**Every Little Mistake $\implies$ 0**

**Lincoln, Nebraska**

**Day I    8 a.m. - 12:30 p.m.**

**June 16, 2012**

**Note:** For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper, carbon paper). Failure to meet any of these requirements will result in an automatic 0 for that problem.

1. In acute triangle $ABC$, let $D, E, F$ denote the feet of the altitudes from $A, B, C$, respectively, and let $\omega$ be the circumcircle of $\triangle AEF$. Let $\omega_1$ and $\omega_2$ be the circles through $D$ tangent to $\omega$ at $E$ and $F$, respectively. Show that $\omega_1$ and $\omega_2$ meet at a point $P$ on $BC$ other than $D$.

2. Find all ordered pairs of positive integers $(m, n)$ for which there exists a set $C = \{c_1, \ldots, c_k\}$ $(k \geq 1)$ of colors and an assignment of colors to each of the $mn$ unit squares of a $m \times n$ grid such that for every color $c_i \in C$ and unit square $S$ of color $c_i$, exactly two direct (non-diagonal) neighbors of $S$ have color $c_i$.

3. Let $f, g$ be polynomials with complex coefficients such that $\gcd(\deg f, \deg g) = 1$. Suppose that there exist polynomials $P(x, y)$ and $Q(x, y)$ with complex coefficients such that $f(x) + g(y) = P(x, y)Q(x, y)$. Show that one of $P$ and $Q$ must be constant.

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4. Let $a_0, b_0$ be positive integers, and define $a_{i+1} = a_i + \lfloor \sqrt{b_i} \rfloor$ and $b_{i+1} = b_i + \lfloor \sqrt{a_i} \rfloor$ for all $i \geq 0$. Show that there exists a positive integer $n$ such that $a_n = b_n$.

5. Let $ABC$ be an acute triangle with $AB < AC$, and let $D$ and $E$ be points on side $BC$ such that $BD = CE$ and $D$ lies between $B$ and $E$. Suppose there exists a point $P$ inside $ABC$ such that $PD \parallel AE$ and $\angle PAB = \angle EAC$. Prove that $\angle PBA = \angle PCA$.

6. A diabolical combination lock has $n$ dials (each with $c$ possible states), where $n, c > 1$. The dials are initially set to states $d_1, d_2, \ldots, d_n$, where $0 \leq d_i \leq c - 1$ for each $1 \leq i \leq n$. Unfortunately, the actual states of the dials (the $d_i$'s) are concealed, and the initial settings of the dials are also unknown. On a given turn, one may advance each dial by an integer amount $c_i$ ($0 \leq c_i \leq c - 1$), so that every dial is now in a state $d'_i \equiv d_i + c_i \pmod{c}$ with $0 \leq d'_i \leq c - 1$. After each turn, the lock opens if and only if all of the dials are set to the zero state; otherwise, the lock selects a random integer $k$ and cyclically shifts the $d_i$'s by $k$ (so that for every $i$, $d_i$ is replaced by $d_{i-k}$, where indices are taken modulo $n$).

   Show that the lock can always be opened, regardless of the choices of the initial configuration and the choices of $k$ (which may vary from turn to turn), if and only if $n$ and $c$ are powers of the same prime.