \angle D|}.$$

Victor Wang

G7*

G7*

Let ABC be a triangle inscribed in circle ω, and let the medians from B and C intersect ω at D and E respectively. Let O_1 be the center of the circle through D tangent to AC at C, and let O_2 be the center of the circle through E tangent to AB at B. Prove that O_1, O_2, and the nine-point center of ABC are collinear.

Michael Kural

G8

G8

Let ABC be a triangle, and let D, A, B, E be points on line AB, in that order, such that $AC = AD$ and $BE = BC$. Let ω_1, ω_2 be the circumcircles of $\triangle ABC$ and $\triangle CDE$, respectively, which meet at a point $F \neq C$. If the tangent to ω_2 at F cuts ω_1 again at G, and the foot of the altitude from G to FC is H, prove that $\angle AGH = \angle BGH$.

David Stoner

G9

G9

Let $ABCD$ be a cyclic quadrilateral inscribed in circle ω whose diagonals meet at F. Lines AB and CD meet at E. Segment EF intersects ω at X. Lines BX and CD meet at M, and lines CX and AB meet at N. Prove that MN and BC concur with the tangent to ω at X.

Allen Liu

G10*

G10*

Let $AB = AC$ in $\triangle ABC$, and let D be a point on segment AB. The tangent at D to the circumcircle ω of BCD hits AC at E. The other tangent from E to ω touches it at F, and $G = BF \cap CD$, $H = AG \cap BC$. Prove that $BH = 2HC$.

David Stoner

G11

G11

Let $\triangle ABC$ be a nondegenerate isosceles triangle with $AB = AC$, and let D, E, F be the midpoints of BC, CA, AB respectively. BE intersects the circumcircle of $\triangle ABC$ again at G, and H is the midpoint of minor arc BC. $CF \cap DG = I, BI \cap AC = J$. Prove that $\angle BJH = \angle ADG$ if and only if $\angle BID = \angle GBC$.

David Stoner

G12*

G12*

Let ABC be a nondegenerate acute triangle with circumcircle ω and let its incircle γ touch AB, AC, BC at X, Y, Z respectively. Let XY hit arcs AB, AC of ω at M, N respectively, and let $P \neq X, Q \neq Y$ be the points on γ such that $MP = MX, NQ = NY$. If I is the center of γ, prove that P, I, Q are collinear if and only if $\angle BAC = 90°$.

David Stoner

G13

G13

In $\triangle ABC$, $AB < AC$. D and P are the feet of the internal and external angle bisectors of $\angle BAC$, respectively. M is the midpoint of segment BC, and ω is the circumcircle of $\triangle APD$. Suppose Q is on the minor arc AD of ω such that MQ is tangent to ω. QB meets ω again at R, and the line through R perpendicular to BC meets PQ at S. Prove SD is tangent to the circumcircle of $\triangle QDM$.

Ray Li

G14

G14

Let O be a point (in the plane) and T be an infinite set of points such that $|P_1P_2| \leq 2012$ for every two distinct points $P_1, P_2 \in T$. Let $S(T)$ be the set of points Q in the plane satisfying $|QP| \leq 2013$ for at least one point $P \in T$.

Now let L be the set of lines containing exactly one point of $S(T)$. Call a line ℓ_0 passing through O *bad* if there does not exist a line $\ell \in L$ parallel to (or coinciding with) ℓ_0.

(a) Prove that L is nonempty.

(b) Prove that one can assign a line $\ell(i)$ to each positive integer i so that for every bad line ℓ_0 passing through O, there exists a positive integer n with $\ell(n) = \ell_0$.

David Yang

Number Theory

N1

N1

Find all ordered triples of non-negative integers (a, b, c) such that $a^2 + 2b + c$, $b^2 + 2c + a$, and $c^2 + 2a + b$ are all perfect squares.

Note: This problem was withdrawn from the ELMO Shortlist and used on ksun48's mock AIME.

Matthew Babbitt

N2*

N2*

For what polynomials $P(n)$ with integer coefficients can a positive integer be assigned to every lattice point in $\mathbb{R}^3$ so that for every integer $n \geq 1$, the sum of the n^3 integers assigned to any $n \times n \times n$ grid of lattice points is divisible by $P(n)$?

Andre Arslan

N3

N3

Prove that each integer greater than 2 can be expressed as the sum of pairwise distinct numbers of the form a^b, where $a \in \{3, 4, 5, 6\}$ and b is a positive integer.

Matthew Babbitt

N4

N4

Find all triples (a, b, c) of positive integers such that if n is not divisible by any integer less than 2013, then $n + c$ divides $a^n + b^n + n$.

Evan Chen

N5*

N5*

Let $m_1, m_2, \ldots, m_{2013} > 1$ be 2013 pairwise relatively prime positive integers and $A_1, A_2, \ldots, A_{2013}$ be 2013 (possibly empty) sets with $A_i \subseteq \{1, 2, \ldots, m_i - 1\}$ for $i = 1, 2, \ldots, 2013$. Prove that there is a positive integer N such that

$$N \leq (2|A_1| + 1)(2|A_2| + 1) \cdots (2|A_{2013}| + 1)$$

and for each $i = 1, 2, \ldots, 2013$, there does *not* exist $a \in A_i$ such that m_i divides $N - a$.

Victor Wang

N6*

N6*

Find all positive integers m for which there exists a function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that

$$f^{f^{f(n)}(n)}(n) = n$$

for every positive integer n, and $f^{2013}(m) \neq m$. Here $f^k(n)$ denotes $\underbrace{f(f(\cdots f}_{k\ f\text{'s}}(n)\cdots))$.

Evan Chen

N7*

N7*

Let p be a prime satisfying $p^2 \mid 2^{p-1} - 1$, n be a positive integer, and $f(x) = \frac{(x-1)^{p^n} - (x^{p^n} - 1)}{p(x-1)}$. Find the largest positive integer N such that there exist polynomials $g, h \in \mathbb{Z}[x]$ and an integer r satisfying $f(x) = (x-r)^N g(x) + p \cdot h(x)$.

Victor Wang

N8

N8

We define the *Fibonacci sequence* $\{F_n\}_{n \geq 0}$ by $F_0 = 0$, $F_1 = 1$, and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$; we define the *Stirling number of the second kind* $S(n,k)$ as the number of ways to partition a set of $n \geq 1$ distinguishable elements into $k \geq 1$ indistinguishable nonempty subsets.

For every positive integer n, let $t_n = \sum_{k=1}^{n} S(n,k) F_k$. Let $p \geq 7$ be a prime. Prove that

$$t_{n+p^{2p}-1} \equiv t_n \pmod{p}$$

for all $n \geq 1$.

Victor Wang

Part II

Solutions

A1*

Find all triples (f, g, h) of injective functions from $\mathbb{R}$ to $\mathbb{R}$ satisfying

$$f(x + f(y)) = g(x) + h(y)$$
$$g(x + g(y)) = h(x) + f(y)$$
$$h(x + h(y)) = f(x) + g(y)$$

for all real numbers x and y. (We say a function F is *injective* if $F(x) \neq F(y)$ whenever $x \neq y$.)

Evan Chen

Answer. For all real numbers x, $f(x) = g(x) = h(x) = x + C$, where C is an arbitrary real number.

Solution 1. Let a, b, c denote the values $f(0)$, $g(0)$ and $h(0)$. Notice that by putting $y = 0$, we can get that $f(x + a) = g(x) + c$, etc. In particular, we can write

$$h(y) = f(y - c) + b$$

and

$$g(x) = h(x - b) + a = f(x - b - c) + a + b$$

So the first equation can be rewritten as

$$f(x + f(y)) = f(x - b - c) + f(y - c) + a + 2b.$$

At this point, we may set $x = y - c - f(y)$ and cancel the resulting equal terms to obtain

$$f(y - f(y) - (b + 2c)) = -(a + 2b).$$

Since f is injective, this implies that $y - f(y) - (b + 2c)$ is constant, so that $y - f(y)$ is constant. Thus, f is linear, and $f(y) = y + a$. Similarly, $g(x) = x + b$ and $h(x) = x + c$.

Finally, we just need to notice that upon placing $x = y = 0$ in all the equations, we get $2a = b + c$, $2b = c + a$ and $2c = a + b$, whence $a = b = c$.

So, the family of solutions is $f(x) = g(x) = h(x) = x + C$, where C is an arbitrary real. One can easily verify these solutions are valid. ■

This problem and solution were proposed by Evan Chen.

Remark. This is not a very hard problem. The basic idea is to view $f(0)$, $g(0)$ and $h(0)$ as constants, and write the first equation entirely in terms of $f(x)$, much like we would attempt to eliminate variables in a standard system of equations. At this point we still had two degrees of freedom, x and y, so it seems likely that the result would be easy to solve. Indeed, we simply select x in such a way that two of the terms cancel, and the rest is working out details.

Solution 2. First note that plugging $x = f(a), y = b; x = f(b), y = a$ into the first gives $g(f(a)) + h(b) = g(f(b)) + h(a) \implies g(f(a)) - h(a) = g(f(b)) - h(b)$. So $g(f(x)) = h(x) + a_1$ for a constant a_1. Similarly, $h(g(x)) = f(x) + a_2, f(h(x)) = g(x) + a_3$.

Now, we will show that $h(h(x)) - f(x)$ and $h(h(x)) - g(x)$ are both constant. For the second, just plug in $x = 0$ to the third equation. For the first, let $x = a_3, y = k$ in the original to get $g(f(h(k))) = h(a_3) + f(k)$. But $g(f(h(k))) = h(h(k)) + a_1$, so $h(h(k)) - f(k) = h(a_3) - a_1$ is constant as desired.

Now $f(x) - g(x)$ is constant, and by symmetry $g(x) - h(x)$ is also constant. Now let $g(x) = f(x) + p, h(x) = f(x) + q$. Then we get:

$$f(x + f(y)) = f(x) + f(y) + p + q$$
$$f(x + f(y) + p) = f(x) + f(y) + q - p$$
$$f(x + f(y) + q) = f(x) + f(y) + p - q$$

Now plugging in (x, y) and (y, x) into the first one gives $f(x+f(y)) = f(y+f(x)) \implies f(x)-x = f(y)-y$ from injectivity, $f(x) = x + c$. Plugging this in gives $2p = q, 2q = p, p+q = 0$ so $p = q = 0$ and $f(x) = x+c, g(x) = x+c, h(x) = x+c$ for a constant c are the only solutions. ■

This second solution was suggested by David Stoner.

A2

Prove that for all positive reals a, b, c,

$$\frac{1}{a+\frac{1}{b}+1}+\frac{1}{b+\frac{1}{c}+1}+\frac{1}{c+\frac{1}{a}+1} \geq \frac{3}{\sqrt[3]{abc}+\frac{1}{\sqrt[3]{abc}}+1}.$$

David Stoner

Solution. Let $a = N\frac{x}{y}$, $b = N\frac{y}{z}$ and $c = N\frac{z}{x}$. Then

$$\begin{aligned}
\sum_{\text{cyc}} \frac{1}{a+\frac{1}{b}+1} &= \sum_{\text{cyc}} \frac{y}{Nx+\frac{1}{N}z+y} \\
&= \sum_{\text{cyc}} \frac{y^2}{Nxy+\frac{1}{N}yz+y^2} \\
&\geq \frac{(x+y+z)^2}{(xy+yz+zx)\left(N+\frac{1}{N}\right)+x^2+y^2+z^2} \\
&= \frac{(x+y+z)^2}{(xy+yz+zx)\left(N+\frac{1}{N}-2\right)+(x+y+z)^2} \\
&= \frac{3}{3+\frac{3(xy+yz+zx)}{(x+y+z)^2}\left(N+\frac{1}{N}-2\right)} \\
&\geq \frac{3}{3+N+\frac{1}{N}-2} \\
&= \frac{3}{N+\frac{1}{N}+1} \\
&= \frac{3}{\sqrt[3]{abc}+\frac{1}{\sqrt[3]{abc}}+1}.
\end{aligned}$$

■

This problem and solution were proposed by David Stoner.

A3

Find all $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$, $f(x) + f(y) = f(x + y)$ and $f(x^{2013}) = f(x)^{2013}$.

Calvin Deng

Answer. $f(x) = x$, $f(x) = -x$, and $f(x) \equiv 0$.

Solution. WLOG $f(1) \geq 0$ (since 2013 is odd); then $f(1) = f(1)^{2013} \implies f(1) \in \{0, 1\}$.

Hence for any reals x, y,

$$\begin{aligned}\sum_{k=0}^{2013} \binom{2013}{k} n^{2013-k} f(x)^k f(y)^{2013-k} &= [f(x) + nf(y)]^{2013} \\ &= f(x + ny)^{2013} \\ &= f((x + ny)^{2013}) \\ &= \sum_{k=0}^{2013} \binom{2013}{k} n^{2013-k} f(x^k y^{2013-k})\end{aligned}$$

for all positive integers n, so viewing this as a polynomial identity in n we get $f(x)^k f(y)^{2013-k} = f(x^k y^{2013-k})$ for $k = 0, 1, \dots, 2013$.

If $f(1) = 1$, then $k = 2$ gives $f(x^2) = f(x)^2 \geq 0$ which is enough to get $f(x) = x$ for all x. Otherwise, if $f(1) = 0$, then $k = 1$ gives $f(x) = 0$ for all x. ∎

This problem and solution were proposed by Calvin Deng.

A4

Positive reals a, b, and c obey $\frac{a^2+b^2+c^2}{ab+bc+ca} = \frac{ab+bc+ca+1}{2}$. Prove that

$$\sqrt{a^2+b^2+c^2} \le 1 + \frac{|a-b|+|b-c|+|c-a|}{2}.$$

Evan Chen

Solution 1. The given condition rearranges as $2(a^2+b^2+c^2)-(ab+bc+ca) = (ab+bc+ca)^2$. Homogenizing, this becomes:

$$|a-b|+|b-c|+|c-a| + \frac{2(ab+bc+ca)}{\sqrt{2(a^2+b^2+c^2)-(ab+bc+ca)}} \ge 2\sqrt{a^2+b^2+c^2}.$$

An application of Holder's inequality gives:

$$\begin{aligned}
\text{LHS}^2 &\ge \frac{\left((a-b)^2+(b-c)^2+(c-a)^2+2(ab+bc+ca)\right)^3}{\left(\sum_{\text{cyc}}(a-b)^4 + 2(ab+bc+ca)\left(2(a^2+b^2+c^2)-(ab+bc+ca)\right)\right)^1} \\
&= \frac{(2a^2+2b^2+2c^2)^3}{2a^4+2b^4+2c^4+4a^2b^2+4a^2c^2+4c^2a^2} \\
&= \frac{8(a^2+b^2+c^2)^3}{2(a^2+b^2+c^2)^2} \\
&= 4(a^2+b^2+c^2)
\end{aligned}$$

Upon taking square roots of both sides we are done. ■

This problem and solution were proposed by Evan Chen.

Solution 2. Let $x = ab+bc+ca$, so $1 \le \frac{a^2+b^2+c^2}{x} = \frac{x+1}{2}$ implies $x \ge 1$. If $\alpha = a-b$, $\beta = b-c$, $\gamma = c-a$, WLOG with $\alpha, \beta \ge 0$ (or equivalently $a \ge b \ge c$), then because $\alpha+\beta+\gamma = 0$, we have

$$2(\alpha^2+\alpha\beta+\beta^2) = \alpha^2+\beta^2+\gamma^2 = 2x\frac{x+1}{2} - 2x = x(x-1),$$

and we want to minimize $|\alpha|+|\beta|+|\gamma| = 2(\alpha+\beta)$. But $(\alpha+\beta)^2 \ge \alpha^2+\alpha\beta+\beta^2$ implies $\alpha+\beta \ge \sqrt{\frac{x(x-1)}{2}}$, with equality attained for some choice of (a,b,c) precisely when $\alpha\beta = 0$ and $(\alpha+\beta)\beta \le x$ (since $c \ge 0$). In particular, $\beta = 0$ works for any fixed $x \ge 1$, so the problem is equivalent to $\sqrt{\frac{x(x+1)}{2}} \le 1 + \sqrt{\frac{x(x-1)}{2}}$ for $x \ge 1$, which is easy after squaring both sides. ■

This second solution was suggested by Victor Wang.

A5*

Let a, b, c be positive reals satisfying $a + b + c = \sqrt[7]{a} + \sqrt[7]{b} + \sqrt[7]{c}$. Prove that $a^a b^b c^c \ge 1$.

Evan Chen

Solution 1. By weighted AM-GM we have that

$$\begin{aligned}1 &= \sum_{\text{cyc}} \left(\frac{\sqrt[7]{a}}{a+b+c} \right) \\ &= \sum_{\text{cyc}} \left(\frac{a}{a+b+c} \cdot \frac{1}{\sqrt[7]{a^6}} \right) \\ &\ge \left(\frac{1}{a^a b^b c^c} \right)^{\frac{6/7}{a+b+c}}.\end{aligned}$$

Rearranging yields $a^a b^b c^c \ge 1$. ■

This problem and solution were proposed by Evan Chen.

Remark. The problem generalizes easily to n variables, and exponents other than $\frac{1}{7}$. Specifically, if positive reals $x_1 + \cdots + x_n = x_1^r + \cdots + x_n^r$ for some real number $r \neq 1$, then $\prod_{i \ge 1} x_i^{x_i} \ge 1$ if and only if $r < 1$. When $r \le 0$, a Jensen solution is possible using only the inequality $a + b + c \ge 3$.

Solution 2. First we claim that $a, b, c < 5$. Assume the contrary, that $a \ge 5$. Let $f(x) = x - \sqrt[7]{x}$. Since $f'(x) > 0$ for $x \ge 5$, we know that $f(a) \ge 5 - \sqrt[7]{5} > 3$. But this means that WLOG $b - \sqrt[7]{b} < -1.5$, which is clearly false since $b - \sqrt[7]{b} \ge 0$ for $b \ge 1$, and $b - \sqrt[7]{b} \ge -\sqrt[7]{b} \ge -1$ for $0 < b < 1$. So indeed $a, b, c < 5$.

Now rewrite the inequality as

$$\sum a \ln a \ge 0 \Leftrightarrow \sum \left(\frac{a^{\frac{1}{7}}}{a^{\frac{1}{7}} + b^{\frac{1}{7}} + c^{\frac{1}{7}}} \right) (a^{\frac{6}{7}} \ln a) \ge 0.$$

Now note that if $g(x) = x^{\frac{6}{7}} \ln x$, then $g''(x) = \frac{35 - 6 \ln x}{49 x^{\frac{8}{7}}} > 0$ for $x \in (0, 5)$. Therefore g is convex and we can use Jensen's Inequality to get

$$\sum \left(\frac{a^{\frac{1}{7}}}{a^{\frac{1}{7}} + b^{\frac{1}{7}} + c^{\frac{1}{7}}} \right) (a^{\frac{6}{7}} \ln a) \ge \left(\sum \frac{a^{\frac{8}{7}}}{a^{\frac{1}{7}} + b^{\frac{1}{7}} + c^{\frac{1}{7}}} \right)^{\frac{6}{7}} \ln \left(\sum \frac{a^{\frac{8}{7}}}{a^{\frac{1}{7}} + b^{\frac{1}{7}} + c^{\frac{1}{7}}} \right).$$

Since $\sum a = \sum a^{\frac{1}{7}}$, it suffices to show that $\sum a^{\frac{8}{7}} \ge \sum a$. But by weighted AM-GM we have

$$6a^{\frac{8}{7}} + a^{\frac{1}{7}} \ge 7a \implies a^{\frac{8}{7}} - a \ge \frac{1}{6}(a - \sqrt[7]{a}).$$

Adding up the analogous inequalities for b, c gives the desired result. ■

This second solution was suggested by David Stoner.

Solution 3. Here we unify the two solutions above.

It's well-known that weighted AM-GM follows from (and in fact, is equivalent to) the convexity of e^x (or equivalently, the concavity of $\ln x$), as $\sum w_i e^{x_i} \ge e^{\sum w_i x_i}$ for reals x_i and nonnegative weights w_i summing to 1. However, it also follows from the convexity of $y \ln y$ (or equivalently, the concavity of ye^y) for $y > 0$. Indeed, letting $y_i = e^{x_i} > 0$, and taking logs, weighted AM-GM becomes

$$\sum w_i y_i \cdot \frac{1}{y_i} \log \frac{1}{y_i} \ge (\sum w_i y_i) \frac{\sum w_i y_i \cdot \frac{1}{y_i}}{\sum w_i y_i} \log \frac{\sum w_i y_i \cdot \frac{1}{y_i}}{\sum w_i y_i},$$

which is clear.

To find Evan's solution, we can use the concavity of $\ln x$ to get $\sum a \ln a^{-s} \le (\sum a) \ln \sum \frac{a \cdot a^{-s}}{\sum a} = 0$. (Here we take $s = 6/7 > 0$.)

For a cleaner version of David's solution, we can use the convexity of $x \ln x$ to get

$$\sum a \ln a^s = \sum a^{1-s} \cdot a^s \ln a^s \ge (\sum a^{1-s}) \frac{\sum a^{1-s} \cdot a^s}{\sum a^{1-s}} \ln \frac{\sum a^{1-s} \cdot a^s}{\sum a^{1-s}} = 0$$

(where we again take $s = 6/7 > 0$).

Both are pretty intuitive (but certainly not obvious) solutions once one realizes direct Jensen goes in the wrong direction. In particular, $s = 1$ doesn't work since we have $a+b+c \le 3$ from the power mean inequality. ■

This third solution was suggested by Victor Wang.

Solution 4. From $e^t \ge 1 + t$ for $t = \ln x^{-\frac{6}{7}}$, we find $\frac{6}{7} \ln x \ge 1 - x^{-\frac{6}{7}}$. Thus

$$\frac{6}{7} \sum a \ln a \ge \sum a - a^{\frac{1}{7}} = 0,$$

as desired. ■

This fourth solution was suggested by chronodecay.

Remark. Polya once dreamed a similar proof of n-variable AM-GM: $x \ge 1 + \ln x$ for positive x, so $\sum x_i \ge n + \ln \prod x_i$. This establishes AM-GM when $\prod x_i = 1$; the rest follows by homogenizing.

A6

Let a, b, c be positive reals such that $a+b+c=3$. Prove that

$$18\sum_{\text{cyc}}\frac{1}{(3-c)(4-c)}+2(ab+bc+ca)\geq 15.$$

David Stoner

Solution. Since $0 \leq a, b, c \leq 3$ we have

$$\frac{1}{(3-c)(4-c)} \geq \frac{2c^2+c+3}{36} \iff c(c-1)^2(2c-9) \leq 0.$$

Then

$$2(ab+bc+ca)+18\sum_{\text{cyc}}\left(\frac{2c^2+c+3}{36}\right)=(a+b+c)^2+\frac{a+b+c+9}{2}=15.$$

■

This problem was proposed by David Stoner. This solution was given by Evan Chen.

A7*

Consider a function $f : \mathbb{Z} \to \mathbb{Z}$ such that for every integer $n \geq 0$, there are at most $0.001n^2$ pairs of integers (x, y) for which $f(x + y) \neq f(x) + f(y)$ and $\max\{|x|, |y|\} \leq n$. Is it possible that for some integer $n \geq 0$, there are more than n integers a such that $f(a) \neq a \cdot f(1)$ and $|a| \leq n$?

David Yang

Answer. No.

Solution. Call an integer conformist if $f(n) = n \cdot f(1)$. Call a pair (x, y) good if $f(x+y) = f(x) + f(y)$ and bad otherwise. Let $h(n)$ denote the number of conformist integers with absolute value at most n.

Let $\epsilon = 0.001$, S be the set of conformist integers, $T = \mathbb{Z} \setminus S$ be the set of non-conformist integers, and $X_n = [-n, n] \cap X$ for sets X and positive integers n (so $|S_n| = h(n)$); clearly $|T_n| = 2n + 1 - h(n)$.

First we can easily get $h(n) = 2n + 1$ ($-n$ to n are all conformist) for $n \leq 10$.

Lemma 1. *Suppose a, b are positive integers such that $h(a) > a$ and $b \leq 2h(a) - 2a - 1$. Then $h(b) \geq 2b(1 - \sqrt{\epsilon}) - 1$.*

Proof. For any integer t, we have

$$\begin{aligned}
|S_a \cap (t - S_a)| &= |S_a| + |t - S_a| - |S_a \cup (t - S_a)| \\
&\geq 2h(a) - (\max (S_a \cup (t - S_a)) - \min (S_a \cup (t - S_a)) + 1) \\
&\geq 2h(a) - (\max(a, t + a) - \min(-a, t - a) + 1) \\
&= 2h(a) - (|t| + 2a + 1) \\
&\geq b - |t|.
\end{aligned}$$

But (x, y) is bad whenever $x, y \in S$ yet $x + y \in T$, so summing over all $t \in T_b$ (assuming $|T_b| \geq 2$) yields

$$\begin{aligned}
\epsilon b^2 \geq g(b) &\geq \sum_{t \in T_b} |S_a \cap (t - S_a)| \\
&\geq \sum_{t \in T_b} (b - |t|) \geq \sum_{k=0}^{\lfloor |T_b|/2 \rfloor - 1} k + \sum_{k=0}^{\lceil |T_b|/2 \rceil - 1} k \geq 2\frac{1}{2}(|T_b|/2)(|T_b|/2 - 1),
\end{aligned}$$

where we use $\lfloor r/2 \rfloor + \lceil r/2 \rceil = r$ (for $r \in \mathbb{Z}$) and the convexity of $\frac{1}{2}x(x-1)$. We conclude that $|T_b| \leq 2 + 2b\sqrt{\epsilon}$ (which obviously remains true without the assumption $|T_b| \geq 2$) and $h(b) = 2b+1-|T_b| \geq 2b(1-\sqrt{\epsilon})-1$. $\square$

Now we prove by induction on n that $h(n) \geq 2n(1 - \sqrt{\epsilon}) - 1$ for all $n \geq 10$, where the base case is clear. If we assume the result for $n - 1$ ($n > 10$), then in view of the lemma, it suffices to show that $2h(n - 1) - 2(n - 1) - 1 \geq n$, or equivalently, $2h(n - 1) \geq 3n - 1$. But

$$2h(n - 1) \geq 4(n - 1)(1 - \sqrt{\epsilon}) - 2 \geq 3n - 1,$$

so we're done. (The second inequality is equivalent to $n(1 - 4\sqrt{\epsilon}) \geq 5 - 4\sqrt{\epsilon}$; $n \geq 11$ reduces this to $6 \geq 40\sqrt{\epsilon} = 40\sqrt{0.001} = 4\sqrt{0.1}$, which is obvious.) ∎

This problem and solution were proposed by David Yang.

A8*

Let a, b, c be positive reals with $a^{2013} + b^{2013} + c^{2013} + abc = 4$. Prove that

$$\left(\sum a(a^2 + bc)\right)\left(\sum\left(\frac{a}{b} + \frac{b}{a}\right)\right) \geq \left(\sum\sqrt{(a+1)(a^3+bc)}\right)\left(\sum\sqrt{a(a+1)(a+bc)}\right).$$

David Stoner

Solution.

Lemma 1. *Let x, y, z be positive reals, not all strictly on the same side of* 1*. Then* $\sum \frac{x}{y} + \frac{y}{x} \geq \sum x + \frac{1}{x}$.

Proof. WLOG $(x-1)(y-1) \leq 0$; then

$$(x+y+z-1)(x^{-1}+y^{-1}+z^{-1}-1) \geq (xy+z)(x^{-1}y^{-1}+z) \geq 4$$

by Cauchy.

Alternatively, if $x, y \geq 1 \geq z$, one may smooth z up to 1 (e.g. by differentiating with respect to z and observing that $x^{-1} + y^{-1} - 1 \leq x + y - 1$) to reduce the inequality to $\frac{x}{y} + \frac{y}{x} \geq 2$. □

Let $s_i = a^i + b^i + c^i$ and $p = abc$. The key is to Cauchy out s_3's from the RHS and use the lemma (in the form $s_1 s_{-1} - 3 \geq s_1 + s_{-1}$) on the LHS to reduce the problem to

$$(s_1 + s_{-1})^2(s_3 + 3p)^2 \geq (3 + s_1)(3 + s_{-1})(s_3 + ps_{-1})(s_3 + ps_1).$$

By AM-GM on the RHS, it suffices to prove

$$\frac{\frac{s_1+s_{-1}}{2} + \frac{s_1+s_{-1}}{2}}{\frac{s_1+s_{-1}}{2} + 3} \geq \frac{s_3 + p\frac{s_1+s_{-1}}{2}}{s_3 + 3p},$$

or equivalently, since $\frac{s_1+s_{-1}}{2} \geq 3$, that $\frac{s_3}{p} \geq \frac{s_1+s_{-1}}{2}$. By the lemma, this boils down to $2\sum_{\text{cyc}} a^3 \geq \sum_{textcyc} a(b^2 + c^2)$, which is obvious. ■

This problem and solution were proposed by David Stoner.

Remark. The condition $a^{2013} + b^{2013} + c^{2013} + abc = 4$ can be replaced with anything that guarantees a, b, c are not all on the same side of 1. One can also propose the following more direct application of the lemma instead: "Let a, b, c be positive reals with $a^{2013} + b^{2013} + c^{2013} + abc = 4$. Prove that

$$\sum\left(\left(\frac{a}{b}\right)^{2012} + \left(\frac{b}{a}\right)^{2012}\right) \geq \sum\left(a^{2011} + \frac{1}{a^{2011}}\right).$$

" This is perhaps more motivated, but also significantly easier. Note that if one replaces the exponents in the inequality with something like 2013 and 2012, then one may use the PQR method to reduce the problem to the case when two of a, b, c are equal. Alternatively, if one changes the condition to $a^{2013}b + b^{2013}c + c^{2013}a + abc = 4$, then it's perfectly fine for the first exponent to be at least 2013 and the second to be at most 2013; however, this makes the lemma much more transparent.

A9

Let a, b, c be positive reals, and let $\sqrt[2013]{\frac{3}{a^{2013}+b^{2013}+c^{2013}}} = P$. Prove that

$$\prod_{\text{cyc}} \left(\frac{(2P + \frac{1}{2a+b})(2P + \frac{1}{a+2b})}{(2P + \frac{1}{a+b+c})^2} \right) \ge \prod_{\text{cyc}} \left(\frac{(P + \frac{1}{4a+b+c})(P + \frac{1}{3b+3c})}{(P + \frac{1}{3a+2b+c})(P + \frac{1}{3a+b+2c})} \right).$$

David Stoner

Solution. WLOG $P = 1$; we prove that any positive a, b, c (even those without $\sum a^{2013} = 3$) satisfy the inequality. The key is that $f(x) = \log(1 + x^{-1}) = \log(1 + x) - \log(x)$ is convex, since $f''(x) = -(1 + x)^{-2} + x^{-2} > 0$ for all x.

By Jensen's inequality, we have

$$\begin{aligned} \frac{1}{2} f(2(2a + b)) + \frac{1}{2} f(2(2a + c)) &\ge f(4a + b + c) \\ \frac{1}{2} f(2(2b + c)) + \frac{1}{2} f(2(2c + b)) &\ge f(3b + 3c) \\ -2f(2(a + b + c)) &\ge -f(3a + 2b + c) - f(3c + 2b + a). \end{aligned}$$

Exponentiating and multiplying everything once (cyclically), we get the desired inequality. ■

This problem and solution were proposed by David Stoner.

C1

Let $n \geq 2$ be a positive integer. The numbers $1, 2, \ldots, n^2$ are consecutively placed into squares of an $n \times n$, so the first row contains $1, 2, \ldots, n$ from left to right, the second row contains $n+1, n+2, \ldots, 2n$ from left to right, and so on. The *magic square value* of a grid is defined to be the number of rows, columns, and main diagonals whose elements have an average value of $\frac{n^2+1}{2}$. Show that the magic-square value of the grid stays constant under the following two operations: (1) a permutation of the rows; and (2) a permutation of the columns. (The operations can be used multiple times, and in any order.)

Ray Li

Solution 1. The set of row sums and column sums is clearly preserved under operations (1) and (2), so we just have to consider the main diagonals. In configuration A, let a_{ij} denote the number in the ith row and jth column; then whenever $i \neq j$ and $k \neq l$, we have $a_{ij} + a_{kl} = a_{il} + a_{kj}$. But this property is invariant as well, so the main diagonal sums remain constant under the operations, and we're done. ■

This problem and solution were proposed by Ray Li.

Solution 2. We present a proof without words for the case $n = 4$, which easily generalizes to other values of n.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 4 & 4 & 4 & 4 \\ 8 & 8 & 8 & 8 \\ 12 & 12 & 12 & 12 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

■

This second solution was suggested by Evan Chen.

C2

Let n be a fixed positive integer. Initially, n 1's are written on a blackboard. Every minute, David picks two numbers x and y written on the blackboard, erases them, and writes the number $(x+y)^4$ on the blackboard. Show that after $n-1$ minutes, the number written on the blackboard is at least $2^{\frac{4n^2-4}{3}}$.

Calvin Deng

Solution. We proceed by strong induction on n. For $n = 1$ this is obvious; now assuming the result up to $n-1$ for some $n > 1$, consider the two numbers on the blackboard after $n-2$ minutes. They must have been created "independently," where the first took $a-1$ minutes and the second took $b-1$ minutes for two positive integers a, b $(a+b=n)$. But 2^x is convex, so

$$2^{\frac{4a^2-4}{3}} + 2^{\frac{4b^2-4}{3}} \geq 2 \cdot 2^{\frac{2(a^2+b^2)-4}{3}} \geq 2 \cdot 2^{\frac{(a+b)^2-4}{3}} = 2^{\frac{(a+b)^2-1}{3}} = 2^{\frac{n^2-1}{3}},$$

completing the induction. ■

This problem and solution were proposed by Calvin Deng.

C3*

Let $a_1, a_2, \ldots, a_9$ be nine real numbers, not necessarily distinct, with average m. Let A denote the number of triples $1 \le i < j < k \le 9$ for which $a_i + a_j + a_k \ge 3m$. What is the minimum possible value of A?

Ray Li

Answer. $A \ge 28$.

Solution 1. Call a 3-set *good* iff it has average at least m, and let S be the family of good sets.

The equality case $A = 28$ can be achieved when $a_1 = \cdots = a_8 = 0$ and $a_9 = 1$. Here $m = \frac{1}{9}$, and the good sets are precisely those containing a_9. This gives a total of $\binom{8}{2} = 28$.

To prove the lower bound, suppose we have exactly N good 3-sets, and let $p = \frac{N}{\binom{9}{3}}$ denote the probability that a randomly chosen 3-set is good. Now, consider a random permutation π of $\{1, 2, \ldots, 9\}$. Then the corresponding partition $\bigcup_{i=0}^{2} \{\pi(3i+1), \pi(3i+2), \pi(3i+3)\}$ has at least 1 good 3-set, so by the linearity of expectation,

$$\begin{aligned} 1 &\le \mathbb{E}\left[\sum_{i=0}^{2} [\{\pi(3i+1), \pi(3i+2), \pi(3i+3)\} \in S]\right] \\ &= \sum_{i=0}^{2} \left[\mathbb{E}[\{\pi(3i+1), \pi(3i+2), \pi(3i+3)\} \in S]\right] \\ &= \sum_{i=0}^{2} 1 \cdot p = 3p. \end{aligned}$$

Hence $N = p\binom{9}{3} \ge \frac{1}{3}\binom{9}{3} = 28$, establishing the lower bound. ■

This problem and solution were proposed by Ray Li.

Remark. One can use double-counting rather than expectation to prove $N \ge 28$. In any case, this method generalizes effortlessly to larger numbers.

Solution 2. Proceed as above to get an upper bound of 28.

On the other hand, we will show that we can partition the $\binom{9}{3} = 84$ 3-sets into 28 groups of 3, such that in any group, the elements $a_1, a_2, \cdots, a_9$ all appear. This will imply the conclusion, since if $A < 28$, then there are at least 57 sets with average at most m, but by pigeonhole three of them must be in such a group, which is clearly impossible.

Consider a 3-set and the following array:

$$\begin{matrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{matrix}$$

Consider a set $|S| = 3$. We obtain the other two 3-sets in the group as follows:

- If S contains one element in each column, then shift the elements down cyclically mod 3.
- If S contains one element in each row, then shift the elements right cyclically mod 3. Note that the result coincides with the previous case if both conditions are satisfied.
- Otherwise, the elements of S are "constrained" in a 2×2 box, possibly shifted diagonally. In this case, we get an L-tromino. Then shift diagonally in the direction the L-tromino points in.

One can verify that this algorithm creates such a partition, so we conclude that $A \ge 28$. ■

This second solution was suggested by Lewis Chen.

C4

Let n be a positive integer. The numbers $\{1, 2, \dots, n^2\}$ are placed in an $n \times n$ grid, each exactly once. The grid is said to be *Muirhead-able* if the sum of the entries in each column is the same, but for every $1 \le i, k \le n-1$, the sum of the first k entries in column i is at least the sum of the first k entries in column $i+1$. For which n can one construct a Muirhead-able array?

Evan Chen

Answer. All $n \neq 3$.

Solution. It's easy to prove $n = 3$ doesn't work since the top row must be 9,8,7 (each column sums to 15) and the first column is either 9,5,1 or 9,4,2.

A construction for even n is not hard to realize:

$$\begin{array}{cccc}
n^2 & n^2-1 & \cdots & n^2-n+1 \\
n^2-n & n^2-n-1 & \cdots & n^2-2n+1 \\
\vdots & \vdots & \ddots & \vdots \\
n^2-(\frac{n}{2}-1)n & n^2-(\frac{n}{2}-1)n & \cdots & n^2-(\frac{n}{2})n+1 \\
n^2-(\frac{n}{2}+1)n+1 & n^2-(\frac{n}{2}+1)n+2 & \cdots & n^2-(\frac{n}{2})n \\
\vdots & \vdots & \ddots & \vdots \\
n+1 & n+2 & \cdots & 2n \\
1 & 2 & \cdots & n
\end{array}$$

And we can just alter the even construction a bit for $n \ge 5$ odd; I'll just write it out for $n = 7$ since it generalizes easily: we modify

$$7\begin{pmatrix}
6 & 6 & 6 & 6 & 6 & 6 & 6 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
7 & 6 & 5 & 4 & 3 & 2 & 1 \\
7 & 6 & 5 & 4 & 3 & 2 & 1 \\
7 & 6 & 5 & 4 & 3 & 2 & 1 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 2 & 3 & 4 & 5 & 6 & 7
\end{pmatrix}$$

to get

$$7\begin{pmatrix}
6 & 6 & 6 & 6 & 6 & 6 & 6 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
7 & 6 & 5 & 4 & 3 & 2 & 1 \\
7 & 6 & 5 & 4 & 3 & 2 & 1 \\
5 & 6 & 7 & 1 & 2 & 3 & 4 \\
6 & 4 & 2 & 7 & 5 & 3 & 1 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 2 & 3 & 4 & 5 & 6 & 7
\end{pmatrix}.$$

If we verify the majorization condition for the original one (without regard to distinctness) then we only have to check it in the new one for $k = 3 = \frac{n-1}{2}$ and $i = 1, 2, 4, 5, 6$ (in particular, we can skip $i = 3 = \frac{n-1}{2}$). ■

This problem and solution were proposed by Evan Chen.

C5

There is a 2012×2012 grid with rows numbered $1, 2, \ldots 2012$ and columns numbered $1, 2, \ldots, 2012$, and we place some rectangular napkins on it such that the sides of the napkins all lie on grid lines. Each napkin has a positive integer thickness. (in micrometers!)

(a) Show that there exist 2012^2 unique integers $a_{i,j}$ where $i, j \in [1, 2012]$ such that for all $x, y \in [1, 2012]$, the sum
$$\sum_{i=1}^{x} \sum_{j=1}^{y} a_{i,j}$$
is equal to the sum of the thicknesses of all the napkins that cover the grid square in row x and column y.

(b) Show that if we use at most $500,000$ napkins, at least half of the $a_{i,j}$ will be 0.

Ray Li

Solution 1. (a) Let $t_{i,j}$ be the total thickness at square (i, j) (row i, column j). For convenience, set $t_{i,j} = 0$ outside the boundary (i.e. if one of i, j is less than 1 or greater than 2012). By induction on $i + j \geq 2$ (over $i, j \in [2012]$), it's easy to see that the $a_{i,j}$ are uniquely defined as $t_{i,j} + t_{i-1,j-1} - t_{i-1,j} - t_{i,j-1}$ (and that this solution also works).

(b) One can easily check that $a_{i,j} = 0$ if no napkin corners lie at intersection of the ith vertical grid line (from the top) and the jth horizontal grid line (from the left). Indeed, if we color squares $(i-1, j-1)$ and (i, j) red, $(i-1, j)$ and $(i, j-1)$ blue, then if there are no such napkin corners, every napkin must hit an equal number of red and blue squares and thus contribute zero to the sum $t_{i,j} + t_{i-1,j-1} - t_{i-1,j} - t_{i,j-1}$. On the other hand, there are at most $4 \cdot 500000$ corners, and $2012^2 > 4000000 = 2(4 \cdot 500000)$ pairs $(i, j) \in [2012]^2$, so we're done. ■

Solution 2. Throughout this proof, rows go from bottom to top, and columns go from left to right.

Suppose we add a napkin with thickness x.

This affects the a-value only at the four corner points of the napkin. Corners are defined to be the bolded points in the following diagram. If the napkin shares an edge with the top boundary or the right boundary, some corners may not be considered for a-value valuation, which is even better for part (b). [Alternatively, for purists out there, define a-values for $i, j = 2013$.]

$$\begin{array}{|ccccc|} 0 & 0 & 0 & 0 & 0 \\ \mathbf{-1} & 0 & 0 & \mathbf{1} & 0 \\ \boxed{0} & \boxed{0} & \boxed{0} & 0 & 0 \\ \boxed{\mathbf{1}} & \boxed{0} & \boxed{0} & \mathbf{-1} & 0 \end{array}$$

Boxes represent squares covered by napkins.

Specifically, the a-values of the bottom-left and top-right corners increment by x, and the bottom-right and top-left corners decrement by x. (Easy verification with diagram. This should be somewhat intuitive as well: think PIE.)

Notably, the process of adding a napkin is additive and reversible. Hence no matter how many napkins are placed on the table, we can just add a-values together.

So a-values exist, and can be consistently labeled. Furthermore, each napkin modifies at most 4 a-values, so with 500,000 napkins at most 2 million a-values are modified, which is less than half of 2012^2. ■

This problem and its solutions were proposed by Ray Li.

C6

A 4×4 grid has its 16 cells colored arbitrarily in three colors. A *swap* is an exchange between the colors of two cells. Prove or disprove that it always takes at most three swaps to produce a line of symmetry, regardless of the grid's initial coloring.

Matthew Babbitt

Answer. No.

Solution. We provide the following counterexample, in the colors red, white, and green:

$$\begin{array}{cccc} W & W & G & W \\ R & W & W & R \\ R & R & R & G \\ R & W & W & G \end{array}$$

Suppose for contradiction that we can get a line of symmetry in 3 or less swaps. Clearly the symmetry must be over a diagonal.

If it is upper left to lower right, then there are 6 pairs of squares that reflect to each other over this diagonal and 4 squares on the diagonal. None of the 6 pairs are matched, so at least one square in each must be part of a swap. Also, there must be an even number of red squares on the diagonal, so one of the diagonal squares must be swapped, for a total of $7 > 3 \cdot 2$. This requires more than 3 swaps. The other diagonal works similarly. ■

This problem was proposed by Matthew Babbitt. This solution was given by Bobby Shen.

Remark. To construct counterexamples, we first put an odd number of one color (so symmetry must be over a diagonal), make no existing matches over the diagonal, and require that one or more of the diagonal squares be part of a swap.

C7*

A $2^{2013}+1$ by $2^{2013}+1$ grid has some black squares filled. The filled black squares form one or more snakes on the plane, each of whose heads splits at some points but never comes back together. In other words, for every positive integer $n > 1$, there do not exist pairwise distinct black squares $s_1, s_2, \dots, s_n$ such that s_i, s_{i+1} share an edge for $i = 1, 2, \dots, n$ (here $s_{n+1} = s_1$). What is the maximum possible number of filled black squares?

David Yang

Answer. If $n = 2^m + 1$ is the dimension of the grid, the answer is $\frac{2}{3}n(n+1) - 1$. In this particular instance, $m = 2013$ and $n = 2^{2013} + 1$.

Solution. Let $n = 2^m + 1$. Double-counting square edges yields $3v + 1 \le 4v - e \le 2n(n+1)$, so because $n \not\equiv 1 \pmod 3$, $v \le 2n(n+1)/3 - 1$. Observe that if $3 \nmid n - 1$, equality is achieved iff (a) the graph formed by black squares is a connected forest (i.e. a tree) and (b) all but two square edges belong to at least one black square.

We prove by induction on $m \ge 1$ that equality can in fact be achieved. For $m = 1$, take an "H-shape" (so if we set the center at $(0,0)$ in the coordinate plane, everything but $(0, \pm 1)$ is black); call this G_1. To go from G_m to G_{m+1}, fill in $(2x, 2y)$ in G_{m+1} iff (x, y) is filled in G_m, and fill in (x, y) with x, y not both even iff $x + y$ is odd (so iff one of x, y is odd and the other is even). Each "newly-created" white square has both coordinates odd, and thus borders 4 (newly-created) black squares. In particular, there are no new white squares on the border (we only have the original two from G_1). Furthermore, no two white squares share an edge in G_{m+1}, since no square with odd coordinate sum is white. Thus G_{m+1} satisfies (b). To check that (a) holds, first we show that $(2x_1, 2y_1)$ and $(2x_2, 2y_2)$ are connected in G_{m+1} iff (x_1, y_1) and (x_2, y_2) are black squares (and thus connected) in G_m (the new black squares are essentially just "bridges"). Indeed, every path in G_{m+1} alternates between coordinates with odd and even sum, or equivalently, new and old black squares. But two black squares (x_1, y_1) and (x_2, y_2) are adjacent in G_m iff $(x_1 + x_2, y_1 + y_2)$ is black and adjacent to $(2x_1, 2y_1)$ and $(2x_2, 2y_2)$ in G_{m+1}, whence the claim readily follows. The rest is clear: the set of old black squares must remain connected in G_{m+1}, and all new black squares (including those on the boundary) border at least one (old) black square (or else G_m would not satisfy (b)), so G_{m+1} is fully connected. On the other hand, G_{m+1} cannot have any cycles, or else we would get a cycle in G_m by removing the new black squares from a cycle in G_{m+1} (as every other square in a cycle would have to have odd coordinate sum). ■

This problem and solution were proposed by David Yang.

C8

There are 20 people at a party. Each person holds some number of coins. Every minute, each person who has at least 19 coins simultaneously gives one coin to every other person at the party. (So, it is possible that A gives B a coin and B gives A a coin at the same time.) Suppose that this process continues indefinitely. That is, for any positive integer n, there exists a person who will give away coins during the nth minute. What is the smallest number of coins that could be at the party?

Ray Li

Solution 1. Call a person giving his 19 coins away a *charity*. For any finite, fixed number of coins there are finitely many states, which implies that the states must cycle infinitely. Hence by doing individual charities one by one, there is a way to make it cycle infinitely (just take the charities that would normally happen at the same time and do them one by one all together before moving on). So this means we can reverse the charities and have it go on infinitely the other way, so call an inverse charity a *theft*. But after $k \le 20$ thefts, the number of coins among the people who have stolen at least once is at least $19 + 18 + \cdots + (20 - k)$ since the kth thief steals at most $k - 1$ coins from people who were already thieves but gains 19. So then we're done since for $k = 20$ this is 190. Of course, one construction is just when person j has $j - 1$ coins. ■

This first solution was suggested by Mark Sellke.

Solution 2. Like above, do the charities in arbitrary order among the ones that are "together." Assume there are at most 189 coins. Then the sum of squares of coins each guy has decreases each time, since if one guy loses 19 coins then the sums of squares decreases by at least 361, while giving 1 coin to everyone else increases it by $19+2$(number of coins they had before), and the number of coins they had before is less than 171 since the giving guy had 19 already, and so the sum of squares decreases since $361 > 19 + 2 \cdot 170$. ■

This second solution was suggested by Mark Sellke.

Remark. Compare with this problem in 102 Combinatorial Problems (paraphrased, St. Petersburg 1988): "119 residents live in a place with 120 apartments. Every day, in each apartment with at least 15 people, all the people move out into pairwise distinct apartments. Must this process terminate?"

This problem was proposed by Ray Li.

C9*

f_0 is the function from $\mathbb{Z}^2$ to $\{0,1\}$ such that $f_0(0,0) = 1$ and $f_0(x,y) = 0$ otherwise. For each $i > 1$, let $f_i(x,y)$ be the remainder when

$$f_{i-1}(x,y) + \sum_{j=-1}^{1}\sum_{k=-1}^{1} f_{i-1}(x+j, y+k)$$

is divided by 2.

For each $i \geq 0$, let $a_i = \sum_{(x,y)\in\mathbb{Z}^2} f_i(x,y)$. Find a closed form for a_n (in terms of n).

Bobby Shen

Solution. a_i is simply the number of odd coefficients of $A_i(x,y) = A(x,y)^i$, where $A(x,y) = (x^2+x+1)(y^2+y+1) - xy$. Throughout this proof, we work in $\mathbb{F}_2$ and repeatedly make use of the Frobenius endomorphism in the form $A_{2^k m}(x,y) = A_m(x,y)^{2^k} = A_m(x^{2^k}, y^{2^k})$ (*). We advise the reader to try the following simpler problem before proceeding: "Find (a recursion for) the number of odd coefficients of $(x^2+x+1)^{2^n-1}$."

First suppose n is not of the form $2^m - 1$, and has $i \geq 0$ ones before its first zero from the right. By direct exponent analysis (after using (*)), we obtain $a_n = a_{\frac{n-(2^i-1)}{2}} a_{2^i-1}$. Applying this fact repeatedly, we find that $a_n = a_{2^{\ell_1}-1}\cdots a_{2^{\ell_r}-1}$, where $\ell_1, \ell_2, \ldots, \ell_r$ are the lengths of the r consecutive strings of ones in the binary representation of n. (When $n = 2^m - 1$, this is trivially true. When $n = 0$, we take $r = 0$ and a_0 to be the empty product 1, by convention.)

We now restrict our attention to the case $n = 2^m - 1$. The key is to look at the exponents of x and y modulo 2—in particular, $A_{2n}(x,y) = A_n(x^2,y^2)$ has only "$(0,0) \pmod 2$" terms for $i \geq 1$. This will allow us to find a recursion.

For convenience, let $U[B(x,y)]$ be the number of odd coefficients of $B(x,y)$, so $U[A_{2^n-1}(x,y)] = a_{2^n-1}$. Observe that

$$\begin{aligned}
A(x,y) &= (x^2+x+1)(y^2+y+1) - xy = (x^2+1)(y^2+1) + (x^2+1)y + x(y^2+1)\\
(x+1)A(x,y) &= (y^2+1) + (x^2+1)y + x^3(y^2+1) + (x^3+x)y\\
(x+1)(y+1)A(x,y) &= (x^2y^2+1) + (x^2y+y^3) + (x^3+xy^2) + (x^3y^3+xy)\\
(x+y)A(x,y) &= (x^2+y^2) + (x^2+1)(y^3+y) + (x^3+x)(y^2+1) + (x^3y+xy^3).
\end{aligned}$$

Hence for $n \geq 1$, we have (using (*) again)

$$\begin{aligned}
U[A_{2^n-1}(x,y)] &= U[A(x,y)A_{2^{n-1}-1}(x^2,y^2)]\\
&= U[(x+1)(y+1)A_{2^{n-1}-1}(x,y)] + U[(y+1)A_{2^{n-1}-1}(x,y)] + U[(x+1)A_{2^{n-1}-1}(x,y)]\\
&= U[(x+1)(y+1)A_{2^{n-1}-1}(x,y)] + 2U[(x+1)A_{2^{n-1}-1}(x,y)].
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
U[(x+1)A_{2^n-1}] &= 2U[(y+1)A_{2^{n-1}-1}] + 2U[(x+1)A_{2^{n-1}-1}] = 4U[(x+1)A_{2^{n-1}-1}]\\
U[(x+1)(y+1)A_{2^n-1}] &= 2U[(xy+1)A_{2^{n-1}-1}] + 2U[(x+y)A_{2^{n-1}-1}] = 4U[(x+y)A_{2^{n-1}-1}]\\
U[(x+y)A_{2^n-1}] &= 2U[(x+1)(y+1)A_{2^{n-1}-1}] + 2U[(x+y)A_{2^{n-1}-1}].
\end{aligned}$$

Here we use the symmetry between x and y, and the identity $(xy+1) = y(x+y^{-1})$.) It immediately follows that

$$\begin{aligned}
U[(x+1)(y+1)A_{2^{n+1}-1}] &= 4U[(x+y)A_{2^n-1}]\\
&= 8U[(x+1)(y+1)A_{2^{n-1}-1}] + 8\frac{U[(x+1)(y+1)A_{2^n-1}]}{4}
\end{aligned}$$

for all $n \geq 1$, and because $x - 4 \mid (x+2)(x-4) = x^2 - 2x - 8$,

$$U[A_{2^{n+2}-1}(x,y)] = 2U[A_{2^{n+1}-1}(x,y)] + 8U[A_{2^n-1}(x,y)]$$

as well. But $U[A_{2^0-1}] = 1$, $U[A_{2^1-1}] = 8$, and

$$U[A_{2^2-1}] = 4U[x+y] + 8U[x+1] = 24,$$

so the recurrence also holds for $n = 0$. Solving, we obtain $a_{2^n-1} = \frac{5\cdot 4^n - 2(-2)^n}{3}$, so we're done. ■

This problem and solution were proposed by Bobby Shen.

Remark. The number of odd coefficients of $(x^2+x+1)^n$ is the Jacobsthal sequence (OEIS A001045) (up to translation). The sequence $\{a_n\}$ in the problem also has a (rather empty) OEIS entry. It may be interesting to investigate the generalization

$$\sum_{j=-1}^{1}\sum_{k=-1}^{1} c_{j,k} f_{i-1}(x+j, y+k)$$

for 9-tuples $(c_{j,k}) \in \{0,1\}^9$. Note that when all $c_{j,k}$ are equal to 1, we get $(x^2+x+1)^n(y^2+y+1)^n$, and thus the square of the Jacobsthal sequence.

Even more generally, one may ask the following: "Let f be an integer-coefficient polynomial in $n \geq 1$ variables, and p be a prime. For $i \geq 0$, let a_i denote the number of nonzero coefficients of f^{p^i-1} (in $\mathbb{F}_p$).

Under what conditions must there always exist an infinite arithmetic progression AP of positive integers for which $\{a_i : i \in AP\}$ satisfies a linear recurrence?"

C10*

Let $N \geq 2$ be a fixed positive integer. There are $2N$ people, numbered $1, 2, \ldots, 2N$, participating in a tennis tournament. For any two positive integers i, j with $1 \leq i < j \leq 2N$, player i has a higher skill level than player j. Prior to the first round, the players are paired arbitrarily and each pair is assigned a unique court among N courts, numbered $1, 2, \ldots, N$.

During a round, each player plays against the other person assigned to his court (so that exactly one match takes place per court), and the player with higher skill wins the match (in other words, there are no upsets). Afterwards, for $i = 2, 3, \ldots, N$, the winner of court i moves to court $i - 1$ and the loser of court i stays on court i; however, the winner of court 1 stays on court 1 and the loser of court 1 moves to court N.

Find all positive integers M such that, regardless of the initial pairing, the players $2, 3, \ldots, N + 1$ all change courts immediately after the Mth round.

Ray Li

Answer. $M \geq N + 1$.

Solution. It is enough to prove the claim for $M = N + 1$. (Why?)

After the kth move ($k \geq 0$), let $a_i^{(k)} \in [0, 2]$ be the number of rookies (players $N + 2, \ldots, 2N$) in court i so that $a_1^{(k)} + \cdots + a_N^{(k)} = N - 1$.

The operation from the perspective of the rookies can be described as follows: $a_i^{(k)} = 2$ for some $i \in \{2, \ldots, N\}$ means we "transfer" a rookie from court i to court $i - 1$ on the $(k + 1)$th move, and $a_1^{(k)} \geq 1$ means we "transfer" a rookie from court 1 to court N on the $(k+1)$th move. Note that if $a_i^{(k)} \geq 1$ for some $k \geq 0$ and $i \in \{2, \ldots, N\}$, we must have $a_i^{(k+r)} \geq 1$ for all $r \geq 0$. (*)

But we also know that all "excesses" can be traced back to "transfers". More precisely, if $a_i^{(k)} = 2$ for some $i \in \{2, \ldots, N - 1\}$ and $k \geq 1$, we must have $a_{i+1}^{(k-1)} = 2$; if $a_N^{(k)} = 2$, we must have $a_1^{(k-1)} \geq 1$; and if $a_1^{(k)} \geq 1$, we must either have (i) $a_2^{(k-1)} = 2$ or (ii) $a_1^{(k-1)} = 2$ and if $k \geq 2$, $a_2^{(k-2)} = 2$.

If $a_i^{(N)} = 2$ for some $i \in \{2, \ldots, N\}$ or $a_1^{(N)} \geq 1$, then by the previous paragraph and (*) we see that $a_i^{(N)} \geq 1$ for $i = 2, \ldots, N$, contradicting the fact that $a_1^{(N)} + \cdots + a_N^{(N)} = N - 1$. (Here possibility (ii) from the previous paragraph forces us to consider the Nth move rather than the $(N - 1)$th move.)

Hence $a_1^{(N)} = 0$, $a_2^{(N)} = \cdots = a_N^{(N)} = 1$, and of course player 1 stabilizes after at most $N - 1$ moves (he always wins), so we get a bound of $\geq 1 + \max(N - 1, N) = N + 1$.

We cannot replace the condition $M \geq N + 1$ with $M \geq N'$ for any $N' < N$. Indeed, any configuration with $(a_1^{(0)}, \ldots, a_N^{(0)}) = (2, 0, 0, 1, 1, 1, 1, \ldots, 1)$ shows that $N + 1$ is the "best bound possible." ■

This problem was proposed by Ray Li. This solution was given by Victor Wang.

Remark. The key idea (which can be easily found by working backwards) is to focus on the rookies. Asking for the minimum number of rounds required for stablization rather than giving the answer directly (here $N + 1$) may make the problem slightly more difficult, but once one conceives the idea of isolating rookies, even this version is not much harder.

G1

Let ABC be a triangle with incenter I. Let U, V and W be the intersections of the angle bisectors of angles A, B, and C with the incircle, so that V lies between B and I, and similarly with U and W. Let X, Y, and Z be the points of tangency of the incircle of triangle ABC with BC, AC, and AB, respectively. Let triangle UVW be the *David Yang triangle* of ABC and let XYZ be the *Scott Wu triangle* of ABC. Prove that the David Yang and Scott Wu triangles of a triangle are congruent if and only if ABC is equilateral.

Owen Goff

Solution. The angles of the triangles are $(\frac{A+B}{2}, \frac{B+C}{2}, \frac{C+A}{2})$ and $(\frac{\frac{A+B}{2}+\frac{B+C}{2}}{2}, \frac{\frac{B+C}{2}+\frac{C+A}{2}}{2}, \frac{\frac{C+A}{2}+\frac{A+B}{2}}{2})$ by quick angle chasing. Since the sets $(x, y, z), (\frac{x+y}{2}, \frac{y+z}{2}, \frac{z+x}{2})$ are equal iff $x = y = z$, we are done. ■

This problem and solution were proposed by Owen Goff.

G2

Let ABC be a scalene triangle with circumcircle Γ, and let D,E,F be the points where its incircle meets BC, AC, AB respectively. Let the circumcircles of $\triangle AEF$, $\triangle BFD$, and $\triangle CDE$ meet Γ a second time at X, Y, Z respectively. Prove that the perpendiculars from A, B, C to AX, BY, CZ respectively are concurrent.

Michael Kural

Solution 1. We claim that this point is the reflection of I the incenter over O the circumcenter. Since $\angle AEI = \angle AFI = \frac{\pi}{2}$, $AFIE$ is cyclic with diameter AI, so $\angle AXI = 90$. Also, if M is the midpoint of AX, then $OM \perp AX$, so clearly the reflection of I over O lies on each of the perpendiculars. ■

Solution 2. Extend BY and CZ, CZ and AZ, and AX and BY to meet at P, Q, R respectively. Note that P is the radical center of the circumcircles of BDF and CDE and Γ, so P lies on the radical axis DI of the two circumcircles (I lies on both circles as we showed before). Then the perpendiculars from P, Q, R to BC, AC, AB concur at I, so by Carnot's theorem

$$PB^2 - PC^2 + QC^2 - QA^2 + RA^2 - RB^2 = 0 \implies AQ^2 - AR^2 + BR^2 - BP^2 + CP^2 - CQ^2 = 0.$$

Again by Carnot's theorem the perpendiculars from A, B, C to QR, PR, PQ concur, which was what we wanted. (In other words, triangles ABC and PQR are *orthologic*.) ■

This problem and its solutions were proposed by Michael Kural.

G3

In $\triangle ABC$, a point D lies on line BC. The circumcircle of ABD meets AC at F (other than A), and the circumcircle of ADC meets AB at E (other than A). Prove that as D varies, the circumcircle of AEF always passes through a fixed point other than A, and that this point lies on the median from A to BC.

Allen Liu

Solution 1. Invert about A. We get triangle ABC with a variable point D on its circumcircle. CD meets AB at E, BD meets AC at F. The pole of EF is the intersection of AD and BC, so it lies on BC, and the fixed pole of BC lies on EF, proving the claim. Also, since pole of BC is the intersection of the tangents from B and C, the point lies on the symmedian, which is the median under inversion. ■

This first solution was suggested by Michael Kural.

Solution 2. Use barycentric coordinates with $A = (1, 0, 0)$, etc. Let $D = (0 : m : n)$ with $m + n = 1$. Then the circle ABD has equation $-a^2yz - b^2zx - c^2xy + (x + y + z)\left(a^2m \cdot z\right)$. To intersect it with side AC, put $y = 0$ to get $(x + z)(a^2mz) = b^2zx \implies \frac{b^2}{a^2m} \cdot x = x + z \implies \left(\frac{b^2}{a^2m} - 1\right) x = z$, so

$$F = (a^2m : 0 : b^2 - a^2m)$$

Similarly,

$$G = (a^2n : c^2 - a^2n : 0).$$

Then, the circle (AFG) has equation

$$-a^2yz - b^2zx - c^2xy + a^2(x + y + z)(my + nz) = 0.$$

Upon picking $y = z = 1$, we easily see there exists a t such that $(t : 1 : 1)$ is on the circle, implying the conclusion. ■

This second solution was suggested by Evan Chen.

Solution 3. Let M be the midpoint of BC. By power of a point, $c \cdot BE + b \cdot CF = a \cdot BD + a \cdot CD = a^2$ is constant. Fix a point D_0; and let $P_0 = AM \cap (AE_0F_0)$. For any other point D, we have $\frac{E_0E}{F_0F} = \frac{b}{c} = \frac{\sin \angle BAM}{\sin \angle CAM} = \frac{P_0E_0}{P_0F_0}$ from the extended law of sines, so triangles P_0E_0E and P_0F_0F are directly similar, whence AEP_0F is cyclic, as desired. ■

This third solution was suggested by Victor Wang.

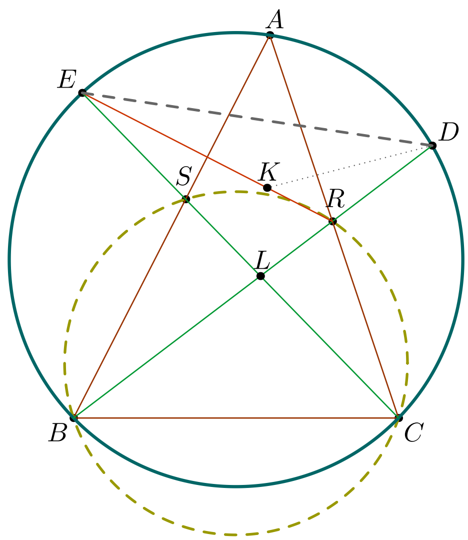
This problem was proposed by Allen Liu.

G4*

Triangle ABC is inscribed in circle ω. A circle with chord BC intersects segments AB and AC again at S and R, respectively. Segments BR and CS meet at L, and rays LR and LS intersect ω at D and E, respectively. The internal angle bisector of $\angle BDE$ meets line ER at K. Prove that if $BE = BR$, then $\angle ELK = \frac{1}{2}\angle BCD$.

Evan Chen

Solution 1.



First, we claim that $BE = BR = BC$. Indeed, construct a circle with radius $BE = BR$ centered at B, and notice that $\angle ECR = \frac{1}{2}\angle EBR$, implying that it lies on the circle.

Now, CA bisects $\angle ECD$ and DB bisects $\angle EDC$, so R is the incenter of $\triangle CDE$. Then, K is the incenter of $\triangle LED$, so $\angle ELK = \frac{1}{2}\angle ELD = \frac{1}{2}\left(\frac{\widehat{ED}+\widehat{BC}}{2}\right) = \frac{1}{2}\frac{\widehat{BED}}{2} = \frac{1}{2}\angle BCD$. ■

This problem and solution were proposed by Evan Chen.

Solution 2. Note $\angle EBA = \angle ECA = \angle SCR = \angle SBR = \angle ABR$, so AB bisects $\angle EBR$. Then by symmetry $\angle BEA = \angle BRA$, so $\angle BCR = \angle BCA = 180 - \angle BEA = 180 - \angle BRA = \angle BRC$, so $BE = BR = BC$. Proceed as above. ■

This second solution was suggested by Michael Kural.

G5

Let ω_1 and ω_2 be two orthogonal circles, and let the center of ω_1 be O. Diameter AB of ω_1 is selected so that B lies strictly inside ω_2. The two circles tangent to ω_2, passing through O and A, touch ω_2 at F and G. Prove that $FGOB$ is cyclic.

Eric Chen

Solution. Invert about ω_1. Then the problem becomes: "ω_1 and ω_2 are orthogonal circles. Show that if A is on ω_1 and outside of ω_2, and its tangents to ω_2 touch ω_2 at F, G, then its antipode B lies on FG."

Now let P be the center of ω_2, and let AP intersect FG at E. Then ω_1 is constant under an inversion with respect to ω_2, so E, the inverse of A, is on ω_1. Then $\angle AEB = \frac{\pi}{2}$, but $AE \perp FG$ so B is on FG and we are done. ■

This problem was proposed by Eric Chen. This solution was given by Michael Kural.

G6

Let $ABCDEF$ be a non-degenerate cyclic hexagon with no two opposite sides parallel, and define $X = AB \cap DE$, $Y = BC \cap EF$, and $Z = CD \cap FA$. Prove that

$$\frac{XY}{XZ} = \frac{BE}{AD}\frac{\sin|\angle B - \angle E|}{\sin|\angle A - \angle D|}.$$

Victor Wang

Solution. Use complex numbers with a, b, c, d, e, f on the unit circle, so $x = \frac{ab(d+e)-de(a+b)}{ab-de}$ and so on. It will be simpler to work with the conjugates of x, y, z, i.e. $\overline{x} = \frac{a+b-d-e}{ab-de}$, etc. Observing that

$$\begin{aligned}\overline{x} - \overline{y} &= \frac{a+b-d-e}{ab-de} - \frac{b+c-e-f}{bc-ef} \\ &= \frac{(a-d)(cb-fe)-(c-f)(ab-de)+(b-e)(bc-ef-ab+de)}{(ab-de)(bc-ef)} \\ &= \frac{(b-e)(fa-cd+(bc-ef-ab+de))}{(ab-de)(bc-ef)},\end{aligned}$$

we find (by "cyclically shifting" the variables by one so that $x - y \to z - x$) that

$$\frac{\overline{x-y}}{\overline{x-z}} = \frac{b-e}{a-d}\frac{af-cd}{bc-ef} = \frac{b-e}{a-d}\frac{a/c-d/f}{b/f-e/c},$$

from which the desired claim readily follows. ■

This problem and solution were proposed by Victor Wang.

G7*

Let ABC be a triangle inscribed in circle ω, and let the medians from B and C intersect ω at D and E respectively. Let O_1 be the center of the circle through D tangent to AC at C, and let O_2 be the center of the circle through E tangent to AB at B. Prove that O_1, O_2, and the nine-point center of ABC are collinear.

Michael Kural

Solution 1. Let M, N be the midpoints of AC, AB, respectively. Also, let BD, CE intersect (O_1) for a second time at X_1, Y_1, and let CE, BD intersect (O_2) for a second time at X_2, Y_2.

Now, by power of a point we have

$$MX_1 \cdot MD = MC^2 = MC \cdot MA = MD \cdot MB,$$

so $MX_1 = MB$, and X_1 is the reflection of B over M. Similarly, X_2 is the reflection of C over N.

(Alternatively, let X_1' be the reflection of B over M, and let D' be the intersection of the circles through X_1' tangent to AC at A, C respectively. Then by radical axes $X_1'D'$ bisects AC and $\angle ADC = 180 - \angle AX_1'C = 180 - \angle ABC$. This implies $D' = D$ and $X_1' = X_1$.)

Now let ZX_1X_2 be the antimedial triangle of ABC, and observe that $\angle X_2Y_1X_1 = \angle CDB = A = \angle CEB = \angle X_2Y_2X_1$. But $A = \angle X_2ZX_1$, so $X_1Y_1 \parallel EB$, $X_2Y_2 \parallel DC$, and $X_1X_2Y_2ZY_1$ is cyclic. Hence the lines through the centers of $(O_1), (ZX_1X_2)$, and $(ABC), (O_2)$ are parallel. In other words, $O_1H \parallel OO_2$ $O_1O \parallel HO_2$ (where O, H are the circumcenter and orthocenter of ABC), so O_1HO_2O is a parallelogram. Thus the midpoint of O_1O_2 is the midpoint N of OH. ■

This problem and solution were proposed by Michael Kural.

Remark. In fact, a -2 dilation about G sends B, D, C, E, O, A to X_1, Y_2, X_2, Y_1, H, Z.

Solution 2. Let (ABC) be the unit circle in the complex plane. Using the spiral similarities $D : CO_1 \to AO$ and $E : BO_2 \to AO$ (since AC is tangent to (O_1) and AB is tangent to (O_2)), it's easy to compute $o_1 = \frac{c(a+c-2b)}{c-b}$ and $o_2 = \frac{b(a+b-2c)}{b-c}$ (after solving for d, e via $\frac{bd(a+c)-ac(b+d)}{bd-ac} = m = \frac{a+c}{2}$), which gives us $o_1 + o_2 = a + b + c = 2n$. ■

This second solution was suggested by Victor Wang.

G8

Let ABC be a triangle, and let D, A, B, E be points on line AB, in that order, such that $AC = AD$ and $BE = BC$. Let ω_1, ω_2 be the circumcircles of $\triangle ABC$ and $\triangle CDE$, respectively, which meet at a point $F \neq C$. If the tangent to ω_2 at F cuts ω_1 again at G, and the foot of the altitude from G to FC is H, prove that $\angle AGH = \angle BGH$.

David Stoner

Solution 1. Let the centers of ω_1 and ω_2 be O_1 and O_2. Extend CA and CB to hit ω_2 again at K and L, respectively. Extend CO_2 to hit ω_2 again at R. Let M be the midpoint of arc $\widehat{AB}$, N the midpoint of arc $\widehat{FC}$ on ω_2, and T the intersection of FC and GM.

It's easy to see that $CK = CL = DE$, so O_2 is the C-excenter of triangle ABC. Hence C, M, and O_2 are collinear. Now $\angle CO_2O_1 = \angle CO_2N = 2\angle CRN = \angle CRF = \angle CFG = \angle CMG$, so MT is parallel to O_1O_2, and thus perpendicular to CF. But M is the midpoint of arc $\widehat{AB}$, so $\angle AGM = \angle MGB$, and we're done. ■

Solution 2. The observation that AO_2 is the perpendicular bisector of DC is not crucial; the key fact is just that $\angle GFC = \angle FEC$, since GF is tangent to ω_2. Indeed, this yields

$$\angle AGH = \angle AGF - \angle HGF = \angle ACF - 90^\circ + \angle GFC = \angle ACF - 90^\circ + \angle FEC.$$

But $\angle ACF = 180^\circ - \angle DCA - \angle FED$, $\alpha = \angle DCA$, and $\beta = \angle CEB = \angle FED - \angle FEC$, so $\angle AGH = 90^\circ - \alpha - \beta = \gamma$, where α, β, γ are half-angles. By symmetry, $\angle BGH = \gamma$ as well, so we're done. ■

This problem and its solutions were proposed by David Stoner.

G9

Let $ABCD$ be a cyclic quadrilateral inscribed in circle ω whose diagonals meet at F. Lines AB and CD meet at E. Segment EF intersects ω at X. Lines BX and CD meet at M, and lines CX and AB meet at N. Prove that MN and BC concur with the tangent to ω at X.

Allen Liu

Solution. Let EF meet BC at P, and let K be the harmonic conjugate of P with respect to BC. View EP as a cevian of $\triangle EBC$. Since the cevians AC, BD and EP concur, it follows that AD passes through K. Similarly, MN passes through K. However, by Brokard's theorem, EF is the pole of K with respect to ω, so KX is tangent to ω. Therefore, the three lines in question concur at K. ■

This problem and solution were proposed by Allen Liu.

G10*

Let $AB = AC$ in $\triangle ABC$, and let D be a point on segment AB. The tangent at D to the circumcircle ω of BCD hits AC at E. The other tangent from E to ω touches it at F, and $G = BF \cap CD$, $H = AG \cap BC$. Prove that $BH = 2HC$.

David Stoner

Solution 1. Let J be the second intersection of ω and AC, and X be the intersection of BF and AC. It's well-known that $DJFC$ is harmonic; perspectivity wrt B implies $AJXC$ is also harmonic. Then $\frac{AJ}{JX} = \frac{AC}{CX} \implies (AJ)(CX) = (AC)(JX)$. This can be rearranged to get

$$(AJ)(CX) = (AJ + JX + XC)(JX) \implies 2(AJ)(CX) = (JX + AJ)(JX + XC) = (AX)(CJ),$$

so

$$\left(\frac{AX}{XC}\right)\left(\frac{CJ}{JA}\right) = 2.$$

But $\frac{CJ}{JA} = \frac{AD}{DB}$, so by Ceva's we have $BH = 2HC$, as desired. ∎

Solution 2. Let J be the second intersection of ω and AC. It's well-known that $DJFC$ is harmonic; thus we have $(DJ)(FC) = (JF)(DC)$. By Ptolemy's, this means

$$(DF)(JC) = (DJ)(FC) + (JF)(DC) = 2(JD)(CF) \implies \left(\frac{JC}{JD}\right)\left(\frac{FD}{FC}\right) = 2.$$

Yet $JC = DB$ by symmetry, so this becomes

$$2 = \left(\frac{DB}{JD}\right)\left(\frac{FD}{FC}\right) = \left(\frac{\sin DJB}{\sin JBD}\right)\left(\frac{\sin FCD}{\sin FDC}\right) = \left(\frac{\sin DCB}{\sin ACD}\right)\left(\frac{\sin FBA}{\sin CBF}\right).$$

Thus by (trig) Ceva's we have $\frac{\sin BAH}{\sin CAH} = 2$, and since $AB = AC$ it follows that $BH = 2HC$, as desired. ∎

This problem and its solutions were proposed by David Stoner.

G11

Let $\triangle ABC$ be a nondegenerate isosceles triangle with $AB = AC$, and let D, E, F be the midpoints of BC, CA, AB respectively. BE intersects the circumcircle of $\triangle ABC$ again at G, and H is the midpoint of minor arc BC. $CF \cap DG = I, BI \cap AC = J$. Prove that $\angle BJH = \angle ADG$ if and only if $\angle BID = \angle GBC$.

David Stoner

Solution. By barycentric coordinates on $\triangle ABC$ it is easy to obtain $G = (a^2+c^2 : -b^2 : a^2+c^2)$. Then, one can compute $I = (a^2+c^2 : a^2+c^2 : b^2+2(a^2+c^2))$, from which it follows that $J = (a^2+c^2 : 0 : b^2+2(a^2+c^2))$.

Now we use complex numbers. Set $D = 0$, $C = 1$, $B = -1$, $A = ri$ for $r \in \mathbb{R}^+$, $K = \frac{r}{3}$, and $H = -\frac{i}{r}$. Now, upon using the vector definition for barycentric coordinates, we obtain $I = \frac{(r^2+5)(ri)+(r^2+5)(-1)+(3r^2+11)(1)}{5r^2+21}$, or

$$I = \frac{2r^2+6}{5r^2+21} + \frac{r(r^2+5)}{5r^2+21}i.$$

Similarly, we can get

$$J = \frac{3r^2+11}{4r^2+16} + \frac{r(r^2+5)}{4r^2+16}i.$$

Claim. $\angle BID = \angle GBC \iff r^6 + 9r^4 - 17r^2 - 153 = 0.$

Proof. Let $V(a+bi) = \frac{b}{a}$ for $a, b \in \mathbb{R}$, and note $V(nz) = V(z)$ for all $n \in \mathbb{R}$. Then,

$$\angle BID = \angle GBC \iff V\left(\frac{D-I}{B-I}\right) = V\left(\frac{G-B}{C-B}\right)$$

Obviously the right-hand side is $\frac{r}{3}$. Meanwhile,

$$\begin{aligned}
\frac{-I}{1-I} &= \frac{I}{I+1} \\
&= \frac{\frac{2r^2+6}{5r^2+21} + \frac{r(r^2+5)}{5r^2+21}i}{\frac{7r^2+27}{5r^2+21} + \frac{r(r^2+5)}{5r^2+21}i} \\
&= \frac{1}{\text{real}}\left[\left((2r^2+6) + r(r^2+5)i\right)\left((7r^2+26) - r(r^2+5)i\right)\right] \\
&= \frac{1}{\text{real}}\left[\left(r^6 + 24r^4 + 121r^2 + 162\right) + (5r^2+21)(r)(r^2+5)i\right]
\end{aligned}$$

Hence, $V\left(\frac{I}{I+1}\right) = \frac{(5r^2+21)(r)(r^2+5)}{r^6+24r^4+121r^2+161}$. This is equal to $r/3$ if and only if

$$r^6 + 24r^4 + 121r^2 + 162 - 3(5r^2+21)(r^2+5) = 0.$$

Expanding gives the conclusion. □ □

Claim. $\angle BJH = \angle ADG \iff 2r^8 + 8r^6 - 28r^r - 136r^2 - 102 = 0.$

Proof. We proceed in the same spirit. It's evident that $V\left(\frac{K-D}{G-D}\right) = V(I)^{-1} = \frac{2r^2+6}{r(r^2+5)}$. On the other hand, we can compute

$$
\begin{aligned}
\frac{-\frac{1}{r}\cdot i - J}{-1-J} &= \frac{rJ+i}{r(1+J)} \\
&= \frac{1}{r}\cdot\frac{\frac{r(3r^2+11)}{4r^2+16}+\frac{r^2(r^2+5)+(4r^2+16)}{4r^2+16}i}{\frac{7r^2+27}{4r^2+16}+\frac{r(r^2+5)}{4r^2+16}i} \\
&= \frac{1}{\text{real}}\left[\left(r(3r^2+11)+(r^4+9r^2+16\right)i\right]\left[(7r^2+27)-r(r^2+5)i\right] \\
&= \frac{1}{\text{real}}\left[r(r^6+35r^4+219r^2+377)+i(4r^6+64r^4+300r^2+432)\right]
\end{aligned}
$$

Hence, $V\left(\frac{H-J}{B-J}\right)=\frac{4r^6+64r^4+300r^2+432}{r(r^6+35r^4+219r^2+377)}$. So, the equality occurs when

$$(r^2+5)(4r^6+64r^4+300r^2+432)-(2r^2+6)(r^6+35r^4+219r^2+377)=0.$$

Expand again. □ □

Now all that's left to do is factor these polynomials! The former one is $(r^4-17)(r^2+9)$, and the latter is $2(r^2+1)(r^2+3)(r^4-17)$. Restricted to positive r we see that both are zero if and only if $r=\sqrt[4]{17}$. Therefore the conditions are equivalent, occuring if and only if $AD=\sqrt[4]{17}$. ■

This problem was proposed by David Stoner. This solution was given by Evan Chen.

G12*

Let ABC be a nondegenerate acute triangle with circumcircle ω and let its incircle γ touch AB, AC, BC at X, Y, Z respectively. Let XY hit arcs AB, AC of ω at M, N respectively, and let $P \neq X, Q \neq Y$ be the points on γ such that $MP = MX, NQ = NY$. If I is the center of γ, prove that P, I, Q are collinear if and only if $\angle BAC = 90^\circ$.

David Stoner

Solution. Let α be the half-angles of $\triangle ABC$, r inradius, and u, v, w tangent lengths to the incircle. Let $T = MP \cap NQ$ so that I is the incenter of $\triangle MNT$. Then $\angle IPT = \angle IXY = \alpha = \angle IYX = \angle IQT$ gives $\triangle TIP \sim \triangle TIQ$, so P, I, Q are collinear iff $\angle TIP = 90^\circ$ iff $\angle MTN = 180^\circ - 2\alpha$ iff $\angle MIN = 180^\circ - \alpha$ iff $MI^2 = MX \cdot MN$.

First suppose I is the center of γ. Since A, I are symmetric about XY, $\angle MAN = \angle MIN$. But P, I, Q are collinear iff $\angle MIN = 180^\circ - \alpha$, so because arcs AN and BM sum to 90°, P, I, Q are collinear iff arcs BM, MA have the same measure. Let $M' = CI \cap \omega$; then $\angle BM'I = \angle BM'C = 90^\circ - \angle BXI$, so $M'XIBZ$ is cyclic and $\angle M'XB = \angle M'IB = 180^\circ - \angle BIC = 45^\circ = \angle AXY$, as desired. (There are many other ways to finish as well.)

Conversely, if P, I, Q are collinear, then by power of a point, $m(m+2t) = MI^2 - r^2 = MX \cdot MN - r^2 = m(m+2t+n) - r^2$, so $mn = r^2$. But we also have $m(n+2t) = uv$ and $n(m+2t) = uw$, so

$$r^2 = mn = \frac{uv - r^2}{2t}\frac{uw - r^2}{2t} = \frac{\frac{uv(u+v)}{u+v+w}}{2r\cos\alpha}\frac{\frac{uw(u+w)}{u+v+w}}{2r\cos\alpha} = \frac{r^2}{4\cos^2\alpha}\frac{(u+v)(u+w)}{vw}.$$

Simplifying using $\cos^2\alpha = \frac{u^2}{u^2+r^2} = \frac{u(u+v+w)}{(u+v)(u+w)}$, we get

$$0 = (u+v)^2(u+w)^2 - 4uvw(u+v+w) = (u(u+v+w) - vw)^2,$$

which clearly implies $(u+v)^2 + (u+w)^2 = (v+w)^2$, as desired. ■

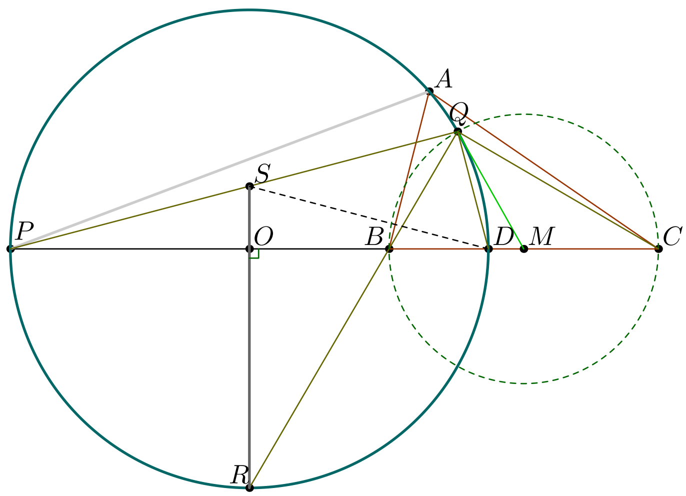
This problem was proposed by David Stoner. This solution was given by Victor Wang.

G13

In $\triangle ABC$, $AB < AC$. D and P are the feet of the internal and external angle bisectors of $\angle BAC$, respectively. M is the midpoint of segment BC, and ω is the circumcircle of $\triangle APD$. Suppose Q is on the minor arc AD of ω such that MQ is tangent to ω. QB meets ω again at R, and the line through R perpendicular to BC meets PQ at S. Prove SD is tangent to the circumcircle of $\triangle QDM$.

Ray Li

Solution.



We begin with a lemma.

Lemma 1. *Let $(A, B; C, D)$ be a harmonic bundle. Then the circles with diameter AB and CD are orthogonal.*

Proof. Let ω be the circle with diameter AB. Then D lies on the pole of C with respect to ω. Hence the inversion at ω sends C to D and vice-versa; so it fixes the circle with diameter CD, implying that the two circles are orthogonal. □ □

It's well known that $(P, D; B, C)$ is harmonic. Let O be the midpoint of PD. If we let Q' be the intersection of the circles with diameter PD and BC, then $\angle OQ'M = \frac{\pi}{2}$, implying that $Q' = Q$. It follows that Q lies on the circle with diameter BC; this is the key observation.

In that case, since $(P, D; B, C)$ is harmonic and $\angle PQD = \frac{\pi}{2}$, we see that QD is an angle bisector (this could also be realized via Apollonian circles). But $\angle BQC = \frac{\pi}{2}$ as well! So we find that $\angle PQB = \angle BQD = \angle DQC = \frac{\pi}{4}$. Then, R is the midpoint of arc PD, so $SP = SD$, insomuch as $SO \perp PD$.

Hence, we can just angle chase as $\angle DQM = \angle SPD = \angle SDP$, implying the conclusion. ■

This problem and solution were proposed by Ray Li.

G14

Let O be a point (in the plane) and T be an infinite set of points such that $|P_1P_2| \le 2012$ for every two distinct points $P_1, P_2 \in T$. Let $S(T)$ be the set of points Q in the plane satisfying $|QP| \le 2013$ for at least one point $P \in T$.

Now let L be the set of lines containing exactly one point of $S(T)$. Call a line ℓ_0 passing through O *bad* if there does not exist a line $\ell \in L$ parallel to (or coinciding with) ℓ_0.

(a) Prove that L is nonempty.

(b) Prove that one can assign a line $\ell(i)$ to each positive integer i so that for every bad line ℓ_0 passing through O, there exists a positive integer n with $\ell(n) = \ell_0$.

David Yang

Solution 1. (a) Instead of unique lines we work with *good directions* (e.g. northernmost points for the direction "north"). Since S is closed and bounded there is a diameter, say AB. Then B is the unique farthest point in the direction of the vector $\overrightarrow{AB}$ (if there was another point C that was the same or farther in that direction then AC would be longer than AB). ■

Solution 2. (b) We can work instead with the convex hull of S, since this does not change if directions are good. Note that bad directions correspond to lines segments that are boundaries of portions of the convex hull (i.e. "sides" of the convex hull). For each direction, consider the corresponding side. Now, consider the area 1 unit in front of the side. For distinct directions, these areas don't intersect, so there must be a countable number of them (more precisely, there are a finite number of squares with area in the interval $(\frac{1}{n+1}, \frac{1}{n}]$ for every positive integer n, and thus we can enumerate the bad directions.) ■

This problem and the above solutions were proposed by David Yang.

Solution 3. (b) Alternatively, take an interior point and look at the angle swept out by each side (positive numbers with finite sum). ■

This third solution was suggested by Mark Sellke.

Remark. We only need S to be a compact (closed and bounded) set in $\mathbb{R}^n$ for (a), and a compact set in $\mathbb{R}^2$ for (b). The current elementary formulation, however, preserves the essence of the problem. Note that the same proof works for (a), while a hyper-cylinder serves as a counterexample for (b) in $\mathbb{R}^n$ (more specifically, the set of points satisfying, say, $x_1^2 + x_2^2 \le 1$ and $0 \le x_3, \dots, x_n \le 1$). Indeed, for each angle $\theta \in [0, 2\pi)$, the hyper-plane with equation $\sin\theta x_1 - \cos\theta x_2 = 0$ is tangent to the cylinder at the set of points of the form $(\cos\theta, \sin\theta, x_3, \dots, x_n)$, yet $[0, 2\pi)$ (which bijects to the real numbers) is uncountable. More precisely, the set of points farthest $\langle\cos\theta, \sin\theta, 0, \dots, 0\rangle$ direction is simply the set of points that maximize $\langle\cos\theta, \sin\theta, 0, \dots, 0\rangle \cdot \langle x_1, x_2, 0, \dots, 0\rangle$ (which is at most 1, by the Cauchy-Schwarz inequality), which is just the set of points of the form $(\cos\theta, \sin\theta, x_3, \dots, x_n)$.

N1

Find all ordered triples of non-negative integers (a, b, c) such that $a^2 + 2b + c$, $b^2 + 2c + a$, and $c^2 + 2a + b$ are all perfect squares.

Note: This problem was withdrawn from the ELMO Shortlist and used on ksun48's mock AIME.

Matthew Babbitt

Answer. We have the trivial solutions $(a, b, c) = (0, 0, 0)$ and $(a, b, c) = (1, 1, 1)$, as well as the solution $(a, b, c) = (127, 106, 43)$ and its cyclic permutations.

Solution. The case $a = b = c = 0$ works. Without loss of generality, $a = \max\{a, b, c\}$. If b and c are both zero, it's obvious that we have no solution. So, via the inequality

$$a^2 < a^2 + 2b + c < (a + 2)^2$$

we find that $a^2 + 2b + c = (a + 1)^2 \implies 2a + 1 = 2b + c$. So,

$$a = b + \frac{c - 1}{2}.$$

Let $c = 2k + 1$ with $k \geq 0$; plugging into the given, we find that

$$b^2 + b + 2 + 5k \quad \text{and} \quad 4k^2 + 6k + 3b + 1$$

are both perfect squares. Multiplying both these quantities by 4, and setting $x = 2b + 1$ and $y = 4k + 3$, we find that

$$x^2 + 5y - 8 \quad \text{and} \quad y^2 + 6x - 11$$

are both even squares.

We may assume $x, y \geq 3$. We now have two cases, both of which aren't too bad:

- If $x \geq y$, then $x^2 < x^2 + 5y - 8 < (x + 3)^2$. Since the square is even, $x^2 + 5y - 8 = (x + 1)^2$. Then, $x = \frac{5y-9}{2}$ and we find that $y^2 + 15y - 38$ is an even square. Since $y^2 < y^2 + 15y - 38 < (y + 8)^2$, there are finitely many cases to check. The solutions are $(x, y) = (3, 3)$ and $(x, y) = (213, 87)$.
- Similarly, if $x \leq y$, then $y^2 < y^2 + 6x - 11 < (y + 3)^2$, so $y^2 + 6x - 11 = (y + 1)^2$. Then, $y = 3x - 6$ and we find that $x^2 + 15x - 38$ (!) is a perfect square. Amusingly, this is the exact same thing (whether this is just a coincidence due to me selecting the equality case to be $x = y$, I'm not sure). Here, the solutions are $(x, y) = (3, 3)$ and $(x, y) = (87, 255)$.

Converting back, we see the solutions are $(0, 0, 0)$, $(1, 1, 1)$ and $(127, 106, 43)$, and permutations. ■

This problem and solution were proposed by Matthew Babbitt.

N2*

For what polynomials $P(n)$ with integer coefficients can a positive integer be assigned to every lattice point in $\mathbb{R}^3$ so that for every integer $n \geq 1$, the sum of the n^3 integers assigned to any $n \times n \times n$ grid of lattice points is divisible by $P(n)$?

Andre Arslan

Answer. All P of the form $P(x) = cx^k$, where c is a nonzero integer and k is a nonnegative integer.

Solution. Suppose $P(x) = x^k Q(x)$ with $Q(0) \neq 0$ and Q is nonconstant; then there exist infinitely many primes p dividing some $Q(n)$; fix one of them not dividing $Q(0)$, and take a sequence of pairwise coprime integers $m_1, n_1, m_2, n_2, \ldots$ with $p \mid Q(m_i), Q(n_i)$ (we can do this with CRT).

Let $f(x, y, z)$ be the number written at (x, y, z). Note that $P(m)$ divides every $mn \times mn \times m$ grid and $P(n)$ divides every $mn \times mn \times n$ grid, so by Bezout's identity, $(P(m), P(n))$ divides every $mn \times mn \times (m, n)$ grid. It follows that p divides every $m_i n_i \times m_i n_i \times 1$ grid. Similarly, we find that p divides every $m_i n_i m_j n_j \times 1 \times 1$ grid whenever $i \neq j$, and finally, every $1 \times 1 \times 1$ grid. Since p was arbitrarily chosen from an infinite set, f must be identically zero, contradiction.

For the other direction, take a solution g to the one-dimensional case using repeated CRT (the key relation $\gcd(P(m), P(n)) = P(\gcd(m, n))$ prevents "conflicts"): start with a positive multiple of $P(1) \neq 0$ at zero, and then construct $g(1), g(-1), g(2), g(-2)$, etc. in that order using CRT. Now for the three-dimensional version, we can just let $f(x, y, z) = g(x)$. ■

This problem and solution were proposed by Andre Arslan.

Remark. The crux of the problem lies in the 1D case. (We use the same type of reasoning to "project" from d dimension to $d-1$ dimensions.) Note that the condition $P(n) \mid g(i) + \cdots + g(i+n-1)$ (for the 1D case) is "almost" the same as $P(n) \mid g(i) - g(i+n)$, so we immediately find $\gcd(P(m), P(n)) \mid g(i) - g(i + \gcd(m, n))$ by Bezout's identity. In particular, when m, n are coprime, we will intuitively be able to get $\gcd(P(m), P(n))$ as large as we want unless P is of the form cx^k (we formalize this by writing $P = x^k Q$ with $Q(0) \neq 0$).

Conversely, if $P = cx^k$, then $\gcd(P(m), P(n)) = P(\gcd(m, n))$ renders our derived restriction $\gcd(P(m), P(n)) \mid g(i) - g(i + \gcd(m, n))$ superfluous. So it "feels easy" to find *nonconstant* g with $P(n) \mid g(i) - g(i+n)$ for all i, n, just by greedily constructing $g(0), g(1), g(-1), \ldots$ in that order using CRT. Fortunately, $g(i) + \cdots + g(i+m-1) - g(i) - \cdots - g(i+n-1) = g(i+n) + \cdots + g(i+n+(m-n)-1)$ for $m > n$, so the inductive approach still works for the stronger condition $P(n) \mid g(i) + \cdots + g(i+n-1)$.

Remark. Note that polynomial constructions cannot work for $P = cx^{d+1}$ in d dimensions. Suppose otherwise, and take a minimal degree $f(x_1, \ldots, x_d)$; then f isn't constant, so $f'(x_1, \ldots, x_d) = f(x_1 + 1, \ldots, x_d + 1) - f(x_1, \ldots, x_d)$ is a working polynomial of strictly smaller degree.

N3

Prove that each integer greater than 2 can be expressed as the sum of pairwise distinct numbers of the form a^b, where $a \in \{3,4,5,6\}$ and b is a positive integer.

Matthew Babbitt

Solution. First, we prove a lemma.

Lemma 1. *Let $a_0 > a_1 > a_2 > \cdots > a_n$ be positive integers such that $a_0 - a_n < a_1 + a_2 + \cdots + a_n$. Then for some $1 \le i \le n$, we have*

$$0 \le a_0 - (a_1 + a_2 + \cdots + a_i) < a_i.$$

Proof. Proceed by contradiction; suppose the inequalities are all false. Use induction to show that $a_0 - (a_1 + \cdots + a_i) \ge a_i$ for each i. This becomes a contradiction at $i = n$. □ □

Let N be the integer we want to express in this form. We will prove the result by strong induction on N. The base cases will be $3 \le N \le 10 = 6 + 3 + 1$.

Let $x_1 > x_2 > x_3 > x_4$ be the largest powers of $3, 4, 5, 6$ less than $N - 3$, in some order. If one of the inequalities of the form

$$3 \le N - (x_1 + \cdots + x_k) < x_k + 3; \quad 1 \le k \le 4$$

is true, then we are done, since we can subtract of $x_1, \ldots, x_k$ from N to get an N' with $3 \le N' < N$ and then apply the inductive hypothesis; the construction for N' cannot use any of $\{x_1, \ldots, x_k\}$ since $N' - x_k < 3$.

To see that this is indeed the case, first observe that $N - 3 > x_1$ by construction and compute

$$x_1 + x_2 + x_3 + x_4 + x_4 \ge (N-3) \cdot \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{6} \right) > N - 3.$$

So the hypothesis of the lemma applies with $a_0 = N - 3$ and $a_i = x_i$ for $1 \le i \le 4$.

Thus, we are done by induction. ■

This problem and solution were proposed by Matthew Babbitt.

Remark. While the approach of subtracting off large numbers and inducting is extremely natural, it is not immediately obvious that one should consider $3 \le N - (x_1 + \cdots + x_k) < x_k + 3$ rather than the stronger bound $3 \le N - (x_1 + \cdots + x_k) < x_k$. In particular, the solution method above does not work if one attempts to get the latter.

N4

Find all triples (a, b, c) of positive integers such that if n is not divisible by any integer less than 2013, then $n + c$ divides $a^n + b^n + n$.

Evan Chen

Answer. $(a, b, c) = (1, 1, 2)$.

Solution. Let p be an arbitrary prime such that $p \geq 2011 \cdot \max\{abc, 2013\}$. By the Chinese Remainder Theorem it is possible to select an integer n satisfying the following properties:

$$\begin{aligned} n &\equiv -c \pmod{p} \\ n &\equiv -1 \pmod{p-1} \\ n &\equiv -1 \pmod{q} \end{aligned}$$

for all primes $q \leq 2011$ not dividing $p - 1$. This will guarantee that n is not divisible by any integer less than 2013. Upon selecting this n, we find that

$$p \mid n + c \mid a^n + b^n + n$$

which implies that

$$a^n + b^n \equiv c \pmod{p}$$

But $n \equiv -1 \pmod{p-1}$; hence $a^n \equiv a^{-1} \pmod{p}$ by Euler's Little Theorem. Hence we may write

$$p \mid ab(a^{-1} + b^{-1} - c) = a + b - abc.$$

But since p is large, this is only possible if $a + b - abc$ is zero. The only triples of positive integers with that property are $(a, b, c) = (2, 2, 1)$ and $(a, b, c) = (1, 1, 2)$. One can check that of these, only $(a, b, c) = (1, 1, 2)$ is a valid solution. ■

This problem and solution were proposed by Evan Chen.

N5*

Let $m_1, m_2, \dots, m_{2013} > 1$ be 2013 pairwise relatively prime positive integers and $A_1, A_2, \dots, A_{2013}$ be 2013 (possibly empty) sets with $A_i \subseteq \{1, 2, \dots, m_i - 1\}$ for $i = 1, 2, \dots, 2013$. Prove that there is a positive integer N such that

$$N \le (2|A_1| + 1)(2|A_2| + 1) \cdots (2|A_{2013}| + 1)$$

and for each $i = 1, 2, \dots, 2013$, there does *not* exist $a \in A_i$ such that m_i divides $N - a$.

Victor Wang

Remark. As Solution 3 shows, the bound can in fact be tightened to $\prod_{i=1}^{2013}(|A_i| + 1)$.

Solution 1. We will show that the smallest integer N such that $N \notin A_i \pmod{m_i}$ is less than the bound provided.

The idea is to use pigeonhole and the "Lagrange interpolation"-esque representation of CRT systems. Define integers t_i satisfying $t_i \equiv 1 \pmod{m_i}$ and $t_i \equiv 0 \pmod{m_j}$ for $j \ne i$. If we find nonempty sets B_i of distinct residues mod m_i with $B_i - B_i \pmod{m_i}$ and $A_i \pmod{m_i}$ disjoint, then by pigeonhole, a positive integer solution with $N \le \frac{m_1 m_2 \cdots m_{2013}}{|B_1| \cdot |B_2| \cdots |B_{2013}|}$ must exist (more precisely, since

$$b_1 t_1 + \dots + b_{2013} t_{2013} \pmod{m_1 m_2 \cdots m_{2013}}$$

is injective over $B_1 \times B_2 \times \dots \times B_{2013}$, some two consecutively ordered solutions must differ by at most $\frac{m_1 m_2 \cdots m_{2013}}{|B_1| \cdot |B_2| \cdots |B_{2013}|}$).

On the other hand, since $0 \notin A_i$ for every i, we know such nonempty B_i must exist (e.g. take $B_i = \{0\}$). Now suppose $|B_i|$ is maximal; then every $x \pmod{m_i}$ lies in at least one of B_i, $B_i + A_i$, $B_i - A_i$ (note that $x - x = 0$ is not an issue when considering $(B_i \cup \{x\}) - (B_i \cup \{x\})$), or else $B_i \cup \{x\}$ would be a larger working set. Hence $m_i \le |B_i| + |B_i + A_i| + |B_i - A_i| \le |B_i|(1 + 2|A_i|)$, so we get an upper bound of $\prod_{i=1}^{2013} \frac{m_i}{|B_i|} \le \prod_{i=1}^{2013}(2|A_i| + 1)$, as desired. ■

Remark. We can often find $|B_i|$ significantly larger than $\frac{m_i}{2|A_i|+1}$ (the bounds $|B_i + A_i|, |B_i - A_i| \le |B_i| \cdot |A_i|$ seem really weak, and $B_i + A_i, B_i - A_i$ might not be that disjoint either). For instance, if $A_i \equiv -A_i \pmod{m_i}$, then we can get (the ceiling of) $\frac{m_i}{|A_i|+1}$.

Remark. By translation and repeated application of the problem, one can prove the following slightly more general statement: "Let $m_1, m_2, \dots, m_{2013} > 1$ be 2013 pairwise relatively prime positive integers and $A_1, A_2, \dots, A_{2013}$ be 2013 (possibly empty) sets with A_i a proper subset of $\{1, 2, \dots, m_i\}$ for $i = 1, 2, \dots, 2013$. Then for every integer n, there exists an integer x in the range $(n, n + \prod_{i=1}^{2013}(2|A_i| + 1)]$ such that $x \notin A_i \pmod{m_i}$ for $i = 1, 2, \dots, 2013$. (We say A is a *proper subset* of B if A is a subset of B but $A \ne B$.)"

Remark. Let f be a non-constant integer-valued polynomial with $\gcd(\dots, f(-1), f(0), f(1), \dots) = 1$. Then by the previous remark, we can easily prove that there exist infinitely many positive integers n such that the smallest prime divisor of $f(n)$ is at least $c \log n$, where $c > 0$ is any constant. (We take m_i the ith prime and $A_i \equiv \{n : m_i \mid f(n)\} \pmod{m_i}$—if $f = \frac{a}{b}x^d + \cdots$, then $|A_i| \le d$ for all sufficiently large i.)

Solution 2. We will mimic the proof of 2010 RMM Problem 1.

Suppose $1, 2, \dots, N$ (for some $N \ge 1$) can be covered by the sets $A_i \pmod{m_i}$.

Observe that for fixed m and $1 \le a \le m$, exactly $1 + \lfloor \frac{N-a}{m} \rfloor$ of $1, 2, \dots, N$ are $a \pmod m$. In particular, we have lower and upper bounds of $\frac{N-m}{m}$ and $\frac{N+m}{m}$, respectively, so PIE yields

$$N \le \sum_i |A_i| \frac{N + m_i}{m_i} - \sum_{i<j} |A_i| \cdot |A_j| \frac{N - m_i m_j}{m_i m_j} \pm \cdots .$$

It follows that

$$N \prod_i \left(1 - \frac{|A_i|}{m_i}\right) \le \prod_i (1 + |A_i|),$$

so $N \le \prod_i \frac{m_i}{m_i-|A_i|}(1+|A_i|)$.

Note that $\frac{m_i}{m_i-|A_i|} \le \frac{2|A_i|+1}{|A_i|+1}$ iff $m_i \ge 2|A_i|+1$, so we're done unless $m_i \le 2|A_i|$ for some i.

In this case, there exists (by induction) $1 \le N \le \prod_{j\ne i}(2|A_j|+1)$ such that $N \notin m_i^{-1}A_j \pmod{m_j}$ for all $j \ne i$. Thus $m_iN \notin A_j \pmod{m_j}$ and we trivially have $m_iN \equiv 0 \notin A_i \pmod{m_i}$, so $m_iN \le \prod_k(2|A_k|+1)$, as desired. ■

This problem and the above solutions were proposed by Victor Wang.

Solution 3. We can in fact get a bound of $\prod(|A_k|+1)$ directly.

Let $t = 2013$. Suppose $1, 2, \dots, N$ are covered by the $A_k \pmod{m_k}$; then

$$z_n = \prod_{1\le k\le t, a\in A_k} \left(1 - e^{\frac{2\pi i}{m_k}(n-a)}\right)$$

is a linear recurrence in $e^{2\pi i \sum_{k=1}^{t} \frac{j_k}{m_k}}$ (where each j_k ranges from 0 to $|A_k|$). But $z_0 \ne 0 = z_1 = \cdots = z_N$, so N must be strictly less than the degree $\prod(|A_k|+1)$ of the linear recurrence. Thus $1, 2, \dots, \prod(|A_k|+1)$ cannot all be covered, as desired. ■

This third solution was suggested by Zhi-Wei Sun.

Remark. Solution 3 doesn't require the m_k to be coprime. Note that if $|A_1| = \cdots = |A_t| = b-1$, then a base b construction shows the bound of $\prod(b-1+1) = b^t$ is "tight" (if we remove the restriction that the m_k must be coprime).

However, Solutions 2 and 3 "ignore" the additive structure of CRT solution sets encapsulated in Solution 1's Lagrange interpolation representation.

N6*

Find all positive integers m for which there exists a function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that

$$f^{f^{f(n)}(n)}(n) = n$$

for every positive integer n, and $f^{2013}(m) \neq m$. Here $f^k(n)$ denotes $\underbrace{f(f(\cdots f}_{k\ f\text{'s}}(n)\cdots))$.

Evan Chen

Answer. All m not dividing 2013; that is, $\mathbb{Z}^+ \setminus \{1, 3, 11, 33, 61, 183, 671, 2013\}$.

Solution. First, it is easy to see that f is both surjective and injective, so f is a permutation of the positive integers. We claim that the functions f which satisfy the property are precisely those functions which satisfy $f^n(n) = n$ for every n.

For each integer n, let $\operatorname{ord}(n)$ denote the smallest integer k such that $f^k(n)$. These orders exist since $f^{f^{f(n)}(n)}(n) = n$, so $\operatorname{ord}(n) \le f^{f(n)}(n)$; in fact we actually have

$$\operatorname{ord}(n) \mid f^{f(n)}(n) \tag{8.1}$$

as a consequence of the division algorithm.

Since f is a permutation, it is immediate that $\operatorname{ord}(n) = \operatorname{ord}(f(n))$ for every n; this implies easily that $\operatorname{ord}(n) = \operatorname{ord}\left(f^k(n)\right)$ for every integer k. In particular, $\operatorname{ord}(n) = \operatorname{ord}\left(f^{f(n)-1}(n)\right)$. But then, applying (8.1) to $f^{f(n)-1}(n)$ gives

$$\begin{aligned}
\operatorname{ord}(n) = \operatorname{ord}\left(f^{f(n)-1}(n)\right) &\mid f^{f\left(f^{f(n)-1}(n)\right)}\left(f^{f(n)-1}(n)\right) \\
&= f^{f^{f(n)}(n)+f(n)-1}(n) \\
&= f^{f(n)-1}\left(f^{f^{f(n)}(n)}(n)\right) \\
&= f^{f(n)-1}(n)
\end{aligned}$$

Inductively, then, we are able to show that $\operatorname{ord}(n) \mid f^{f(n)-k}(n)$ for every integer k; in particular, $\operatorname{ord}(n) \mid n$, so $f^n(n) = n$. To see that this is actually sufficient, simply note that $\operatorname{ord}(n) = \operatorname{ord}(f(n)) = \cdots$, which implies that $\operatorname{ord}(n) \mid f^k(n)$ for every k.

In particular, if $m \mid 2013$, then $\operatorname{ord}(m) \mid m \mid 2013$ and $f^{2013}(m) = m$. The construction for the other values of m is left as an easy exercise. ■

This problem and solution were proposed by Evan Chen.

Remark. There are many ways to express the same ideas.

For instance, the following approach ("unraveling indices") also works: It's not hard to show that f is a bijection with finite cycles (when viewed as a permutation). If $C = (n_0, n_1, \dots, n_{\ell-1})$ is one such cycle with $f(n_i) = n_{i+1}$ for all i (extending indices mod ℓ), then $f^{f^{f(n)}(n)}(n) = n$ holds on C iff $\ell \mid f^{f(n_i)}(n_i) = n_{i+n_{i+1}}$ for all i. But $\ell \mid n_j \implies \ell \mid n_{j-1+n_j} = n_{j-1}$ for fixed j, so the latter condition holds iff $\ell \mid n_i$ for all i. Thus $f^{2013}(n) = n$ is forced unlesss $n \nmid 2013$.

N7*

Let p be a prime satisfying $p^2 \mid 2^{p-1}-1$, n be a positive integer, and $f(x) = \frac{(x-1)^{p^n}-(x^{p^n}-1)}{p(x-1)}$. Find the largest positive integer N such that there exist polynomials $g, h \in \mathbb{Z}[x]$ and an integer r satisfying $f(x) = (x-r)^N g(x) + p \cdot h(x)$.

Victor Wang

Answer. The largest possible N is $2p^{n-1}$.

Solution 1. Let $F(x) = \frac{x}{1} + \cdots + \frac{x^{p-1}}{p-1}$.

By standard methods we can show that $(x-1)^{p^n} - (x^{p^{n-1}}-1)^p$ has all coefficients divisible by p^2. But $p^2 \mid 2^{p-1}-1$ means p is odd, so working in $\mathbb{F}_p$, we have

$$(x-1)f(x) = \sum_{k=1}^{p-1} \frac{1}{p}\binom{p}{k}(-1)^{k-1}x^{p^{n-1}k} = \sum_{k=1}^{p-1}\binom{p-1}{k-1}(-1)^{k-1}\frac{x^{p^{n-1}k}}{k}$$
$$= \sum_{k=1}^{p-1} \frac{x^{p^{n-1}k}}{k^{p^{n-1}}} = F(x)^{p^{n-1}},$$

where we use Fermat's little theorem, $\binom{p-1}{k-1} \equiv (-1)^{k-1} \pmod p$ for $k = 1, 2, \ldots, p-1$, and the well-known fact that $P(x^p) - P(x)^p$ has all coefficients divisible by p for any polynomial P with integer coefficients.

However, it is easy to verify that $p^2 \mid 2^{p-1}-1$ if and only if $p \mid F(-1)$, i.e. -1 is a root of F in $\mathbb{F}_p$. Furthermore, $F'(x) = \frac{x^{p-1}-1}{x-1} = (x+1)(x+2)\cdots(x+p-2)$ in $\mathbb{F}_p$, so -1 is a root of F with multiplicity 2; hence $N \geq 2p^{n-1}$. On the other hand, since F' has no double roots, F has no integer roots with multiplicity greater than 2. In particular, $N \leq 2p^{n-1}$ (note that the multiplicity of 1 is in fact $p^{n-1}-1$, since $F(1) = 0$ by Wolstenholme's theorem but 1 is not a root of F'). ■

This problem and solution were proposed by Victor Wang.

Remark. The rth derivative of a polynomial P evaluated at 1 is simply the coefficient $[(x-1)^r]P$ (i.e. the coefficient of $(x-1)^r$ when P is written as a polynomial in $x-1$) divided by $r!$.

Solution 2. This is asking to find the greatest multiplicity of an integer root of f modulo p; I claim the answer is $2p^{n-1}$.

First, we shift x by 1 and take the negative (since this doesn't change the greatest multiplicity) for convenience, redefining f as $f(x) = \frac{(x+1)^{p^n} - x^{p^n} - 1}{px}$.

Now, we expand this. We can show, by writing out and cancelling, that p^1 fully divides $\binom{p^n}{k}$ only when p^{n-1} divides k; thus, we can ignore all terms except the ones with degree divisible by p^{n-1} (since they still go away when taking it mod p), leaving $f(x) = \frac{1}{px}(\binom{p^n}{p^{n-1}}x^{p^n-p^{n-1}} + \cdots + \binom{p^n}{p^n-p^{n-1}}x^{p^{n-1}})$.

We can also show, by writing out/cancelling, that $\frac{1}{p}\binom{p^n}{kp^{n-1}} = \frac{1}{p}\binom{p}{k}$ modulo p. Simplifying using this, the expression above becomes $f(x) = \frac{1}{px}(\binom{p}{1}x^{p^n-p^{n-1}} + \cdots + \binom{p}{p-1}x^{p^{n-1}}) = \frac{1}{px}((x^{p^{n-1}}+1)^p - (x^{p^n}+1))$.

Now, we ignore the $1/x$ for the moment (all it does is reduce the multiplicity of the root at $x=0$ by 1) and just look at the rest, $P(x) = \frac{1}{p}((x^{p^{n-1}}+1)^p - (x^{p^n}+1))$.

Substituting $y = x^{p^{n-1}}$, this becomes $\frac{1}{p}((y+1)^p - (y^p+1))$; since $\frac{1}{p}\binom{p}{k} = \frac{1}{k}\binom{p-1}{k-1}$, this is equal to $P(x) = \frac{1}{1}\binom{p-1}{0}y^{p-1} + \cdots + \frac{1}{p-1}\binom{p-1}{p-2}y$. (We work mod p now; the ps can be cancelled before modding out.)

We now show that $P(x)$ has no integer roots of multiplicity greater than 2, by considering the root multiplicities of y times its reversal, or $Q(x) = \frac{1}{p-1}\binom{p-1}{p-2}y^{p-1} + \cdots + \frac{1}{1}\binom{p-1}{0}y$.

Note that some polynomial P has a root of multiplicity m at x iff P and its first $m-1$ derivatives all have zeroes at x. (We're using the formal derivatives here - we can prove this algebraically over $\mathbb{Z}$ mod p, if

$m < p$.) The derivative of Q is $\binom{p-1}{p-2}y^{p-2} + \cdots + \binom{p-1}{0}$, or $(y+1)^{p-1} - y^{p-1}$, which has as a root every residue except 0 and -1 by Fermat's little theorem; the second derivative is a constant multiple of $(y+1)^{p-2} - y^{p-2}$, which has no integer roots by Fermat's little theorem and unique inverses. Therefore, no integer root of Q has multiplicity greater than 2; we know that the factorization of a polynomial's reverse is just the reverse of its factorization, and integers have inverses mod p, so $P(x)$ doesn't have integer roots of multiplicity greater than 2 either.

Factoring $P(x)$ completely in y (over some extension of $\mathbb{F}_p$), we know that two distinct factors can't share a root; thus, at most 2 factors have any given integer root, and since their degrees (in x) are each p^{n-1}, this means no integer root has multiplicity greater than $2p^{n-1}$.

However, we see that $y = 1$ is a double root of P. This is because plugging in gives $P(1) = \frac{1}{p}((1+1)^p - (1^p + 1)) = \frac{1}{p}(2^p - 2)$; by the condition, p^2 divides $2^p - 2$, so this is zero mod p. Since 1 is its own inverse, it's a root of Q as well, and it's a root of Q's derivative so it's a double root (so $(y-1)^2$ is part of Q's factorization). Reversing, $(y-1)^2$ is part of P's factorization as well.

Applying a well-known fact, $y - 1 = x^{p^{n-1}} - 1 = (x-1)^{p^{n-1}}$ modulo p, so 1 is a root of P with multiplicity $2p^{n-1}$.

Since adding back in the factor of $1/x$ doesn't change this multiplicity, our answer is therefore $2p^{n-1}$. ■

This second solution was suggested by Alex Smith.

N8

We define the *Fibonacci sequence* $\{F_n\}_{n\geq 0}$ by $F_0 = 0$, $F_1 = 1$, and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$; we define the *Stirling number of the second kind* $S(n,k)$ as the number of ways to partition a set of $n \geq 1$ distinguishable elements into $k \geq 1$ indistinguishable nonempty subsets.

For every positive integer n, let $t_n = \sum_{k=1}^n S(n,k)F_k$. Let $p \geq 7$ be a prime. Prove that

$$t_{n+p^{2p}-1} \equiv t_n \pmod p$$

for all $n \geq 1$.

Victor Wang

Solution. Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. By convention we extend the definition to all $n,k \geq 0$ so that $S(0,0) = 1$ and for $m > 0$, $S(m,0) = S(0,m) = 0$. It will also be convenient to define the *falling factorial* $(x)_n = x(x-1)\cdots(x-n+1)$, where we take $(x)_0 = 1$. Then we can extend our sequence to t_0 by defining $t_n = \sum_{k=0}^n S(n,k)F_k$ instead (the $k=0$ term vanishes for positive n).

A simple combinatorial interpretation yields the polynomial identity $\sum_{k=0}^n S(n,k)(x)_k = x^n$ (it is enough to establish the result just for positive integer x). Inspired by the methods of umbral calculus (we try to "exchange" $(x)_k, x^n$ with F_k, t_n), we consider the linear map $T : \mathbb{Z}[x] \to \mathbb{Z}$ satisfying $T((x)_k) = F_k = \frac{\alpha^k - \beta^k}{\alpha - \beta}$. Because the $(x)_k$ (for $k \geq 0$) form a basis of $\mathbb{Z}[x]$ (the standard one is $\{x^k\}_{k\geq 0}$), this uniquely determines such a map, and $t_n = T(x^n)$. Hence if $\ell = p^{2p} - 1$, we need to show that $p \mid T(x^n(1 - x^\ell))$ for all $n \geq 0$, or equivalently, that $p \mid T((x^\ell - 1)f(x))$ for all $f \in \mathbb{Z}[x]$.

Throughout this solution we will work in $\mathbb{F}_p$ and use the fact that $P(x^p) - P(x)^p$ has all coefficients divisible by p for any $P \in \mathbb{Z}[x]$. It is well-known (e.g. by Binet's formula) that $p \mid F_{n+p^2-1} - F_n$ for all $n \geq 0$ since $p \neq 2, 5$. But by a simple induction on $n \geq 0$ we find that $T((x)_n f(x)) = F_{n-1}T(f(x+n)) + F_n T(xf(x+n-1))$ for all $f \in \mathbb{Z}[x]$, so taking $n = p(p^2-1)$ yields $T((x^p - x)^{p^2-1}f(x)) = F_{-1}T(f(x)) + F_0 T(xf(x-1)) = T(f(x))$, where we use the fact that $x(x-1)\cdots(x-p+1) = x^p - x$, $F_{-1} = F_1 - F_0 = 1$, and $F_0 = 0$.

Since $T([(x^p - x)^{p^2-1} - 1]f(x)) = 0$, it suffices to show that $(x^p - x)^{p^2-1} - 1 \mid x^{p^{2p}-1} - 1$ (still in $\mathbb{F}_p$, of course). It will be convenient to work modulo $(x^p - x)^{p^2-1} - 1$. First note that

$$\begin{aligned}(x^p - x)^{p^2-1} - 1 &\mid (x^p - x)^{p^2} - (x^p - x) = x^{p^3} - x^{p^2} - x^p + x \\ &\mid (x^{p^3} - x^{p^2} - x^p + x)^p + (x^{p^3} - x^{p^2} - x^p + x) = x^{p^4} - 2x^{p^2} + x,\end{aligned}$$

so it's enough to prove that $x^{p^4} - 2x^{p^2} + x \mid x^{p^{2p}} - x$ (since $\gcd(x, (x^p - x)^{p^2-1} - 1) = 1$). But $(x^{p^4} - 2x^{p^2} + x)^{p^2} - (x^{p^4} - 2x^{p^2} + x) = x^{p^6} - 3x^{p^4} + 3x^{p^2} - x$; by a simple induction, we have $x^{p^4} - 2x^{p^2} + x \mid \sum_{k=0}^m (-1)^k \binom{m}{k} x^{p^{2m-2k}}$ for $m \geq 2$; for $m = p$ we obtain $x^{p^4} - 2x^{p^2} + x \mid x^{p^{2p}} - x$, as desired. ∎

This problem and solution were proposed by Victor Wang.

Remark. This is based off of the classical Bell number congruence $B_{n+\frac{p^p-1}{p-1}} \equiv B_n \pmod p$, where $B_n = \sum_{k=0}^n S(n,k)$ is the number of ways to partition a set of n distinguishable elements into indistinguishable nonempty sets (we take $S(0,0) = 1$ and for $m > 0$, $S(m,0) = S(0,m) = 0$, to deal with zero indices). We can replace $\{F_n\}_{n\geq 0}$ with any recurrence $\{a_n\}$ satisfying $a_n = a_{n-1} + a_{n-2}$, but Fibonacci numbers will still appear in the main part of the solution. There is a similar solution working in $\mathbb{F}_{p^2}$ (using Binet's formula more directly); we encourage the reader to find it. There is also an instructive solution using the generating function $\sum_{n\geq 0} a^k S(n,k)x^n = \frac{(ax)^k}{(1-x)(1-2x)\cdots(1-kx)}$ (which holds for all $k \geq 0$, and has a simple combinatorial interpretation) for $a = \alpha, \beta$ and working in $\mathbb{F}_{p^2}$ again; we also encourage the reader to explore this line of attack and realize its connections to umbral calculus.