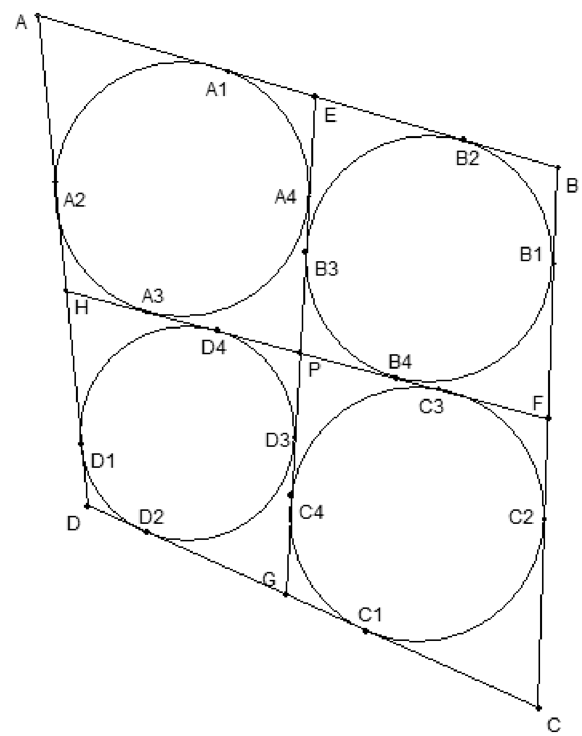
37th English Language Master's Open

1. Let $ABCD$ be a convex quadralateral. Let E, F, G, H be points on segments AB, BC, CD, DA, respectively, and let P be intersection of EG and FH. Given that quadrilaterals $HAEP, EBFP, FCGP, GDHP$ all have inscribed circles, prove that $ABCD$ also has an inscribed circle.



Solution: Let us label the points of tangency of the four given incircles as shown in the diagram.

Then, to prove that $ABCD$ has an inscribed circle, it suffices to show that $AB + CD = AD + BC$. Since common tangents from a point to a circle share the same length, we get

$$AB + CD = AD + BC$$

$$\Leftrightarrow (AA_1 + A_1B_2 + B_2B) + (CC_1 + C_1D_2 + D_2D) = (AA_2 + A_2D_1 + D_1D) + (BB_1 + B_1C_2 + C_2C)$$

$$\Leftrightarrow \; A_1B_2 + C_1D_2 = A_2D_1 + B_1C_2.$$

We first want to show that $A_2D_1 = A_4D_3$. If $AD||EG$, then this is true because A_2, D_1, D_3, A_4 form the corners of a rectangle. Otherwise, consider the intersection of EG and AD. Note that, of the incircles of $AEPH$ and $HPGD$, one is an incircle and the other an excircle of the triangle with the intersection point as a vertex.

Consequently, A_4 is the reflection of D_3 over the midpoint of HP and we have

$$A_2D_1 = A_2H + HD_1 = HA_4 + HD_3 = PD_3 + A_4P = A_3P + PD_4 = A_3D_4.$$

Similarly, $B_1C_2 = B_3C_4$, $A_1B_2 = A_3B_4$, and $D_2C_1 = D_4C_3$.

Combining, we get

$$\begin{aligned} A_2D_1 + B_1C_2 =& A_4D_3 + B_3C_4 \\ =& PA_4 + PD_3 + PB + 3 + PC_4 \\ =& PA_3 + PD_4 + PB_4 + PC_3 \\ =& A_3B_4 + D_4C_3 \\ =& A_1B_2 + C_1D_2, \end{aligned}$$

so we are done.

This problem was proposed by Evan O'Dorney.

2. Wanda the Worm likes to eat Pascal's triangle. One day, she starts at the top of the triangle and eats $\binom{0}{0} = 1$. Each move, she travels to an adjacent positive integer and eats it, but she can never return to a spot that she has previously eaten. If Wanda can never eat numbers a, b, c such that $a + b = c$, proof that it is possible for her to eat $100,000$ numbers in the first 2011 rows given that she is not restricted to traveling only in the first 2011 rows.

   (Here, the $n+1^{\text{st}}$ row of Pascal's triangle consists of entries of the form $\binom{n}{k}$ for integers $0 \leq k \leq n$. Thus, the entry $\binom{n}{k}$ is considered adjacent to the entries $\binom{n-1}{k-1}$, $\binom{n-1}{k}$, $\binom{n}{k-1}$, $\binom{n}{k+1}$, $\binom{n+1}{k}$, $\binom{n+1}{k+1}$.)

   Solution: We will prove by induction on n that it is possible for Wanda to eat 3^n numbers in the first 2^n rows of Pascal's triangle. Our inductive hypothesis includes the following conditions on the first 2^n rows of Pascal's triangle when all the entries are taken modulo 2:

   - Row 2^n contains only odd numbers.
   - The 2^n rows contain a total of 3^n odd numbers.

- The triangle of rows has 120 degree rotational symmetry.
- There is a path for Wanda to munch that starts at any corner of these rows, contains all the odd numbers, and ends at any other corner.

Our base case is $n = 1$; it is not difficult to check that all of these conditions hold. Wanda's path in these two rows is $\binom{0}{0} \to \binom{0}{1} \to \binom{1}{1}$.

Now, assume that these hold for the first 2^m rows of Pascal's triangle. We will show that they also hold for the first 2^{m+1} rows. Note that a single 1 surrounded by $2^m - 1$ 0's to either side generated the first 2^m rows since each element is equal to the sum of the two numbers directly above it. However, by our inductive hypothesis, all of the entries in the 2^m row were 1's. Hence, the first and last entires of the 2^{m+1} row are also both 1, and the remainder of the entires are 0. Consequently, we note that these 1's and 0's generate two other copies of the first 2^m rows of Pascal's triangle, along with an inverted triangle of all 0's in the middle.

Now it suffices to check that our conditions hold:

- As row 2^{m+1} simply contains two side-by-side copies of the 2^mth row modulo 2, it also consists all of 1's.
- The first 2^{m+1} rows contain three copies of the first 2^m rows along with a triangle of 0's, so they contain $3(3^m) = 3^{m+1}$ odd numbers.
- As each of the three 2^m row triangles had rotational symmetry, so does the larger one.
- By our inductive hypothesis, Wanda can travel from $\binom{0}{0}$ to $\binom{2^m-1}{0}$ and eat all the odd numbers in those rows. She can then travel to $\binom{2^m}{0}$, eat all the numbers in the lower-left triangle and end at $\binom{2^{m+1}-1}{2^m-1}$, travel to $\binom{2^{m+1}-1}{2^m}$, eat all the odd numbers in the lower-right triangle, and finally end at $\binom{2^{m+1}-2}{2^{m+1}-1}$. Due to rotational symmetry, she can also start and end at any corner.

We have now proved our induction.

Note that if Wanda only eats odd numbers, then she will never eat three numbers a, b, c such that $a + b = c$. We have $2^{10} < 2011 < 2048 = 2^{11}$.

It suffices to check that there are sufficient odd numbers in the first 2011 rows. We have showed that there are 3^{11} odd numbers in the first 2048 rows. Also, row n has n elements and thus contains at most n odd numbers. Hence, there are at least

$$3^{11} - 2048 - 2047 - \ldots - 2012 = 3^{11} - \frac{1}{2}(2048 + 2012)(2048 - 2011) > 1000000$$

odd numbers in the first 2011 rows.

This problem was proposed by Linus Hamilton.

3. Determine whether there exists a sequence $\{a_n\}_{n=0}^{\infty}$ of real numbers such that the following holds:

   - For all $n \geq 0$, $a_n \neq 0$.
   - There exist real numbers x and y such that $a_{n+2} = xa_{n+1} + ya_n$ for all $n \geq 0$.
   - For all positive real numbers r, there exists positive integers i and j such that $|a_i| < r < |a_j|$.

Solution: The answer is *yes*.

Let $x_n = \underbrace{2^{2^{\cdots^{2}}}}_{2n\ 2\text{'s}}$. Then, let $\theta = \frac{\pi}{2}\left(\frac{1}{x_1} + \frac{1}{x_2} + \ldots\right)$, and let $r = 2$ and $a_n = r^n \cos(n\theta)$.

We will prove that this sequences satisfies the three given conditions.

First, note that

$$a_{n+2} = 2r\cos(\theta)a_{n+1} - r^2 a_n$$

for all n by the addition formula for cosine, so the recursion condition is satisfied by setting $x = 2r\cos(\theta)$ and $y = -r^2$.

Second, we note that if there exists any integer n such that $a_n = 0$, then we would have $n\theta = \pi(k + \frac{1}{2})$ for some $k \in \{0, 1, 2, \ldots\}$, implying that $\frac{\theta}{\pi}$ is rational. However, we have

$$\frac{\theta}{\pi} = \frac{1}{2}\left(\frac{1}{x_1} + \frac{1}{x_2} + \ldots\right),$$

which has a non-periodic binary expansion and is therefore irrational. Hence, we know hat the second condition is satisfied.

Third, consider the subsequence

$$\begin{aligned} b_n &= a_{x_n} \\ &= r^{x_n}\cos(x_n\theta) \\ &= r^{x_n} cos\left(\frac{\pi}{2}\sum_{k=1}^{\infty}\frac{x_n}{x_k}\right). \end{aligned}$$

Where $a = \sum_{k=1}^{n} \frac{x_n}{x_k}$ is an odd integer, we note that

$$\begin{aligned}
|b_n| < |r|^{x_n} &\left|\cos\left(\frac{\pi}{2}\sum_{k=1}^{\infty}\frac{x_n}{x_k}\right)\right| \\
&= |r|^{x_n}\left|\cos\left(\frac{\pi}{2}\left(a + \sum_{k=n+1}^{\infty}\frac{x_n}{x_k}\right)\right)\right| \\
&= |r|^{x_n}\left|\sin\left(\frac{\pi}{2}\sum_{k=n+1}^{\infty}\frac{x_n}{x_k}\right)\right| \\
&\le 2^{x_n}\frac{\pi}{2}\sum_{k=n+1}^{\infty}\frac{x_n}{x_k} \\
&\le 2^{x_n}\frac{\pi}{2}\sum_{k=n+!}^{\infty}\frac{x_n}{x_{n+1}\cdot 2^{k-n-1}} \\
&= 2^{x_n}\cdot\pi\cdot\frac{x_n}{x_{n+1}},
\end{aligned}$$

which becomes arbitrarily small as n approaches infinity.

Consequently, $\{a_n\}$ has a subsequence with arbitrarily small magnitude. By Kronecker's Theorem, there is also a sequence $n_1, n_2, \ldots$ with $\{\frac{n_1\theta}{2\pi}\} \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$ for $i = 1, 2, \ldots$. Then, the sequence $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ tends to infinity. Thus, $\{a_n\}$ has both a subsequence with magnitude tending to 0 and a subsequence with magnitude tending to infinity, so the third property also holds.

This problem was proposed by Alex Zhu.

4. Find all functions $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$, where $\mathbb{R}^+$ denotes the positive reals, such that whenever $a > b > c > d > 0$ are reel numbers with $ad = bc$,

$$f(a+d) + f(b-c) = f(a-d) + f(b+c).$$

Solution: Since $f(a+d) - f(a-d)$ only depends on ad, we can have a function g mapping positive reals to reals such that whenever $a > d$,

$$g(ad) = f(a+d) - f(a-d).$$

Also,

$$\begin{aligned} g(kad + k(k+1)d^2) &= g((a + (k+1)d)(kd)) \\ &= f(a + (2k+1)d) - f(a+d) \\ &\text{and} \\ g((k+1)ad + k(k+1)d^2) &= g((a+kd)((k+1)d)) \\ &= f(a + (2k+1)d) - f(a-d) \\ &= g(ad) + g(kad + k(k+1)d^2) \end{aligned}$$

for any constant $k > 0$.

Let $a = 2d$, and let $x = d^2$. Then we have the following:

$$g((k^2 + 3k + 2)x) = g(2x) + g((k^2 + 3k)x)$$

However, $(k^2+3k)x$ is surjective over the positive reals as $k > 0$, so if we let $y = (k^2+3k)x$, we obtain

$$g(x + y) = g(x) + g(y)$$

for all positive real numbers x and y. Consequently, for any positive real number x, we can always find a unique $\lambda > 0$ such that $\lambda(\lambda + 1) = x$. Thus,

$$g(x) = g(\lambda(\lambda + 1)) = f(2\lambda + 1) - f(1) \geq -f(1)$$

Because g is bounded below and satisfies Cauchy's Functional Equation, there exists a real number a such that $g(x) = ax$ for all $x > 0$. That gives, for $u > 1$,

$$f(u) = f(1) + g((u^2 - 1)/4) = \frac{a}{4}u^2 + \frac{-a + 4f(1)}{4}$$

and for $u < 1$

$$f(u) = f(1) - g((1 - u^2)/4) = \frac{a}{4}u^2 + \frac{-a + 4f(1)}{4}$$

Thus there exist constants c, d such that for $u \neq 1$, $f(u) = cu^2 + d$. Finally,

$$f(1) = f(4-3) = f(4+3) + f(6-2) - f(6+2) = 49c + d + 16c + d - 64c - d = c + d$$

Thus equations of the form $f(u) = cu^2 + d$ for all $u > 0$ are the only possible solutions. It is not hard to see that this is a solution to the functional equation if and only if c and d are nonnegative real numbers which are not both zero.

This problem was proposed by Calvin Deng.

5. Let $p > 13$ be a prime of the the form $2q+1$, where q is prime. Find the number of ordered pairs of integers (m, n) such that $0 \leq m < n < p-1$ and

$$3^m + (-12)^m \equiv 3^n + (-12)^n \pmod{p}.$$

Solution:

Lemma 1: -4 is a primitive root modulo p.

Proof of Lemma 1: Note that $\text{ord}_p(-4) | p-1 = 2q$, so $\text{ord}_p(-4)$ is one of $1, 2, q, 2q$. Because $-4 \not\equiv 1 \pmod{p}$ we have $\text{ord}_p(-4) \neq q$). As $16 = 4^n \not\equiv 1 \pmod{p}$, we have $\text{ord}_p(-4) \neq 2$.

Also, we have

$$\left(\frac{-4}{p}\right) = \left(\frac{-1}{p}\right) \cdot \left(\frac{4}{p}\right) = -1 \cdot 1 \cdot -1,$$

since $\left(\frac{-1}{p}\right) = 1$ follows from $p > 13 \Rightarrow \frac{p-1}{2} = q$ being odd.

Thus, following from the fact that -4 is not a quadratic residue modulo p, we have that

$$2 \not| \frac{p-1}{\text{ord}_p(-4)} = \frac{2q}{\text{ord}_p(-4)}$$

$$\Rightarrow 2 \cdot \text{ord}_p(-4) \not| 2q$$

$$\Rightarrow \text{ord}_p(-4) \not| q$$

Consequently, $\text{ord}_p(-4) = 2q$, as desired.

Lemma 2: The order of 3 modulo p is exactly q.

Proof of Lemma 2: Note that $\text{ord}_p(3) | p-1 = 2q$, so $\text{ord}_p(3)$ is one of $1, 2, q, 2q$. Because $3 \not\equiv 1 \pmod{p}$, we have $\text{ord}_p(3) \neq 1$. As $3^2 = 9 \not\equiv 1 \pmod{p}$, we have $\text{ord}_p(3) \neq 2$. Then, we have

$$\begin{aligned} p &= 2q + 1 \\ &\equiv 2 \cdot 1 + 1 \text{ or } 2 \cdot 2 + 1 \pmod{3} \\ &\equiv 0 \text{ or } 2 \pmod{3} \end{aligned}$$

Because $p > 13$, we know that $q \neq 3$ and $p \neq 3$, giving

$$\left(\frac{p}{3}\right) = \left(\frac{2}{3}\right) = 1.$$

Also, by quadratic reciprocity, we have

$$\begin{aligned}
\left(\frac{3}{p}\right) \cdot \left(\frac{p}{3}\right) &= (-1)^{\frac{(3-1)(p-1)}{4}} \\
\left(\frac{3}{p}\right)(-1) &= (-1)^q \\
&= -1 \\
\Rightarrow \left(\frac{3}{p}\right) &= 1.
\end{aligned}$$

We now know that 3 is a quadratic residue modulo p, so $\text{ord}_p(3) \neq 2q$, giving us $\text{ord}_p(3) = q$, as desired.

Lemma 3: -12 is a primitive root modulo p.

Proof of Lemma 3: Note that $\text{ord}_p(-12) | p-1 = 2q$, so $\text{ord}_p(-12)$ is one of $1, 2, q, 2q$. Because $-12 \not\equiv 1 \pmod{p}$, we have $\text{ord}_p(-12) \neq 1$. As $(-12)^2 = 144 \not\equiv 1 \pmod{p}$, we have $\text{ord}_p(-12) \neq 2$. Then, after substituting for the values found in Lemma 1 and Lemma 2, we obtain

$$\begin{aligned}
\left(\frac{-12}{p}\right) &= \left(\frac{-1}{p}\right) \cdot \left(\frac{4}{p}\right) \cdot \left(\frac{3}{p}\right) \\
&= (-1) \cdot (1) \cdot (1) \\
&= -1.
\end{aligned}$$

Thus, -12 is not a quadratic residue modulo p, giving us

$$\begin{aligned}
2 \nmid \frac{p-1}{\text{ord}_p(-12)} &= \frac{2q}{\text{ord}_p(-12)} \\
\Rightarrow \text{ord}_p(-12) &\nmid q
\end{aligned}$$

It follows that $\text{ord}_p(-12) = 2q$, as desired.

Main Proof: We now simplify the given equation:

$$\begin{aligned}
3^m + (-12)^m &\equiv 3^n + (-12)^n && \pmod{p} \\
&\equiv 3^{n-m} \cdot 3^m + 3^{n-m} \cdot 3^m \cdot (-4)^n && \pmod{p} \\
1 + (-4)^m &\equiv 3^{n-m} + 3^{n-m} \cdot (-1)^n && \pmod{p} \\
1 - 3^{n-m} &\equiv 3^{n-m} \cdot (-4)^n - (-4)^m && \pmod{p} \\
&\equiv (-4)^m \cdot ((-12)^{n-m} - 1) && \pmod{p}.
\end{aligned}$$

We ignore for the moment the condition that $m < n$ and count all pairs $m, n \in \mathbb{Z}_{p-1} = \mathbb{Z}_{2q}$. So, if $n \not\equiv m \pmod{2q}$, then $(-12)^{n-m} - 1 \not\equiv 0 \pmod{p}$, giving us

$$(-4)^m \equiv (1 - 3^{n-m})((-12)^{n-m} - 1)^{-1} \pmod{p}.$$

Because -4 is primitive modulo p, we have that any non-zero residue of $(-4)^m \pmod{p}$ uniquely determines the residue of $m \pmod{2q}$. So each non-zero residue of $n-m \pmod{2q}$ uniquely determines $m \pmod{2q}$, so long as

$$3^{n-m} - 1 \not\equiv 0 \pmod{p} \Leftrightarrow q \not| n - m.$$

Consequently, for each $(n-m) \in \{1, 2, \ldots, q-1, q+1, q+2, \ldots, 2q-1\} \pmod{2q}$, we unitely determine the ordered pair $(m, n) \in \mathbb{Z}_{2q}^2$. However, on taking remainders on divison by $2q$ of m, n, we must have $m < n$. Thus, for each $x \not\equiv q, 0$, the solutions for $n - m \equiv x \pmod{2q}$ and $n - m \equiv -1 \pmod{2q}$ give exactly 1 solution (m, n) with $m < n$.

Thus, we have a total of $\frac{2q-2}{2} = q - 1$ solutions.

This problem was proposed by Alex Zhu.

6. Consider the infinite grid of lattice points in $\mathbb{Z}^3$. Little D and Big Z play a game, where Little D first loses a shoe on an unmunched point in the grid. Then, Big Z munches a shoe-free plane perpendicular to one of the coordinate axes. They continue to alternature turns in this fashion, with Little D's goal to loose a shoe on each of n consecutive lattice points on a line parrallel to one of the coordinate axes. Determine all n for which Little D can accomplish his goal.

Solution: We claim that Little D can accomplish this for all n.

We will start by separating out the three coordinate axes: thus, if Little D loses a shoe at the point (i, j, k) for integers i, j, and k, he plays on i on the x-axis, j on the y-axis, and k on the z-axis in the same move. Meanwhile, when Big Z munches a plane, he plays on only one point on one of the coordinate axes. Hence, since Big Z can only munch a shoe-free plane, he cannot munch point l on a particular axis if Little D has already placed a shoe there.

We will call a string of points marked (by shoes) on one of these coordinate axes *unbounded* if Big Z has not munched any point on that axis within $2n + 1$ of at least one endpoint of the string.

Lemma: For any integers m and l, Little D can create l unbounded strings of m consecutive points on a single coordinate axis.

Proof of lemma: We will prove this by induction on m.

Our base case is $m = 1$. Then, we note that if Little D makes $\lceil 1.5l \rceil$ triplets of moves over the three axes, making sure that he distributes any marked points in the same axis at least $5n$ apart, then Big Z can bound at most $\lceil 1.5l \rceil$ strings because he can only bound at most one string on each move. However, this leaves $3x$ unbounded strings of length 1; by the pigeonhole principle, at least l of these must be in the same cooordinate axis.

Now, suppose that this is true for some m. We will show that it is also true for $m+1$. Without loss of generality, we note from our induction hypothesis that Little D can construct $\lceil \frac{x(m+1)}{2} \rceil$ unbounded strings of length m on the x-axis. Consequently, he can create $m+3$ strings of length $m+1$ in $m+1$ moves: in each move, he lengthens one unbounded string of the x-axis, while on each of the y and z-axes he builds up a new string of $m+1$ marked points. However, Big Z can, in these $m+1$ moves, bound at most $m+1$ of these strings. Hence, Little D can construct at least 2 strings of length $m+1$ for every $m+1$ strings of length m used up. It follows that he can achieve x unbounded strings of length $m+1$. We have now proved our desired induction.

Main Proof: Now, without loss of generality, by our lemma Little D can mark n consecutive points on the x-axis. Then, he has established n consecutive yz-planes that Big Z can never much. Suppose that one of the points he has played on is (i, j, k) for some $i, j, k \in \mathbb{Z}$. Then, Big Z can never munch any part of the line $y = j, z = k$ in those n consecutive xy planes. Hence Little D can lose shoes in the remainder of those n points over his next few moves, at which point he has achieved his goal.

This problem was proposed by David Yang.