EGMO 2019 Solution Notes

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This is a compilation of solutions for the 2019 EGMO. The ideas of the solution are a mix of my own work, the solutions provided by the competition organizers, and solutions found by the community. However, all the writing is maintained by me.

These notes will tend to be a bit more advanced and terse than the “official” solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered “standard”, then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like “let $\mathbb{R}$ denote the set of real numbers” are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. Find all triples (a, b, c) of real numbers such that $ab + bc + ca = 1$ and

$$a^2b + c = b^2c + a = c^2a + b.$$

2. Let n be a positive integer. Dominoes are placed on a $2n \times 2n$ board in such a way that every cell of the board is (orthogonally) adjacent to exactly one cell covered by a domino. For each n, determine the largest number of dominoes that can be placed in this way.

3. Let ABC be a triangle such that $\angle CAB > \angle ABC$, and let I be its incenter. Let D be the point on segment BC such that $\angle CAD = \angle ABC$. Let ω be the circle tangent to AC at A and passing through I. Let X be the second point of intersection of ω and the circumcircle of ABC. Prove that the angle bisectors of $\angle DAB$ and $\angle CXB$ intersect at a point on line BC.

4. Let ABC be a triangle with incentre I. The circle through B tangent to AI at I meets side AB again at P. The circle through C tangent to AI at I meets side AC again at Q. Prove that PQ is tangent to the incircle of ABC.

5. Let $n \geq 2$ be an integer, and let $a_1, a_2, \dots, a_n$ be positive integers. Show that there exist positive integers $b_1, b_2, \dots, b_n$ satisfying the following three conditions:

(a) $a_i \leq b_i$ for $i = 1, 2, \dots, n$;

(b) the remainders of b_1, b_2, $\dots$, b_n on division by n are pairwise different,

(c) $b_1 + \dots + b_n \leq n\left(\frac{n-1}{2} + \left\lfloor \frac{a_1+\dots+a_n}{n} \right\rfloor\right)$.

6. On a circle, Alina draws 2019 chords, the endpoints of which are all different. A point is considered marked if it is either

(i) one of the 4038 endpoints of a chord; or

(ii) an intersection point of at least two chords.

Of the 4038 points meeting criterion (i), Alina labels 2019 points with a 0 and the other 2019 points with a 1. She labels each point meeting criterion (ii) with an arbitrary integer (not necessarily positive).

Along each chord, Alina considers the segments connecting two consecutive marked points. (A chord with k marked points has $k - 1$ such segments.) She labels each such segment in yellow with the sum of the labels of its two endpoints and in blue with the absolute value of their difference. Alina finds that the $N + 1$ yellow labels take each value $0, 1, \dots, N$ exactly once. Show that at least one blue label is a multiple of 3.

§1 Solutions to Day 1

§1.1 EGMO 2019/1, proposed by Netherlands

Available online at https://aops.com/community/p12141493.

Problem statement

Find all triples (a, b, c) of real numbers such that $ab + bc + ca = 1$ and

$$a^2b + c = b^2c + a = c^2a + b.$$

Answer: $(\pm 1/\sqrt{3}, \pm 1/\sqrt{3}, \pm 1/\sqrt{3})$ where the signs correspond, and $(\pm 1, \pm 1, 0)$ and permutations where the signs correspond. These work and we prove that is all.

We begin by eliminating the condition via homogenization: the first equality now reads

$$\begin{aligned} a^2b + c\,[ab + bc + ca] &= b^2c + a\,[ab + bc + ca] \\ \iff c^2(b + a) &= c(b^2 + a^2) \\ \iff c = 0 \text{ or } a^2 + b^2 &= c(a + b). \end{aligned}$$

Cyclic variations hold. So we have two cases.

- If any of the variables is zero, say $a = 0$, then the other two are nonzero. So from $b^2 = bc$ we get $b = c$ giving $(\pm 1, \pm 1, 0)$.
- Now assume all three variables are nonzero, so $a^2 + b^2 = c(a + b)$. If we sum cyclically we get

  $$2(a^2 + b^2 + c^2) = 2(ab + bc + ca) \iff (a - b)^2 + (b - c)^2 + (c - a)^2 = 0$$

  which forces $a = b = c$ and gives the last solution.

§1.2 EGMO 2019/2, proposed by Luxembourg

Available online at https://aops.com/community/p12141498.

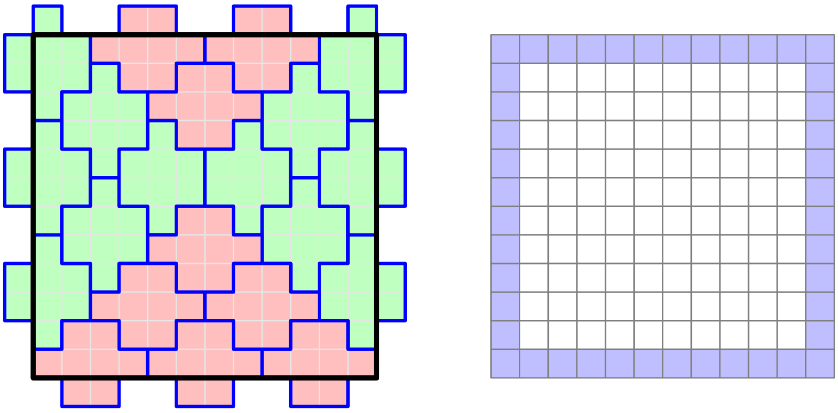
Problem statement

Let n be a positive integer. Dominoes are placed on a $2n \times 2n$ board in such a way that every cell of the board is (orthogonally) adjacent to exactly one cell covered by a domino. For each n, determine the largest number of dominoes that can be placed in this way.

The answer is $\binom{n+1}{2}$ and a construction is shown below.

For each domino, its *aura* consists of all the cells which are adjacent to a cell of the domino. There are up to eight squares in each aura, but some auras could be cut off by the boundary of the board, which means that there could be as few as five squares. So we want to estimate how many auras get cut off.

We denote by a, b, c, k the number of auras with 5, 6, 7, 8 cells on the boundary, so we want an upper bound on $a+b+c+k$. Note also that $\boxed{a \le 4}$ since such a cell uses a corner of the grid.



The big observation about the auras on the edge:

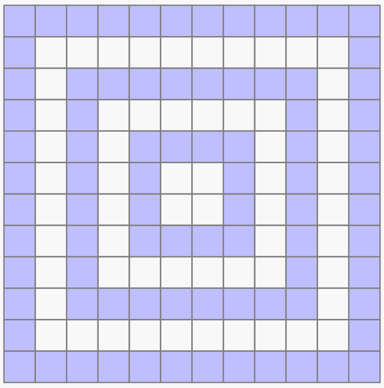
Claim — The auras of type a, b, c have 4, 4, and 3-4 cells on the boundary of the grid, respectively. (The boundary is the $4(2n-1)$ cells touching an edge of the board.)

Consequently, we have a bound $4a+4b+3c \le 4(2n-1)$. On the other hand, we obviously have $5a+6b+7c+8k = 4n^2$. Therefore,

$$\begin{aligned} 4n^2 + 2(2n-1) &\ge (5a+6b+7c+8k) + (2a+2b+1.5c) \\ &= 8(a+b+c+k) + 0.5c - a \\ &\ge 8(a+b+c+k) + 0 - 4 \\ \implies \frac{n(n+1)}{2} + \frac{1}{4} &\ge a+b+c+k \end{aligned}$$

which implies the desired bound after taking the floor of left-hand side.

Remark. In fact IMO 1999/3 shows the reverse bound: we now give a proof that every tiling in this problem has *exactly* $\frac{1}{2}n(n+1)$ dominoes. Color the board as shown below into "rings".



Every aura covers exactly four blue cells. Done.

§1.3 EGMO 2019/3, proposed by Poland

Available online at https://aops.com/community/p12141505.

Problem statement

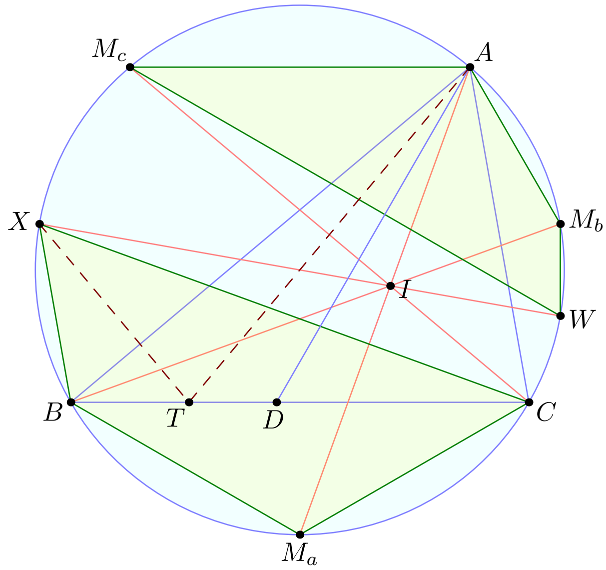
Let ABC be a triangle such that $\angle CAB > \angle ABC$, and let I be its incenter. Let D be the point on segment BC such that $\angle CAD = \angle ABC$. Let ω be the circle tangent to AC at A and passing through I. Let X be the second point of intersection of ω and the circumcircle of ABC. Prove that the angle bisectors of $\angle DAB$ and $\angle CXB$ intersect at a point on line BC.

Here is a cross-ratio/trig solution: rare for me to do one of these, but this problem called out to me that way. As usual, let $\alpha = \angle BAC$, etc.

Let T be the foot of the bisector of $\angle BAD$ on $\overline{BD}$, so that

$$\frac{TB}{TC} = \frac{AB \sin \angle BAT}{AC \sin \angle CAT} = \frac{AB \sin \frac{\alpha-\beta}{2}}{AC \sin \frac{\alpha+\beta}{2}}.$$

Also, call (ABC) by Γ and let ray XI meet Γ again at W, meaning that $\angle WXA = \angle IXA = \angle IAC = \frac{\beta}{2}$, since we assume $\overline{AC}$ was tangent to (IAX). Thus arc $\widehat{AW}$ also has measure β.



Now,

$$\begin{aligned} \frac{XB}{XC} &= -(BC;XM_a)_\Gamma \overset{I}{=} -(M_bM_c;WA)_\Gamma \\ &= -\frac{\sin \frac{1}{2}\mathrm{m}\widehat{M_bW}}{\sin \frac{1}{2}\mathrm{m}\widehat{M_cW}} \div \frac{\sin \frac{1}{2}\mathrm{m}\widehat{M_bA}}{\sin \frac{1}{2}\mathrm{m}\widehat{M_cA}} = \frac{\sin \frac{\alpha-\beta}{2}}{\sin \frac{\alpha+\gamma}{2}} \div \frac{\sin \frac{\gamma}{2}}{\sin \frac{\beta}{2}} \end{aligned}$$

Therefore,

$$\frac{XB}{XC} \div \frac{TB}{TC} = \frac{\sin \frac{\alpha+\beta}{2} \sin \frac{\gamma}{2}}{\sin \frac{\alpha+\gamma}{2} \sin \frac{\beta}{2}} \div \frac{AB}{AC} = \frac{2\cos \frac{\gamma}{2} \sin \frac{\gamma}{2}}{2\cos \frac{\beta}{2} \sin \frac{\beta}{2}} \div \frac{AB}{AC} = 1$$

the last step being the law of sines $AB/AC = \sin\gamma/\sin\beta$.

§2 Solutions to Day 2

§2.1 EGMO 2019/4, proposed by Poland

Available online at https://aops.com/community/p12146839.

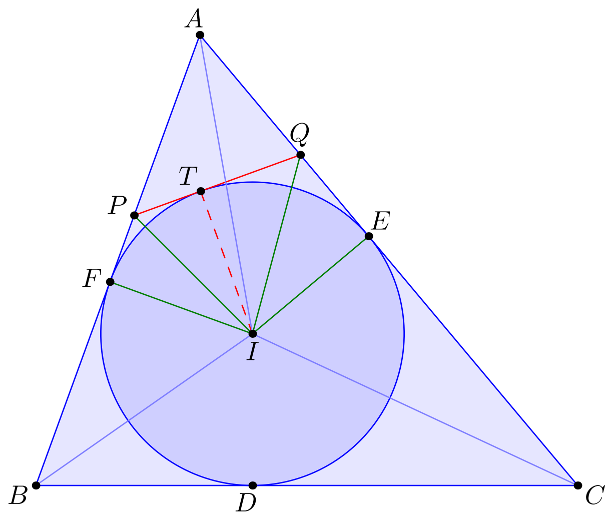
Problem statement

Let ABC be a triangle with incentre I. The circle through B tangent to AI at I meets side AB again at P. The circle through C tangent to AI at I meets side AC again at Q. Prove that PQ is tangent to the incircle of ABC.

Let E and F be the tangency points of the incircle on AC and AB. By angle chasing,

$$\angle PIF = \angle AIF - \angle AIP = \left(90^\circ - \frac{1}{2}\angle A\right) - \frac{1}{2}\angle B = \frac{1}{2}\angle C.$$

Similarly, $\angle EIQ = \frac{1}{2}\angle B$.



Let T_p, T_q denote the second tangency of P, Q to the incircle. Then $\angle EIT_q = \angle B$ and $\angle T_pIF = \angle C$. Since $\angle EIF = \angle B + \angle C$, it follows $T_p = T_q$.

Remark. Notice we really only need $\angle PIQ = 90^\circ - \frac{1}{2}\angle A$ by essentially the same argument.

§2.2 EGMO 2019/5, proposed by Merlijn Staps (NLD)

Available online at https://aops.com/community/p12146863.

Problem statement

Let $n \ge 2$ be an integer, and let $a_1, a_2, \dots, a_n$ be positive integers. Show that there exist positive integers $b_1, b_2, \dots, b_n$ satisfying the following three conditions:

(a) $a_i \le b_i$ for $i = 1, 2, \dots, n$;

(b) the remainders of $b_1, b_2, \dots, b_n$ on division by n are pairwise different,

(c) $b_1 + \dots + b_n \le n\left(\frac{n-1}{2} + \left\lfloor \frac{a_1 + \dots + a_n}{n} \right\rfloor\right)$.

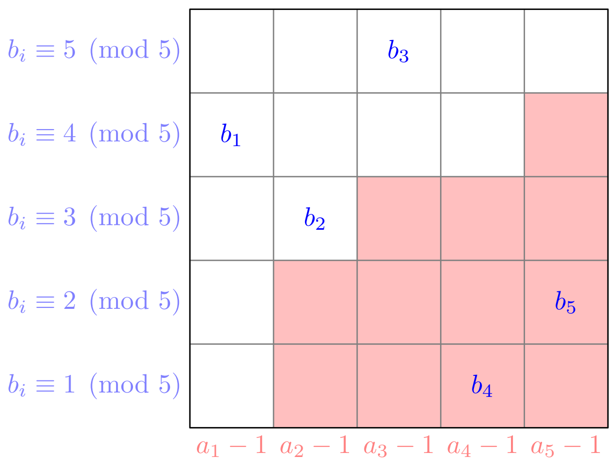
Note that if $a_i > n$, we can replace a_i with $a_i - n$ and b_i with $b_i - n$, and nothing changes. So we may as well assume $a_i \in \{1, \dots, n\}$ for each i.

We choose our b_i's in the following way. Draw an $n \times n$ grid and in the ith column fill in the bottom $a_i - 1$ cells red. We can select b_i by marking n cells, one in each row or column. If we chose jth lowest row in the ith column, then we would set $b_i = j$ on non-red cells and $b_i = j + n$ on red cells.

In this way, define the *penalty* p as the number of selected cells which are red. Then

$$b_1 + \dots + b_n = (1 + 2 + \dots + n) + n \cdot p = n \cdot \frac{n-1}{2} + n \cdot (p+1).$$

and we seek to minimize the penalty p.



But the expected penalty of a *random* permutation is the red area divided by n, which is

$$\mathbb{E}[p] = \frac{(a_1 - 1) + \dots + (a_n - 1)}{n}$$

and so there exists a choice for which the penalty is at most $\lfloor \mathbb{E}[p] \rfloor$. This gives the required result.

Remark. The visual aid can be excised from the solution; which can then be rewritten more tersely as follows. After assuming $1 \le a_i \le n$ for each n, pick a uniformly random permutation σ on $\{1, \dots, n\}$ and define

$$b_i = \begin{cases} n + \sigma(i) & a_i > \sigma(i) \\ \sigma(i) & \text{otherwise.} \end{cases}$$

As before $\mathbb{E}[e_\sigma] = \sum_{i=1}^n \frac{a_i - 1}{n}$ and the rest is the same.

§2.3 EGMO 2019/6, proposed by United Kingdom

Available online at https://aops.com/community/p12146850.

Problem statement

On a circle, Alina draws 2019 chords, the endpoints of which are all different. A point is considered marked if it is either

(i) one of the 4038 endpoints of a chord; or

(ii) an intersection point of at least two chords.

Of the 4038 points meeting criterion (i), Alina labels 2019 points with a 0 and the other 2019 points with a 1. She labels each point meeting criterion (ii) with an arbitrary integer (not necessarily positive).

Along each chord, Alina considers the segments connecting two consecutive marked points. (A chord with k marked points has $k-1$ such segments.) She labels each such segment in yellow with the sum of the labels of its two endpoints and in blue with the absolute value of their difference. Alina finds that the $N+1$ yellow labels take each value $0, 1, \dots, N$ exactly once. Show that at least one blue label is a multiple of 3.

Only the labels mod 3 matter at all.

Assume for contradiction no blue labels are divisible by 3. Let e_{ij} denote the number of segments joining $i \pmod 3$ to $j \pmod 3$. By double-counting (noting that points in (ii) are counted an even number of times but points in (i) are counted once) we derive that

$$\begin{aligned} e_{01} + e_{02} &\equiv 2019 \pmod 2 \\ e_{01} + e_{12} &\equiv 2019 \pmod 2 \\ e_{02} + e_{12} &\equiv 0 \pmod 2 \end{aligned}$$

which gives

$$e_{02} \equiv e_{12} \equiv 1 - e_{01} \pmod 2.$$

However, one can check this is incompatible with the hypothesis that the yellow labels are 0, 1, ..., N.

Remark. In addition, we can replace the points/segments by any graph G for which there are

- an odd number of leaves (or just odd-degree vertices) labeled 0,
- an odd number of leaves (or just odd-degree vertices) labeled 1,
- and the remaining vertices have even degree.

Thus the geometry of the problem is smoke and mirrors, too.