Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper, carbon paper). Failure to meet any of these requirements will result in an automatic 0 for that problem.

1. In acute triangle $ABC$, let $D, E, F$ denote the feet of the altitudes from $A, B, C$, respectively, and let $\omega$ be the circumcircle of $\triangle AEF$. Let $\omega_1$ and $\omega_2$ be the circles through $D$ tangent to $\omega$ at $E$ and $F$, respectively. Show that $\omega_1$ and $\omega_2$ meet at a point $P$ on $BC$ other than $D$.

2. Find all ordered pairs of positive integers $(m, n)$ for which there exists a set $C = \{c_1, \ldots, c_k\}$ ($k \geq 1$) of colors and an assignment of colors to each of the $mn$ unit squares of a $m \times n$ grid such that for every color $c_i \in C$ and unit square $S$ of color $c_i$, exactly two direct (non-diagonal) neighbors of $S$ have color $c_i$.

3. Let $f, g$ be polynomials with complex coefficients such that $\gcd(\deg f, \deg g) = 1$. Suppose that there exist polynomials $P(x, y)$ and $Q(x, y)$ with complex coefficients such that $f(x) + g(y) = P(x, y)Q(x, y)$. Show that one of $P$ and $Q$ must be constant.
Every Little Mistake ⇒ 0
Lincoln, Nebraska
Day II 8 a.m. - 12:30 p.m.
June 17, 2012

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4. Let \( a_0, b_0 \) be positive integers, and define \( a_{i+1} = a_i + \lfloor \sqrt{b_i} \rfloor \) and \( b_{i+1} = b_i + \lfloor \sqrt{a_i} \rfloor \) for all \( i \geq 0 \). Show that there exists a positive integer \( n \) such that \( a_n = b_n \).

5. Let \( ABC \) be an acute triangle with \( AB < AC \), and let \( D \) and \( E \) be points on side \( BC \) such that \( BD = CE \) and \( D \) lies between \( B \) and \( E \). Suppose there exists a point \( P \) inside \( ABC \) such that \( PD \parallel AE \) and \( \angle PAB = \angle EAC \). Prove that \( \angle PBA = \angle PCA \).

6. A diabolical combination lock has \( n \) dials (each with \( c \) possible states), where \( n, c > 1 \). The dials are initially set to states \( d_1, d_2, \ldots, d_n \), where \( 0 \leq d_i \leq c - 1 \) for each \( 1 \leq i \leq n \). Unfortunately, the actual states of the dials (the \( d_i \)'s) are concealed, and the initial settings of the dials are also unknown. On a given turn, one may advance each dial by an integer amount \( c_i \) (\( 0 \leq c_i \leq c - 1 \)), so that every dial is now in a state \( d_i' \equiv d_i + c_i \pmod{c} \) with \( 0 \leq d_i' \leq c - 1 \). After each turn, the lock opens if and only if all of the dials are set to the zero state; otherwise, the lock selects a random integer \( k \) and cyclically shifts the \( d_i \)'s by \( k \) (so that for every \( i \), \( d_i \) is replaced by \( d_i-k \), where indices are taken modulo \( n \)).

Show that the lock can always be opened, regardless of the choices of the initial configuration and the choices of \( k \) (which may vary from turn to turn), if and only if \( n \) and \( c \) are powers of the same prime.