EGMO 2021 Solution Notes

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This is a compilation of solutions for the 2021 EGMO. The ideas of the solution are a mix of my own work, the solutions provided by the competition organizers, and solutions found by the community. However, all the writing is maintained by me.

These notes will tend to be a bit more advanced and terse than the “official” solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered “standard”, then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like “let $\mathbb{R}$ denote the set of real numbers” are typically omitted entirely.

Corrections and comments are welcome!

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§0 Problems

1. The number 2021 is fantabulous. For any positive integer m, if any element of the set $\{m, 2m+1, 3m\}$ is fantabulous, then all the elements are fantabulous. Does it follow that the number 2021^{2021} is fantabulous?

2. Find all functions $f \colon \mathbb{Q} \to \mathbb{Q}$ such that the equation
$$f(xf(x)+y) = f(y) + x^2$$
holds for all rational numbers x and y.

3. Let ABC be a triangle with an obtuse angle at A. Let E and F be the intersections of the external bisector of angle A with the altitudes of ABC through B and C respectively. Let M and N be the points on the segments EC and FB respectively such that $\angle EMA = \angle BCA$ and $\angle ANF = \angle ABC$. Prove that the points E, F, N, M lie on a circle.

4. Let ABC be a triangle with incenter I and let D be an arbitrary point on the side BC. Let the line through D perpendicular to BI intersect CI at E. Let the line through D perpendicular to CI intersect BI at F. Prove that the reflection of A across the line EF lies on the line BC.

5. A plane has a special point O called the origin. Let P be a set of 2021 points in the plane such that
   - no three points in P lie on a line and
   - no two points in P lie on a line through the origin.

   A triangle with vertices in P is *fat* if O is strictly inside the triangle. Find the maximum number of fat triangles.

6. Does there exist a nonnegative integer a for which the equation
$$\left\lfloor \frac{m}{1} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + \cdots + \left\lfloor \frac{m}{m} \right\rfloor = n^2 + a$$
has more than one million different solutions (m, n) where m and n are positive integers?

§1 Solutions to Day 1

§1.1 EGMO 2021/1, proposed by Angelo Di Pasquale (AUS)

Available online at https://aops.com/community/p21455194.

Problem statement

The number 2021 is fantabulous. For any positive integer m, if any element of the set $\{m, 2m+1, 3m\}$ is fantabulous, then all the elements are fantabulous. Does it follow that the number 2021^{2021} is fantabulous?

Write $a \iff b$ to mean a fantabulous iff b fantabulous. Notice that for any integer n, we have

$$2n \iff 4n+1 \iff 12n+3 \iff 6n+1 \iff 3n \iff n.$$

Together with the given $2n+1 \iff n$, it follows that if any integer is fantabulous then all of them are.

§1.2 EGMO 2021/2, proposed by Patrik Bak (SVK)

Available online at https://aops.com/community/p21455202.

Problem statement

Find all functions $f\colon \mathbb{Q} \to \mathbb{Q}$ such that the equation

$$f(xf(x)+y) = f(y) + x^2$$

holds for all rational numbers x and y.

The answers are $f(x) \equiv +x$ and $f(x) \equiv -x$ which work. To show they are the only ones, we follow an approach similar to `SnowPanda`.

Claim — If $f(z) = 0$ then $z = 0$. In other words, f has at most one root, at 0.

Proof. Take $P(z, 0)$. □

Claim — The function f is linear.

Proof. Let $a, b \in \mathbb{Q}$ be any two nonzero rational numbers, so $f(a), f(b) \neq 0$. Choose nonzero integers m and n such that

$$\frac{n}{m} = \frac{af(a)}{bf(b)}.$$

Then for any $y \in \mathbb{Q}$ we have

$$f(y + maf(a)) = f(y) + m \cdot a^2$$
$$f(y + nbf(b)) = f(y) + n \cdot b^2.$$

The two left-hand sides were equal by construction, so we get

$$\frac{af(a)}{bf(b)} = \frac{n}{m} = \frac{a^2}{b^2}.$$

Thus $f(a)/a = f(b)/b$, as needed. □

Once f is linear we here quickly recover the solution set.

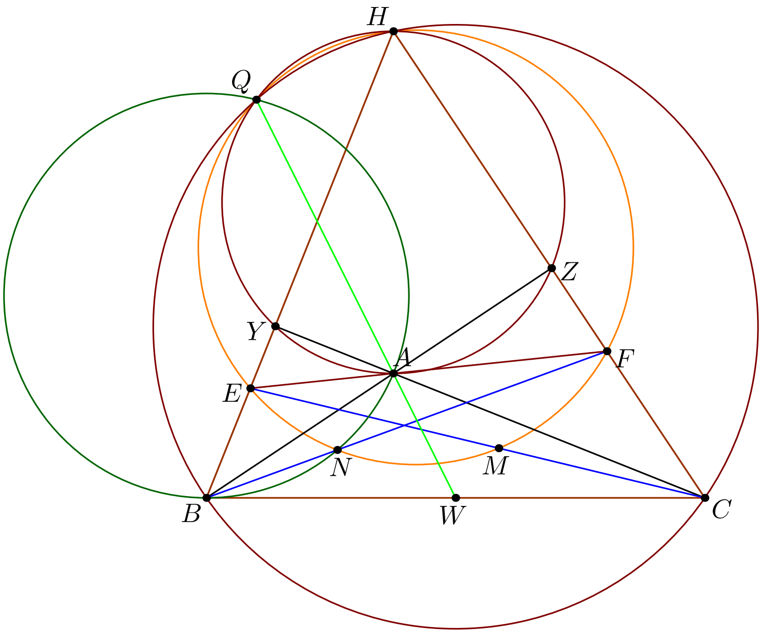
§1.3 EGMO 2021/3, proposed by Anton Trygub (UKR)

Available online at https://aops.com/community/p21455206.

Problem statement

Let ABC be a triangle with an obtuse angle at A. Let E and F be the intersections of the external bisector of angle A with the altitudes of ABC through B and C respectively. Let M and N be the points on the segments EC and FB respectively such that $\angle EMA = \angle BCA$ and $\angle ANF = \angle ABC$. Prove that the points E, F, N, M lie on a circle.

Call the altitudes $\overline{BZ}$ and $\overline{CY}$, and H the orthocenter. Let W be the midpoint of $\overline{BC}$. Then according to **IMO Shortlist 2005 G5**, the line AW is concurrent with (HYZ), (HEF), (HBC) at a point Q.



Since $WA \cdot WQ = WB^2$, it follows that (AQB) is tangent to $\overline{BC}$, ergo $N \in (AQB)$. Then

$$\measuredangle QNF = \measuredangle QNB = \measuredangle QAB = \measuredangle QAZ = \measuredangle QHZ = \measuredangle QHF$$

and hence N lies on $(HQEF)$. Similarly, so does M.

§2 Solutions to Day 2

§2.1 EGMO 2021/4, proposed by Sampson Wong (AUS)

Available online at https://aops.com/community/p21455215.

Problem statement

Let ABC be a triangle with incenter I and let D be an arbitrary point on the side BC. Let the line through D perpendicular to BI intersect CI at E. Let the line through D perpendicular to CI intersect BI at F. Prove that the reflection of A across the line EF lies on the line BC.

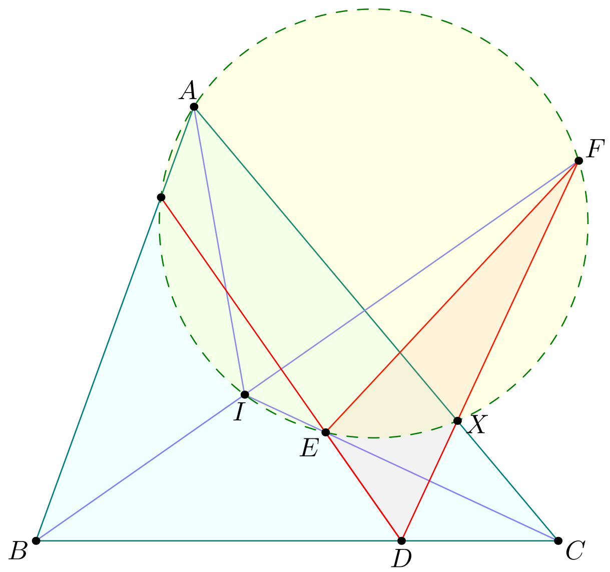
We begin as follows:

Claim — $AEIF$ is cyclic.

Proof. Let $X = \overline{CA} \cap \overline{DF}$. Then

$$\begin{aligned} \measuredangle FXA = \measuredangle DXC = \measuredangle CDX = 90^\circ - \frac{1}{2}C = \measuredangle AIB = \measuredangle FIA \\ \measuredangle EXF = \measuredangle EXD = \measuredangle XDE = \measuredangle FDE = \measuredangle EIF \end{aligned}$$

which implies $AFXI$ and $FXIE$ are cyclic, respectively. $\square$



Now, the dilation of the Simson line of A to $\triangle IEF$ should be collinear, but the reflections of A about lines BI and CI lie on line BC by definition. This solves the problem.

§2.2 EGMO 2021/5, proposed by Veronika Schreitter (AUT)

Available online at https://aops.com/community/p21455223.

Problem statement

A plane has a special point O called the origin. Let P be a set of 2021 points in the plane such that

- no three points in P lie on a line and
- no two points in P lie on a line through the origin.

A triangle with vertices in P is *fat* if O is strictly inside the triangle. Find the maximum number of fat triangles.

For every pair of points P and Q, draw a directed arrow $P \to Q$ if $\angle POQ$ is labeled clockwise. This gives a tournament on 2021 vertices, and a triangle is fat if its three edges form a directed cycle.

Consequently, by **Canada 2006/4**, there are at most $\frac{1}{6}n(n+1)(2n+1)$ fat triangles, where $n = 1010$. Equality occurs if the set of points are the vertices of a regular 2021-gon containing O.

§2.3 EGMO 2021/6, proposed by Veronika Schreitter (AUT)

Available online at https://aops.com/community/p21455228.

Problem statement

Does there exist a nonnegative integer a for which the equation

$$\left\lfloor \frac{m}{1} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + \dots + \left\lfloor \frac{m}{m} \right\rfloor = n^2 + a$$

has more than one million different solutions (m, n) where m and n are positive integers?

The answer is yes. In fact, we prove the following general version:

Claim — Consider any function $f \colon \mathbb{N} \to \mathbb{N}$ satisfying $f(m) = o(m^2)$. Then for some a, the equation

$$f(m) = n^2 + a$$

has at least 1000000 solutions (m, n).

Proof. We consider triples (m, n, a) satisfying the equation with the additional property that

$$n = \left\lfloor \sqrt{f(m)} \right\rfloor^2 \implies a = f(m) - n^2 \in [0, 2\sqrt{f(m)}].$$

Now, let M be large enough that $\max_{m=1}^{M} f(m) < \frac{M^2}{2 \cdot 10^{12}}$. Then there are M triples, but $a < \frac{M}{10^6}$ in each triple. So some a appears 10^6 times. □