EGMO 2020 Solution Notes

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This is a compilation of solutions for the 2020 EGMO. The ideas of the solution are a mix of my own work, the solutions provided by the competition organizers, and solutions found by the community. However, all the writing is maintained by me.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

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§0 Problems

1. The positive integers a_0, a_1, a_2, …, a_{3030} satisfy

$$2a_{n+2} = a_{n+1} + 4a_n \text{ for } n = 0, 1, 2, \dots, 3028.$$

Prove that at least one of the numbers a_0, a_1, a_2, …, a_{3030} is divisible by 2^{2020}.

2. Find all lists $(x_1, x_2, \dots, x_{2020})$ of non-negative real numbers such that the following three conditions are all satisfied:

- $x_1 \le x_2 \le \cdots \le x_{2020}$;
- $x_{2020} \le x_1 + 1$;
- there is a permutation $(y_1, y_2, \dots, y_{2020})$ of $(x_1, x_2, \dots, x_{2020})$ such that

$$\sum_{i=1}^{2020} ((x_i + 1)(y_i + 1))^2 = 8 \sum_{i=1}^{2020} x_i^3.$$

3. Let $ABCDEF$ be a convex hexagon such that $\angle A = \angle C = \angle E$ and $\angle B = \angle D = \angle F$ and the (interior) angle bisectors of $\angle A$, $\angle C$, and $\angle E$ are concurrent. Prove that the (interior) angle bisectors of $\angle B$, $\angle D$, and $\angle F$ must also be concurrent.

4. A permutation of the integers $1, 2, \dots, m$ is called *fresh* if there exists no positive integer $k < m$ such that the first k numbers in the permutation are $1, 2, \dots, k$ in some order. Let f_m be the number of fresh permutations of the integers $1, 2, \dots, m$. Prove that $f_n \ge n \cdot f_{n-1}$ for all $n \ge 3$.

5. Triangle ABC has circumcircle Γ and obeys $\angle BCA > 90^\circ$. There is a point P in the interior of the line segment AB such that $PB = PC$ and the length of PA equals the radius of Γ. The perpendicular bisector of $\overline{PB}$ intersects Γ at the points D and E. Prove P is the incenter of triangle CDE.

6. Find all integers $m > 1$ for which the sequence $(a_n)_{n \ge 1}$ defined recursively by

$$a_{n+2} = m(a_{n+1} + a_n) - a_{n-1}$$

with initial conditions $a_1 = a_2 = 1$ and $a_3 = 4$ contains only perfect squares.

§1 Solutions to Day 1

§1.1 EGMO 2020/1, proposed by Dzmitry Badziahin (AUS)

Available online at https://aops.com/community/p14780286.

Problem statement

The positive integers a_0, a_1, a_2, …, a_{3030} satisfy

$$2a_{n+2} = a_{n+1} + 4a_n \text{ for } n = 0, 1, 2, \dots, 3028.$$

Prove that at least one of the numbers a_0, a_1, a_2, …, a_{3030} is divisible by 2^{2020}.

The idea is this:

- All terms a_0, …, a_{3030} are integers (divisible by $2^0 = 1$);
- Hence all terms a_1, …, a_{3029} are divisible by 2,
- Hence all terms a_1, …, a_{3028} are divisible by 4,
- Hence all terms a_2, …, a_{3027} are divisible by 8,
- …and so on.

The 2021st item in this list reads as $2^{2020} \mid a_{1010}$.

One can also phrase this with induction. Replace 1010 by N in the obvious way and proceed by induction on $N \geq 0$ with the base case $N = 0$ being vacuous. Notice that for any index $k \neq 0, 3N-1, 3N$ we have

$$a_k = 2a_{k+1} - 4a_{k-1} = 2(2a_{k+2} - 4a_k) - 4a_{k-1} = 4(a_{k+2} - 2a_k - a_{k-1})$$

so it follows that $(a_1, a_2, \dots, a_{3N-2})$ are all divisible by 4 and satisfy the same relations. But then $(a_1/4, a_2/4, \dots, a_{3N-2}/4)$ has length $3(N-1)+1$ and so one of them is divisible by 4^{N-1}; hence some term of our original sequence is divisible by 4^N.

§1.2 EGMO 2020/2, proposed by Patrik Bak (SVK)

Available online at https://aops.com/community/p14780288.

Problem statement

Find all lists $(x_1, x_2, \dots, x_{2020})$ of non-negative real numbers such that the following three conditions are all satisfied:

- $x_1 \le x_2 \le \dots \le x_{2020}$;
- $x_{2020} \le x_1 + 1$;
- there is a permutation $(y_1, y_2, \dots, y_{2020})$ of $(x_1, x_2, \dots, x_{2020})$ such that

$$\sum_{i=1}^{2020} ((x_i+1)(y_i+1))^2 = 8 \sum_{i=1}^{2020} x_i^3.$$

The main point of the problem is to prove an inequality.

Claim — If a and b are real numbers with $|a-b| \le 1$, then

$$(a+1)^2(b+1)^2 \ge 4(a^3+b^3).$$

Moreover, equality holds only when $\{a,b\} = \{1,0\}$ or $\{a,b\} = \{2,1\}$.

Proof. Write

$$\begin{aligned} a^3+b^3 &= (a+b)(a^2-ab+b^2) = (a+b)[(a-b)^2+ab] \\ &\le (a+b)(1+ab) \\ &\le \left(\frac{(a+b)+(1+ab)}{2}\right)^2 = \frac{(a+1)^2(b+1)^2}{4}. \end{aligned}$$ □

From this it follows that $(x_n)_n$ should be either $(1,\dots,1,2,\dots,2)$ or $(0,\dots,0,1,\dots,1)$.

Remark. There is a sense in which the problem *must* be an inequality, because given a *fixed* permutation of the indices of y_i by x_i, one gets an equation which *a priori* ought to cut out a codimension 1 surface in $\mathbb{R}^{2020}$. So the fact that the answer set appears finite is an indication that some inequality at play here.

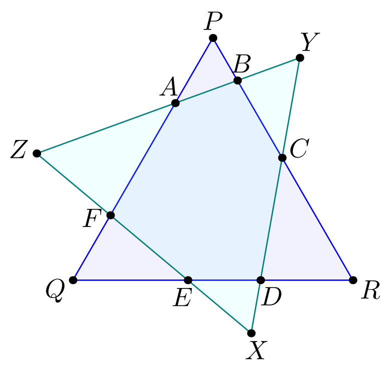
§1.3 EGMO 2020/3, proposed by Anton Trygub (UKR)

Available online at https://aops.com/community/p14780291.

Problem statement

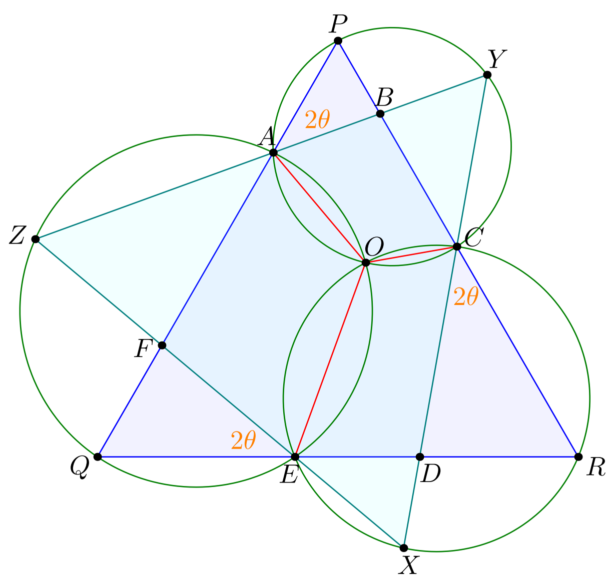
Let $ABCDEF$ be a convex hexagon such that $\angle A = \angle C = \angle E$ and $\angle B = \angle D = \angle F$ and the (interior) angle bisectors of $\angle A$, $\angle C$, and $\angle E$ are concurrent. Prove that the (interior) angle bisectors of $\angle B$, $\angle D$, and $\angle F$ must also be concurrent.

In general, if hexagon $ABCDEF$ has $\angle A = \angle C = \angle E$ and $\angle B = \angle D = \angle F$, then its sides can be extended to form two equilateral triangles PQR and XYZ, as shown.



The problem is solved upon proving the following claim.

Claim — The angle bisectors of $\angle A$, $\angle C$, $\angle E$ are concurrent if and only if the unique spiral similarity sending PQR to YZX is a rotation; equivalently, the two triangles are congruent.



Proof of "if" direction. Let O be the center of the rotation. Then O is equidistant form $\overline{YZ}$ and $\overline{PQ}$ by rotation; similarly for the other pairs of sides. So the bisectors meet at O. □

Proof of "only if" direction. Let O be the concurrency point. Let $\angle PAB = \angle RCD = \angle QEF = 2\theta$ Since $\angle APC = \angle AYC = 60^\circ$ the quadrilateral $PACY$ is cyclic. But $\angle PAO + \angle PCO = (90^\circ + \theta) + (90^\circ - \theta) = 180^\circ$, so $PAOCY$ is cyclic.

Now $\angle PAO = \angle YCO \implies OP = OY$, and $\angle POY = 2\theta$. So O is the center of rotation from $\triangle PQR$ to $\triangle YXZ$ with angle 2θ. $\square$

§2 Solutions to Day 2

§2.1 EGMO 2020/4, proposed by Patrik Bak (SVK)

Available online at https://aops.com/community/p14780296.

Problem statement

A permutation of the integers $1, 2, \dots, m$ is called *fresh* if there exists no positive integer $k < m$ such that the first k numbers in the permutation are $1, 2, \dots, k$ in some order. Let f_m be the number of fresh permutations of the integers $1, 2, \dots, m$. Prove that $f_n \ge n \cdot f_{n-1}$ for all $n \ge 3$.

For every fresh permutation on $(1, 2, \dots, n-1)$ we generate n fresh permutations on $(1, 2, \dots, n)$ in the following way:

- Insert n in the kth position for $k = 1, 2, \dots, n-1$;
- Replace $n-1$ with n and append $n-1$ to the end.

For example, 3142 would generate 53142, 35142, 31542, 31452 and 31524.

All permutations generated this way are distinct. Indeed, the only thing to note is that the second type of permutation yields a non-fresh permutation when n is deleted (because $n-1$ is at the end, and $n \ge 3$). This implies the result.

§2.2 EGMO 2020/5, proposed by Agnijo Banerjee (UNK)

Available online at https://aops.com/community/p14780281.

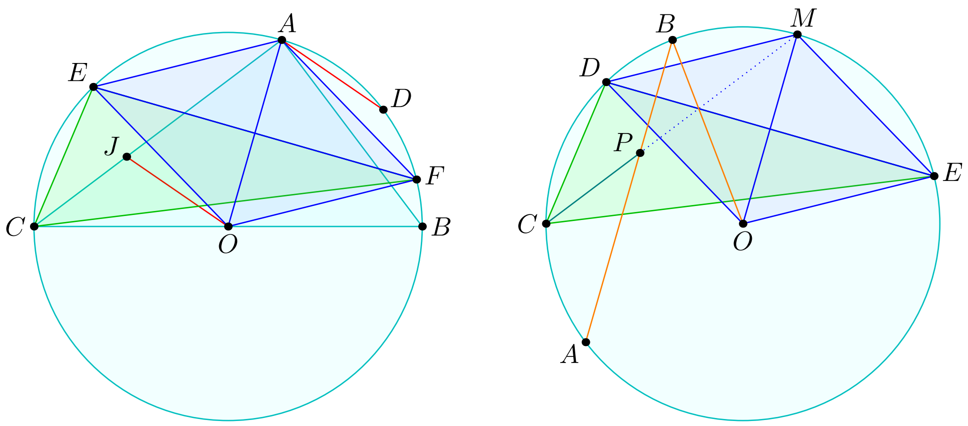
Problem statement

Triangle ABC has circumcircle Γ and obeys $\angle BCA > 90^\circ$. There is a point P in the interior of the line segment AB such that $PB = PC$ and the length of PA equals the radius of Γ. The perpendicular bisector of $\overline{PB}$ intersects Γ at the points D and E. Prove P is the incenter of triangle CDE.

Let O be the center of Γ and M the arc midpoint of $\overline{DE}$.

Claim — Quadrilateral $APMO$ is a rhombus.

Proof. Since $PA = MO$ and both are perpendicular to $\overline{DE}$, it follows $APMO$ is a parallelogram. In fact though, because $AO = MO$, we get the rhombus. □



IMO 2002/2 EGMO 2020/5

Since $PC = PB$, and $PA = PM$, it follows that C, P, M are collinear now.

Claim — We have $MP = MD = ME$ equal to the radius of Γ.

Proof. Note $PBMO$ is an isosceles trapezoid. Since $\overline{DE}$ is the perpendicular bisector of $\overline{PB}$, it is the perpendicular bisector of $\overline{OM}$ too. Hence $MDOE$ is a rhombus (with 60° internal angles), the end. □

Since $MP = MD = ME$ we are done by Fact 5.

Remark. The same figure appears in IMO 2002/2, drawn on the left for contrast.

§2.3 EGMO 2020/6, proposed by Mads Bjerge Christensen (DEN)

Available online at https://aops.com/community/p14780304.

Problem statement

Find all integers $m > 1$ for which the sequence $(a_n)_{n\ge 1}$ defined recursively by

$$a_{n+2} = m(a_{n+1} + a_n) - a_{n-1}$$

with initial conditions $a_1 = a_2 = 1$ and $a_3 = 4$ contains only perfect squares.

The answer is $m = 2$ and $m = 10$. The verification they work is left as an exercise. Now compute the first several terms:

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 1 \\ a_3 &= 4 \\ a_4 &= 5m - 1 \\ a_5 &= 5m^2 + 3m - 1 \\ a_6 &= 5m^3 + 8m^2 - 2m - 4. \end{aligned}$$

We will now prove:

Claim — If $a_4 \cdot a_6$ is a square then $m \in \{2, 10\}$.

Proof. A computation gives

$$\begin{aligned} 16a_4a_6 &= 400m^4 + 560m^3 - 288m^2 - 288m + 64 \\ &= \left(20m^2 + 14m - \frac{121}{10}\right)^2 + \frac{508}{10}m - \frac{8241}{100}. \end{aligned}$$

Let $A = 200m^2 + 140m - 121 \equiv -1 \pmod{20}$. Then $42A + 441 > 5080m - 8241$ for all m and hence

$$A^2 < \underbrace{A^2 + 5080m - 8241}_{=1600a_3a_5} < (A + 21)^2.$$

But the inner term is the square of a multiple of 20 so it must equal $(A+1)^2$. Thus, we have

$$2(\underbrace{200m^2 + 140m - 121}_{=A}) + 1 = 5080m - 8241 \implies m \in \{2, 10\}$$

as desired. □

Remark. In general, if $f(x) \in \mathbb{Z}[x]$ has even degree and the leading coefficient is a square, then $f(x)$ should be a square finitely often (unless f is itself the square of a polynomial). The proof proceeds along the same lines, by bounding f between two squares. See China TST 2001 for example which asked students to determine all x for which

$$f(x) = x^6 + 15x^5 + 85x^4 + 225x^3 + 274x^2 + 120x + 1$$

> was equal to a perfect square.