EGMO 2017 Solution Notes

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This is a compilation of solutions for the 2017 EGMO. The ideas of the solution are a mix of my own work, the solutions provided by the competition organizers, and solutions found by the community. However, all the writing is maintained by me.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

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§0 Problems

1. Let $ABCD$ be a convex quadrilateral with $\angle DAB = \angle BCD = 90^\circ$ and $\angle ABC > \angle CDA$. Let Q and R be points on segments BC and CD, respectively, such that line QR intersects lines AB and AD at points P and S, respectively. It is given that $PQ = RS$. Let the midpoint of BD be M and the midpoint of QR be N. Prove that the points M, N, A and C lie on a circle.

2. Find the smallest positive integer k for which there exists a coloring of the positive integers $\mathbb{Z}_{>0}$ with k colors and a function $f \colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ with the following two properties:
   (i) For all positive integers m, n of the same color, $f(m+n) = f(m) + f(n)$.
   (ii) There are positive integers m, n such that $f(m+n) \neq f(m) + f(n)$.

3. There are 2017 lines in the plane such that no three of them go through the same point. Turbo the snail sits on a point on exactly one of the lines and starts sliding along the lines in the following fashion: she moves on a given line until she reaches an intersection of two lines. At the intersection, she follows her journey on the other line turning left or right, alternating her choice at each intersection point she reaches. She can only change direction at an intersection point. Can there exist a line segment through which she passes in both directions during her journey?

4. Let $n \geq 1$ be an integer and let $t_1 < t_2 < \cdots < t_n$ be positive integers. In a group of $t_n + 1$ people, some games of chess are played. Two people can play each other at most once. Prove that it is possible for the following two conditions to hold at the same time:
   (i) The number of games played by each person is one of $t_1, t_2, \dots, t_n$.
   (ii) For every i with $1 \leq i \leq n$, there is someone who has played exactly t_i games of chess.

5. An n-tuple $(a_1, a_2, \dots, a_n)$ of positive integers is *expensive* if
$$(a_1 + a_2)(a_2 + a_3) \dots (a_{n-1} + a_n)(a_n + a_1) = 2^{2k-1}$$
for some positive integer k.
   (a) Find all integers $n \geq 2$ for which there exists an expensive n-tuple.
   (b) Prove that each odd integer $m \geq 1$ appears in an expensive n-tuple for some $n \geq 2$.

6. Let ABC be an acute-angled triangle in which no two sides have the same length. The reflections of the centroid G and the circumcenter O of ABC in its sides BC, CA, AB are denoted by G_1, G_2, G_3 and O_1, O_2, O_3, respectively. Show that the circumcircles of triangles G_1G_2C, G_1G_3B, G_2G_3A, O_1O_2C, O_1O_3B, O_2O_3A and ABC have a common point.

§1 Solutions to Day 1

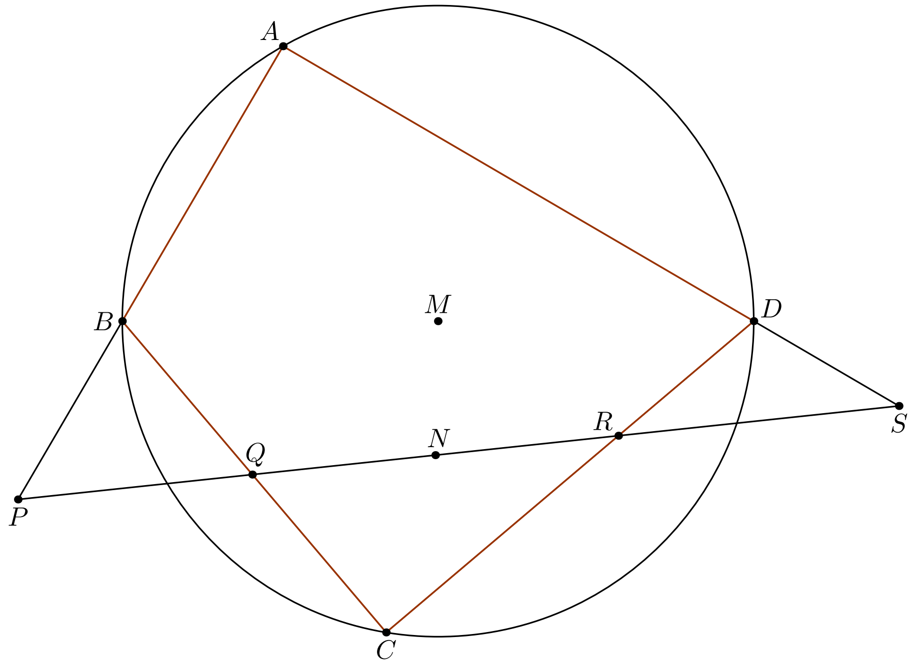
§1.1 EGMO 2017/1, proposed by Mark Mordechai Etkind (ISR)

Available online at https://aops.com/community/p8024554.

Problem statement

Let $ABCD$ be a convex quadrilateral with $\angle DAB = \angle BCD = 90^\circ$ and $\angle ABC > \angle CDA$. Let Q and R be points on segments BC and CD, respectively, such that line QR intersects lines AB and AD at points P and S, respectively. It is given that $PQ = RS$. Let the midpoint of BD be M and the midpoint of QR be N. Prove that the points M, N, A and C lie on a circle.

The condition is equivalent to N being the midpoint of both $\overline{PS}$ and $\overline{QR}$ simultaneously. (Thus triangles BAD and BCD play morally dual roles.)



The rest is angle chasing. We have

$$\begin{aligned}
\measuredangle ANC &= \measuredangle ANP + \measuredangle QNC \\
&= 2\measuredangle ASP + 2\measuredangle QRC \\
&= 2\measuredangle DSR + 2\measuredangle DRS = 2\measuredangle RDS \\
&= 2\measuredangle ADC = \measuredangle AMC.
\end{aligned}$$

§1.2 EGMO 2017/2, proposed by Merlijn Staps (NLD)

Available online at https://aops.com/community/p8024575.

Problem statement

Find the smallest positive integer k for which there exists a coloring of the positive integers $\mathbb{Z}_{>0}$ with k colors and a function $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ with the following two properties:

(i) For all positive integers m, n of the same color, $f(m+n) = f(m) + f(n)$.

(ii) There are positive integers m, n such that $f(m+n) \neq f(m) + f(n)$.

Answer: $k = 3$.

Construction for $k = 3$: let

$$f(n) = \begin{cases} n/3 & n \equiv 0 \pmod 3 \\ n & \text{else} \end{cases}$$

and color the integers modulo 3.

Now we prove that for $k = 2$ a function f obeying (i) must be linear, even if $f\colon \mathbb{Z}_{>0} \to \mathbb{R}_{>0}$. Call the colors blue/red and WLOG $f(1) = 1$.

First, we obviously have:

Claim — $f(2n) = 2f(n)$ for every n.

Now we proceed by induction in the following way. Assume that $f(1) = 1$, $f(2) = 2$, $\dots$, $f(2n) = 2n$. For brevity let $m = 2n + 1$ be red and assume for contradiction that $f(m) \neq m$.

The proof now proceeds in four steps. First:

- The number $m-2$ must be blue. Indeed if $m-2$ was red we would have $f(2m-2) = f(m) + f(m-2)$ which is a contradiction as $f(2m-2) = 2f(m-1) = 2m-2$ and $f(m-2) = m-2$.
- The number 2 must be red. Indeed if it was blue then $f(m) = f(2) + f(m-2) = m$ contradiction.

Observe then that $f(m+2) = f(m) + 2$ since m and 2 are both red. Now we consider two cases:

- If $m+2$ is red, then $f(2m+2) = f(m+2) + f(m) = (f(m)+2) + f(m)$. But $f(2m+2) = f(4n+4) = 4f(n+1) = 2m+2$, contradiction.
- If $m+2$ is blue, then $2f(m) = f(2m) = f(m+2) + f(m-2) = f(m) + 2 + (m-2)$. So then $f(m) = m$ again a contradiction.

So $f(m) = m$, which completes the induction.

§1.3 EGMO 2017/3, proposed by Márk Di Giovanni (HUN)

Available online at https://aops.com/community/p8024557.

Problem statement

There are 2017 lines in the plane such that no three of them go through the same point. Turbo the snail sits on a point on exactly one of the lines and starts sliding along the lines in the following fashion: she moves on a given line until she reaches an intersection of two lines. At the intersection, she follows her journey on the other line turning left or right, alternating her choice at each intersection point she reaches. She can only change direction at an intersection point. Can there exist a line segment through which she passes in both directions during her journey?

Color the regions of the plane black and white in alternating colors. Then:

Claim — Turbo will always move in one orientation around black regions and the other orientation around white regions.

This completes the proof, even if Turbo may pick which of left/right she follows each time!

Remark. An example of a cyclic path: take a pentagon and extend its sides to form a star. Then Turbo can trace around the exterior of the star.

§2 Solutions to Day 2

§2.1 EGMO 2017/4, proposed by Gerhard Wöginger (LUX)

Available online at https://aops.com/community/p8029369.

Problem statement

Let $n \geq 1$ be an integer and let $t_1 < t_2 < \cdots < t_n$ be positive integers. In a group of $t_n + 1$ people, some games of chess are played. Two people can play each other at most once. Prove that it is possible for the following two conditions to hold at the same time:

(i) The number of games played by each person is one of $t_1, t_2, \ldots, t_n$.

(ii) For every i with $1 \leq i \leq n$, there is someone who has played exactly t_i games of chess.

Phrased in graph theory, the problem asks to produce a simple graph G on $t_n + 1$ vertices such that all degrees are in the set $\{t_1, \ldots, t_n\}$ and each degree appears at least once.

We proceed by induction on n. If $n = 1$, take a clique on $t_1 + 1$ vertices. For $n = 2$, take a clique on t_1 vertices and an empty graph on $t_2 + 1 - t_1$ vertices, and join them all together. For the inductive step:

- Take an example for the $(n-2)$ tuple $(t_2 - t_1, \ldots, t_{n-1} - t_1)$, which has $t_{n-1} - t_1 + 1$ vertices.
- Then add in $t_n - t_{n-1}$ isolated vertices.
- Finally add in t_1 universal vertices.

> **Remark.** The universal vertices are "forced", so the only parameter is the number of universal vertices to add in.

§2.2 EGMO 2017/5, proposed by Harun Hindija (BIH)

Available online at https://aops.com/community/p8029376.

Problem statement

An n-tuple $(a_1, a_2, \dots, a_n)$ of positive integers is *expensive* if

$$(a_1 + a_2)(a_2 + a_3) \dots (a_{n-1} + a_n)(a_n + a_1) = 2^{2k-1}$$

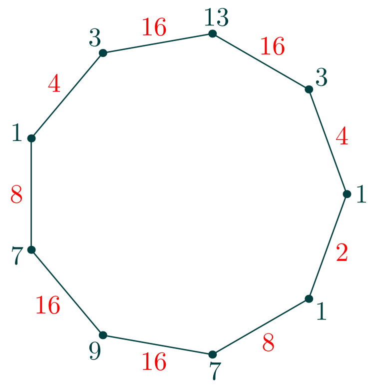
for some positive integer k.

(a) Find all integers $n \ge 2$ for which there exists an expensive n-tuple.

(b) Prove that each odd integer $m \ge 1$ appears in an expensive n-tuple for some $n \ge 2$.

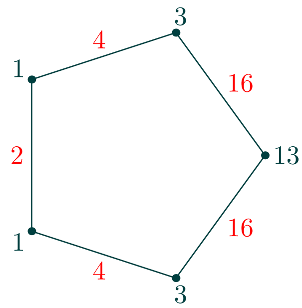
For part (a), the answer is odd n, with construction given by setting $a_1 = \dots = a_n = 1$. We show by induction that even n all fail, with $n = 2$ being plain.

Construct a regular n-gon whose vertices are labeled a_i and edges are labeled with $a_i + a_j$.



Then we observe that it's impossible for an edge to exceed both of its neighbors, since if $a + b = 2^e$ then $\min(a + 1, b + 1) > 2^{e-1}$. Consequently we can take two adjacent edges $a + b = b + c$ with the same label; hence $a = c$. Then delete b, c and induct down.

For part (b), for odd integers m, let $f(m)$ denote $2^k - m$ where 2^k is the smallest power of two exceeding m. Note $f(m) \le m$ with equality if and only if $m = 1$. So repeatedly apply f until we wrap around; this gives twice a square of a power of two. Example when $m = 13$:



Remark. By applying the process of (a) in reverse, one essentially finds an inductive characterization of all expensive n-tuples.

§2.3 EGMO 2017/6, proposed by Charles Leytem (LUX)

Available online at https://aops.com/community/p8029388.

Problem statement

Let ABC be an acute-angled triangle in which no two sides have the same length. The reflections of the centroid G and the circumcenter O of ABC in its sides BC, CA, AB are denoted by G_1, G_2, G_3 and O_1, O_2, O_3, respectively. Show that the circumcircles of triangles G_1G_2C, G_1G_3B, G_2G_3A, O_1O_2C, O_1O_3B, O_2O_3A and ABC have a common point.

Here is an approach with complex numbers. Let P be an arbitrary point. Let P_B and P_C be the reflections of P across AB and AC and let Q_B and Q_C be the second intersections of lines AP_B and AP_C with the circumcircle. Then we will compute the intersection of (AP_BP_C) and $(AQ_BQ_C) \equiv (ABC)$.

We have $p_B = a + c - ac\overline{p}$, $p_C = a + b - ab\overline{p}$. To compute q_B note that

$$\begin{aligned} a + q_B &= p_B + aq_B\overline{p_B} \\ &= a + c - ac\overline{p} + aq_B\left(1/a + 1/c - p/ac\right) \\ \implies ac\overline{p} - c &= (a/c - p/c)q_B \\ \implies q_B &= c^2\frac{a\overline{p}-1}{a-p}. \end{aligned}$$

Similarly $q_C = b^2\frac{a\overline{p}-1}{a-p}$. Then, the desired intersection is

$$\begin{aligned} \frac{p_Bq_C - p_Cq_B}{p_B - p_C + q_C - q_B} &= \frac{\left(\frac{a\overline{p}-1}{a-p}\right)\left(b^2(a+c-ac\overline{p}) - c^2(a+b-ab\overline{p})\right)}{(b-c)(a\overline{p}-1) + (b^2-c^2)\cdot\frac{a\overline{p}-1}{a-p}} \\ &= \frac{b^2(a+c-ac\overline{p}) - c^2(a+b-ab\overline{p})}{(b-c)(a-p) + (b^2-c^2)} \\ &= \frac{(b-c)(a(b+c)+bc) - (b-c)abc\overline{p}}{(a-p)+(b+c)} \\ &= \frac{ab+bc+ca-abc\overline{p}}{a+b+c-p} \end{aligned}$$

which is in any case symmetric in a, b, c. Moreover taking $p = \frac{1}{3}(a+b+c)$ and $p = 0$ give the same numbers (and indeed any p on the Euler line).