

EGMO 2016 Solution Notes

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This is a compilation of solutions for the 2016 EGMO. The ideas of the solution are a mix of my own work, the solutions provided by the competition organizers, and solutions found by the community. However, all the writing is maintained by me.

These notes will tend to be a bit more advanced and terse than the “official” solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered “standard”, then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like “let \mathbb{R} denote the set of real numbers” are typically omitted entirely.

Corrections and comments are welcome!

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§0 Problems

1. Let n be an odd positive integer, and let x_1, x_2, \dots, x_n be nonnegative real numbers. Show that

$$\min(x_i^2 + x_{i+1}^2) \leq \max(2x_j x_{j+1})$$

where $1 \leq i, j \leq n$ and $x_{n+1} = x_1$.

2. Let $ABCD$ be a cyclic quadrilateral, and let diagonals AC and BD intersect at X . Let C_1, D_1 and M be the midpoints of segments CX, DX and CD , respectively. Lines AD_1 and BC_1 intersect at Y , and line MY intersects diagonals AC and BD at different points E and F , respectively. Prove that line XY is tangent to the circle through E, F and X .
3. Let m be a positive integer. Consider a $4m \times 4m$ array of square unit cells. Two different cells are *related* to each other if they are in either the same row or in the same column. No cell is related to itself. Some cells are colored blue, such that every cell is related to at least two blue cells. Determine the minimum number of blue cells.
4. Two circles ω_1 and ω_2 , of equal radius intersect at different points X_1 and X_2 . Consider a circle ω externally tangent to ω_1 at T_1 and internally tangent to ω_2 at point T_2 . Prove that lines X_1T_1 and X_2T_2 intersect at a point lying on ω .
5. Let k and n be integers such that $k \geq 2$ and $k \leq n \leq 2k - 1$. Place rectangular tiles, each of size $1 \times k$ or $k \times 1$ on a $n \times n$ chessboard so that each tile covers exactly k cells and no two tiles overlap. Do this until no further tile can be placed in this way. For each such k and n , determine the minimum number of tiles that such an arrangement may contain.
6. Let S be the set of all positive integers n such that n^4 has a divisor in the range $n^2 + 1, n^2 + 2, \dots, n^2 + 2n$. Prove that there are infinitely many elements of S of each of the forms $7m, 7m + 1, 7m + 2, 7m + 5, 7m + 6$ and no elements of S of the form $7m + 3$ and $7m + 4$, where m is an integer.

§1 Solutions to Day 1

§1.1 EGMO 2016/1, proposed by Netherlands

Available online at <https://aops.com/community/p6171538>.

Problem statement

Let n be an odd positive integer, and let x_1, x_2, \dots, x_n be nonnegative real numbers. Show that

$$\min(x_i^2 + x_{i+1}^2) \leq \max(2x_j x_{j+1})$$

where $1 \leq i, j \leq n$ and $x_{n+1} = x_1$.

It suffices to exhibit a single pair (i, j) where

$$x_i^2 + x_{i+1}^2 \leq 2x_j x_{j+1}.$$

Since n is odd we can find three adjacent x_k 's, call them a, b , and c (with b in middle) such that $a \geq b \geq c$. Then indeed

$$2ab \geq b^2 + c^2$$

so we're done.

Remark. When n is even, counterexamples to the problem statement alternate up and down, such as $(1, 100, 1, 100, 1, 100, \dots)$. This provides a reason why one should consider $a \geq b \geq c$ to exploit the parity of n .

§1.2 EGMO 2016/2, proposed by Belarus

Available online at <https://aops.com/community/p6171514>.

Problem statement

Let $ABCD$ be a cyclic quadrilateral, and let diagonals AC and BD intersect at X . Let C_1, D_1 and M be the midpoints of segments CX , DX and CD , respectively. Lines AD_1 and BC_1 intersect at Y , and line MY intersects diagonals AC and BD at different points E and F , respectively. Prove that line XY is tangent to the circle through E , F and X .

We present two approaches.

¶ **First approach through isogonality lemma.** Note ABC_1D_1 is cyclic. By the “isogonality lemma” applied to $\triangle YC_1D_1$, it follows that YX and YM are isogonal with respect to $\triangle YC_1D_1$. Then

$$\angle EXY = \angle XC_1Y + \angle C_1YX = \angle AD_1X + \angle MYA = \angle YFX$$

and we’re done (angles are directed).

Remark. Note that the points C and D can be more or less deleted immediately. The “isogonality lemma” people has mentioned has appeared in other places too. In addition to the Taiwan TST 2014 round 2 problem 6, it also shows itself in ELMO 2012/5, BrMO2 2013/2, SL 2009 G4, SL 2012 G2.

¶ **Solution with complex numbers, by Sanjana Das.** Quadrilateral ABC_1D_1 is also cyclic ($\angle AC_1D_1 = \angle ACD = \angle ABD_1$). We want $\angle YXE = \angle XFE$, or

$$\angle(XY, AC) = \angle(BD, MY).$$

Now use complex numbers with unit circle (ABC_1D_1) (using c and d to refer to C_1 and D_1 for brevity).

We want to check that $\frac{(x-y)(m-y)}{(a-c)(b-d)}$ is real. First, if $Z = AB \cap C_1D_1$, then $XY \perp OZ$ where O is the circumcenter of (ABC_1D_1) by Brokard’s Theorem, so we want to show

$$\frac{(ab(c+d) - cd(a+b)) \cdot (y-m)}{(a-c)(b-d)(ab-cd)} \in i\mathbb{R}.$$

Now we have $m = c + d - x$, which means

$$\begin{aligned} y - m &= x + y - c - d \\ &= \frac{ac(b+d) - bd(a+c)}{ac - bd} + \frac{ad(b+c) - bc(a+d)}{ad - bc} - c - d \\ &= \frac{ab(2acd + 2bcd + c^3 + d^3 - ac^2 - ad^2 - bc^2 - bd^2 - c^2d - cd^2)}{(ac - bd)(ad - bc)} \\ &= -\frac{ab(a+b-c-d)(c-d)^2}{(ac - bd)(ad - bc)}. \end{aligned}$$

This means our final expression is

$$-\frac{ab(ab(c+d) - cd(a+b))(a+b-c-d)(c-d)^2}{(ac-bd)(ad-bc)(a-c)(b-d)(ab-cd)},$$

which is visibly the negative of its conjugate and is therefore imaginary.

§1.3 EGMO 2016/3, proposed by Mexico

Available online at <https://aops.com/community/p6171553>.

Problem statement

Let m be a positive integer. Consider a $4m \times 4m$ array of square unit cells. Two different cells are *related* to each other if they are in either the same row or in the same column. No cell is related to itself. Some cells are colored blue, such that every cell is related to at least two blue cells. Determine the minimum number of blue cells.

We'll prove that the answer is at least $6m$. The construction is to take diagonal copies of the matrix

$$\begin{bmatrix} & 1 & 1 & 1 \\ 1 & & & \\ 1 & & & \\ 1 & & & \end{bmatrix}.$$

We present two proofs this is optimal.

¶ **Bipartite graph (Kevin Sun).** Consider the bipartite subgraph

$$H \subseteq K_{4m,4m}$$

with vertices being rows/columns, and edges corresponding blue squares.

Assume for contradiction there are fewer than $6m$ edges. Then there are at least $|V(H)| - |E(H)| = 8m - 6m = 2m$ connected components in H , so some connected component has at most three vertices.

We consider two cases:

- Note that if there are any isolated vertices v (connected components with no edges), then it follows H needs at least $8m$ edges, because every vertex on the opposite shore from v has blue-degree at least 2.
- Thus assume no isolated vertices exist. On the other hand, every blue edge is incident to at least two more blue edges, so no connected component can have fewer than three edges. As the connected components are bipartite, it follows each connected component has at least four vertices, which is impossible.

¶ **Brute force (mine).** We prove that for a general $n \times n$ board, a good configuration has at least $\frac{3}{2}n$ blue squares. Consider any minimal good configuration with k squares, and assume $k \leq \frac{3}{2}n$ (and we will prove $k \geq \frac{3}{2}n$). We observe that no column and row can be empty in a good configuration with $k \leq 2n$ squares.

Now, assume there are x_i columns with exactly i blue squares, and a_i rows with exactly

i blue squares (for $i = 1, \dots, n$.) Permute the board to arrive at the following:

	x_1 cols	x_2 cols	$x_3 + x_4 + \dots + x_n$ cols
a_1 rows	0	0	a_1
a_2 rows	0	P	Q
$a_3 + \dots + a_n$ rows	x_1	R	S

In each subgrid, the letter denotes the number of blue squares in that region. Consequently, we have $P + Q = 2a_2$, $P + R = 2x_2$, and $a_1 + Q + S = 3x_3 + 4x_4 + \dots + nx_n$. We also know that $n = \sum a_i = \sum x_i$ and $k = \sum ia_i = \sum ix_i$.

Without loss of generality we assume that $a_1 \leq x_1$. Then, it would suffice to prove that

$$\sum ia_i \geq \frac{3}{2} \sum a_i \iff a_1 \leq a_2 + 3a_3 + 5a_4 + \dots$$

We observe that

$$\begin{aligned} 0 &\leq S = 3x_3 + \dots + nx_n - a_1 - Q \\ &= (a_1 + 2a_2 + \dots + na_n) - (x_1 + 2x_2) - (a_1 - Q) \\ \implies x_1 &\leq (3a_3 + 4a_4 + \dots + na_n) - (Q - 2a_2 + 2x_2) \\ &= (3a_3 + 4a_4 + \dots + na_n) - R \\ &\leq 3a_3 + 4a_4 + \dots + na_n \\ &\leq a_2 + 3a_3 + 5a_4 + \dots + (2n - 3)a_n. \end{aligned}$$

Since we assumed $a_1 \leq x_1$, we are done.

§2 Solutions to Day 2

§2.1 EGMO 2016/4, proposed by Charles Leytem (LUX)

Available online at <https://aops.com/community/p6177803>.

Problem statement

Two circles ω_1 and ω_2 , of equal radius intersect at different points X_1 and X_2 . Consider a circle ω externally tangent to ω_1 at T_1 and internally tangent to ω_2 at point T_2 . Prove that lines X_1T_1 and X_2T_2 intersect at a point lying on ω .

We present two solutions, one by homothety and another by inversion.

¶ **First solution (homothety).** Consider the composition of homotheties

$$\omega_1 \xrightarrow{T_1} \omega \xrightarrow{T_2} \omega_2.$$

This is a negative homothety, and since ω_1 and ω_2 have equal radius, it follows that the composition is just reflection about the midpoint of $\overline{X_1X_2}$. In particular, it sends X_1 to X_2 , and thus $\overline{X_1T_1} \cap \overline{X_2T_2}$ is just the image of X_1 under the first homothety.

¶ **Second solution (inversion).** Invert around X_1 . Then we have the following:

- ω_1^* and ω_2^* are lines intersecting at X_2^* , and X_1 is a point on the external angle bisector of the two lines.
- The circle ω^* becomes tangent to the two lines (in a region adjacent to X_1).
- Let $P = \overline{T_1^*X_1} \cap \omega^*$. Then one wishes to show $X_1X_2^*PT_2^*$ are concyclic.

To show this, notice that

$$\angle T_2^*X_2^*X_1 = \angle T_1^*T_2^*X_2^* = \angle T_2^*PT_1^* = \angle T_2^*PX_1.$$

§2.2 EGMO 2016/5, proposed by Netherlands

Available online at <https://aops.com/community/p6177809>.

Problem statement

Let k and n be integers such that $k \geq 2$ and $k \leq n \leq 2k - 1$. Place rectangular tiles, each of size $1 \times k$ or $k \times 1$ on a $n \times n$ chessboard so that each tile covers exactly k cells and no two tiles overlap. Do this until no further tile can be placed in this way. For each such k and n , determine the minimum number of tiles that such an arrangement may contain.

The answer is

$$\begin{cases} n & n = k \text{ or } n = 2k - 1 \\ 2(n - k + 1) & k < n < 2k - 1. \end{cases}$$

Here are constructions.

- For $n = k$ it's trivial.
- For $k < n < 2k - 1$, observe that one can construct it when $n = k + 1$ by simply filling in the perimeter of the $(k + 1) \times (k + 1)$ square with four dominoes, then one can inductively obtain the next values $n = k + 2, k + 3, \dots$ by adding a new domino to each row and column, hence increasing the count by 2. An example with $(k, n) = (5, 8)$ is shown below.

$$\begin{bmatrix} w & w & w & w & w & x & a & b \\ y & & & & & x & a & b \\ y & & & & & x & a & b \\ y & & & & & x & a & b \\ y & & & & & x & a & b \\ y & z & z & z & z & z & & \\ c & c & c & c & c & & & \\ d & d & d & d & d & & & \end{bmatrix}$$

- For $n = 2k - 1$ one places a domino in each column, alternating between upper-most possible and lower-most possible. An example with $(k, n) = (3, 5)$ is shown below.

$$\begin{bmatrix} a & c & e \\ a & c & e \\ a & b & c & d & e \\ & b & d & & \\ & b & d & & \end{bmatrix}$$

Now we now prove this is optimal.

Call a row (resp column) *bare* if it contains no horizontal (resp vertical) domino contained completely inside it. The main observation is that:

Claim — The bare columns are consecutive, as are the bare rows.

Proof. Consider a vertical domino D (say) in some column C , and assume it is in the left half of the board. Then the column to the left of C must also contain a domino,

otherwise we could add the domino exactly to the left of D since there is not enough space to fit a horizontal domino to D 's left.

Inductively, then, every column left of C is not bare. Similarly, if C had been on the right half of the board, then all columns to the right of C are not bare. \square

To finish, we consider two cases.

- If there are no bare columns, then we need at least n dominoes, since each column contains a domino.

Similarly, if there are no bare rows, we also need at least n dominoes.

- In the case there exist at least one bare row or bare column, note that given k (consecutive) bare columns, then intersecting them with a bare row gives a place where we can add an additional domino, contradiction. So there are at most k bare columns, and hence at least $n - (k - 1) = n - k + 1$ vertical dominoes (one per non-bare column).

Similarly, there are at least $n - k + 1$ horizontal dominoes. So there are at least $2(n - k + 1)$ dominoes total.

This shows one always needs at least $\min(n, 2(n - k + 1))$ dominoes. This resolves every case except $k = n$, (in which case the bound is true but not achievable). However, the case $k = n$ is trivial — just note that all the dominoes must have the same orientation when $k = n$, so we are necessarily in the first case.

§2.3 EGMO 2016/6, proposed by Netherlands

Available online at <https://aops.com/community/p6177824>.

Problem statement

Let S be the set of all positive integers n such that n^4 has a divisor in the range $n^2 + 1, n^2 + 2, \dots, n^2 + 2n$. Prove that there are infinitely many elements of S of each of the forms $7m, 7m + 1, 7m + 2, 7m + 5, 7m + 6$ and no elements of S of the form $7m + 3$ and $7m + 4$, where m is an integer.

Start from the implication

$$n^2 + k \mid n^4 \iff n^2 + k \mid k^2.$$

Since $1 \leq k \leq 2n$, in fact the quotient $\frac{k^2}{n^2+k}$ can only take values from 1 to 3. In other words, S is the set of integers n for which at least one equation

$$\begin{aligned} n^2 + k &= k^2 \\ 2(n^2 + k) &= k^2 \\ 3(n^2 + k) &= k^2 \end{aligned}$$

has at least one solution $1 \leq k \leq 2n$.

The first equation has no solutions with $k \geq 1$ since we can put $(k-1)^2 < n^2 < k^2$. The other two are Pell equations, and one can check that $n^2 \equiv 2 \pmod{7}$ has no solutions at all for $k \pmod{7}$ in either case. The assertion about infinitely many solutions then follows by using the Pell recursion, and taking modulo 7.