EGMO 2014 Solution Notes

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This is a compilation of solutions for the 2014 EGMO. The ideas of the solution are a mix of my own work, the solutions provided by the competition organizers, and solutions found by the community. However, all the writing is maintained by me.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

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§0 Problems

1. Determine all real constants t such that whenever a, b and c are the lengths of sides of a triangle, then so are $a^2 + bct$, $b^2 + cat$, $c^2 + abt$.

2. Let D and E be points in the interiors of sides AB and AC, respectively, of a triangle ABC, such that $DB = BC = CE$. Let the lines CD and BE meet at F. Prove that the incenter I of triangle ABC, the orthocenter H of triangle DEF and the midpoint M of the arc BAC of the circumcircle of triangle ABC are collinear.

3. We denote the number of positive divisors of a positive integer m by $d(m)$ and the number of distinct prime divisors of m by $\omega(m)$. Let k be a positive integer. Prove that there exist infinitely many positive integers n such that $\omega(n) = k$ and $d(n)$ does not divide $d(a^2 + b^2)$ for any positive integers a, b satisfying $a + b = n$.

4. Determine all positive integers $n \geq 2$ for which there exist integers $x_1, x_2, \ldots, x_{n-1}$ satisfying the condition that if $0 < i < n$, $0 < j < n$, $i \neq j$ and n divides $2i + j$, then $x_i < x_j$.

5. Let n be a positive integer. We have n boxes where each box contains a non-negative number of pebbles. In each move we are allowed to take two pebbles from a box we choose, throw away one of the pebbles and put the other pebble in another box we choose. An initial configuration of pebbles is called *solvable* if it is possible to reach a configuration with no empty box, in a finite (possibly zero) number of moves. Determine all initial configurations of pebbles which are not solvable, but become solvable when an additional pebble is added to a box, no matter which box is chosen.

6. Solve over $\mathbb{R}$ the functional equation

$$f(y^2 + 2xf(y) + f(x)^2) = (y + f(x))(x + f(y)).$$

§1 Solutions to Day 1

§1.1 EGMO 2014/1, proposed by United Kingdom

Available online at https://aops.com/community/p3459747.

Problem statement

Determine all real constants t such that whenever a, b and c are the lengths of sides of a triangle, then so are a^2+bct, b^2+cat, c^2+abt.

¶ **Answer.** We will show the answer is exactly

$$2/3 \le t \le 2.$$

¶ **Proof.** Apply the substitution $a=y+z$, $b=z+x$, $c=x+y$. We need the inequality

$$\begin{aligned} a^2+bct &< b^2+cat+c^2+abt \\ (y+z)^2+(x+y)(x+z)t &< (x+y)^2+(x+z)^2+(y+z)(2x+y+z)t \\ \iff [x^2-(xy+xz+yz+y^2+z^2)]t &< 2x^2+2xy+2xz-2yz \end{aligned}$$

to be true for all $(x,y,z)\in\mathbb{R}^3_{>0}$ (checking only one inequality is enough by symmetry). Writing this is a quadratic in x, we want

$$Q(x) := (2-t)x^2+(2+t)(y+z)x+((y^2+yz+z^2)t-2yz)>0.$$

Claim — Except when $t=2$, the quadratic Q always has two real roots.

Proof. The discriminant is

$$\begin{aligned} D &= [(2+t)(y+z)]^2-4(2-t)\left((y^2+yz+z^2)t-2yz\right) \\ &= \left((2+t)^2-4(2-t)t\right)(y^2+z^2)+\left(2(2+t)^2-4(2-t)(t-2)\right)yz \\ &= \left(5t^2-4t+4\right)(y^2+z^2)+\left(6t^2-8t+24\right)yz>0 \end{aligned}$$

which means Q always has two real roots (aside for the single exceptional case $t=2$, in which Q is not a quadratic). □

Now, we make a few closing observations.

- It is clear we need $t\le 2$, since a negative leading coefficient will cause the inequality to fail for $x \gg y,z$.
- The case $t=2$ obviously has $Q>0$ always.
- For $t<2$, as Q always has two real roots, the assertion is true if and only if all coefficients of Q are nonnegative.

- When $t \geq \frac{2}{3}$, we have $(2+t)(y+z) > 0$ obviously, and from

$$\frac{y^2 + yz + z^2}{3} \geq yz$$

the constant coefficient is nonnegative as well. Thus when $\boxed{\frac{2}{3} \leq t \leq 2}$ we indeed have $Q(x) > 0$ for $x, y, z > 0$.

- If $t < \frac{2}{3}$, then by letting $y = z$ fails the condition.

§1.2 EGMO 2014/2, proposed by Ukraine

Available online at https://aops.com/community/p3459750.

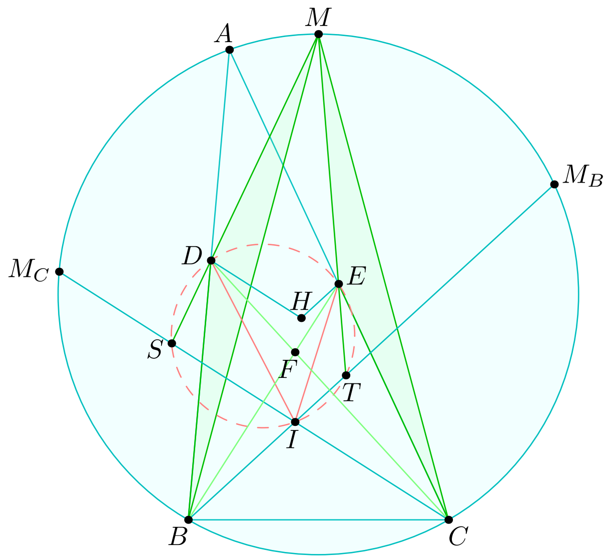
Problem statement

Let D and E be points in the interiors of sides AB and AC, respectively, of a triangle ABC, such that $DB = BC = CE$. Let the lines CD and BE meet at F. Prove that the incenter I of triangle ABC, the orthocenter H of triangle DEF and the midpoint M of the arc BAC of the circumcircle of triangle ABC are collinear.

¶ First solution (Cynthia Du). Let BI and CI meet the circumcircle again at M_B, M_C. Observe that we have the spiral congruence

$$\triangle MDB \sim \triangle MEC$$

from $\measuredangle MBD = \measuredangle MBA = \measuredangle MCA = \measuredangle MCE$ and $BD = EC$, $BM = CM$. That is, M is the Miquel point of $BDEC$.



Let $T = \overline{ME} \cap \overline{BI}$ and $S = \overline{MD} \cap \overline{CI}$. First, since $\overline{BI}$ is the perpendicular bisector of $\overline{CD}$ we have that

$$\measuredangle DIT = \measuredangle CIT = \measuredangle CIB = 90^\circ - \frac{1}{2}\measuredangle A = \measuredangle MCB = \measuredangle MED = \measuredangle TED$$

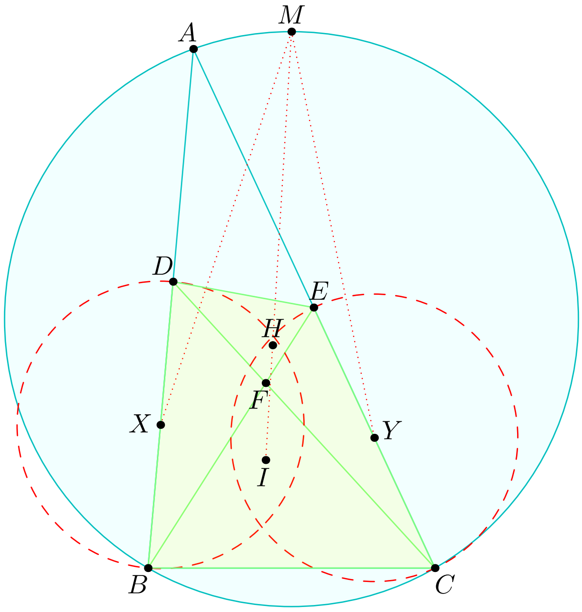
and so D, I, T, E is cyclic. Similarly S lies on this circle too. But $\measuredangle SDE = \measuredangle EDM = \measuredangle MED = \measuredangle TED$ so in fact $\overline{ST} \parallel \overline{DE}$ (isosceles trapezoid).

Then $\triangle IST$ and $\triangle HDE$ are homothetic, so $\overline{IH}$, $\overline{DS}$, and $\overline{ET}$ concur (at M).

¶ Second solution (Evan Chen). Observe that we have the spiral congruence

$$\triangle MDB \sim \triangle MEC$$

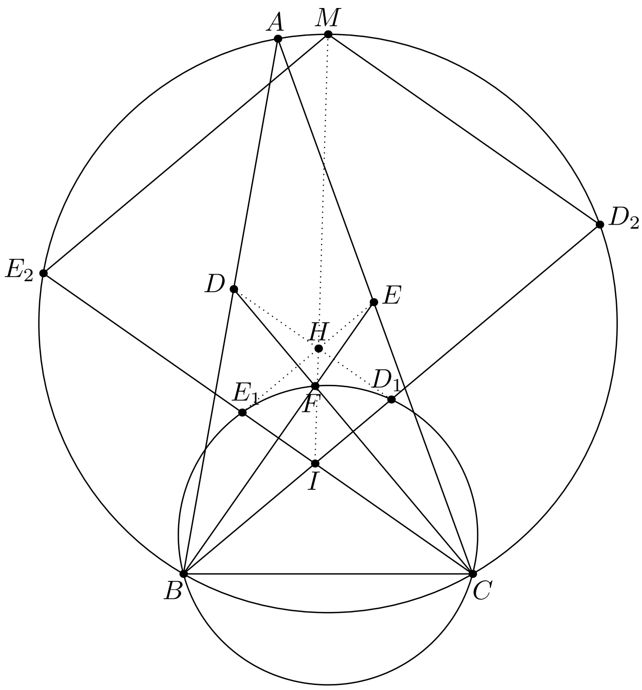
from $\measuredangle MBD = \measuredangle MBA = \measuredangle MCA = \measuredangle MCE$ and $BD = EC$, $BM = CM$. That is, M is the Miquel point of $BDEC$.



Let X and Y be the midpoints of $\overline{BD}$ and $\overline{CE}$. Then $MX = MY$ by our congruence.

Consider now the circles with diameters $\overline{BD}$ and $\overline{CE}$. We now claim that H, I, M all lie on the radical axis of these circles. Note that I is the orthocenter of $\triangle BFC$ and H is the orthocenter of $\triangle DEF$, so this follows from the so-called Steiner line of $BCDE$ (perpendicular to Gauss line $\overline{XY}$). For M, we observe $MX^2 - XB^2 = MY^2 - YC^2$ thus completing the proof.

¶ Third solution (homothety, official solution). Extend DH and EH to meet BI and CI at D_1 and E_1. Note $DD_1 \perp BE$, $CI \perp BE$, so $DD_1 \parallel CI$. Similarly $EE_1 \parallel BI$. So HE_1ID_1.



Angle chase to show that B, E_1, F, C are cyclic – $\angle DCE_1 = \angle DCI$ is computable in terms of ABC and

$$\angle E_1BF = \angle E_1BE = \angle E_1EB = \angle HEF = \angle HDF = \angle HDC = \angle DCE_1 = \angle FCE_1.$$

Thus B, D_1, F, C are also cyclic. So B, D_1, E_1, C are cyclic.

Extend BI and CI to meet the circumcircle again at D_2 and E_2. Direct computation gives that ME_2ID_2 is also a parallelogram. We also get E_1D_1 is parallel to E_2D_2 (both are antiparallel to BC through $\angle BIC$). So we have homothetic paralellograms and that finishes the problem.

§1.3 EGMO 2014/3, proposed by Japan

Available online at https://aops.com/community/p3459754.

Problem statement

We denote the number of positive divisors of a positive integer m by $d(m)$ and the number of distinct prime divisors of m by $\omega(m)$. Let k be a positive integer. Prove that there exist infinitely many positive integers n such that $\omega(n) = k$ and $d(n)$ does not divide $d(a^2 + b^2)$ for any positive integers a, b satisfying $a + b = n$.

Let $n = 2^{p-1}t$, where $t \equiv 5 \pmod 6$, $\omega(t) = k - 1$, and $p \gg t$ is a sufficiently large prime. Let $a + b = n$ and $a^2 + b^2 = c$. We claim that $p \nmid d(c)$, which solves the problem since $p \mid d(n)$.

First, note that $3 \nmid a^2 + b^2$, since $3 \nmid n$. Next, note that $c < 2n^2 < 5^{p-1}$ (since $p \gg t$) so no exponent of an odd prime in c exceeds $p - 2$. Moreover, $c < 2^{3p-1}$.

So, it remains to check that $\nu_2(c) \notin \{p - 1, 2p - 1\}$. On the one hand, if $\nu_2(a) < \nu_2(b)$, then $\nu_2(a) = p - 1$ and $\nu_2(c) = 2\nu_2(a) = 2p - 2$. On the other hand, if $\nu_2(a) = \nu_2(b)$ then $\nu_2(a) \le p - 2$, and $\nu_2(c) = 2\nu_2(a) + 1$ is odd and less than $2p - 1$.

Remark. Personally, I find the condition to be artificial, but the construction is kind of fun.

I also think the scores on this problem during the real contest are low mostly because of the difficulty of problem 2.

§2 Solutions to Day 2

§2.1 EGMO 2014/4, proposed by Netherlands

Available online at https://aops.com/community/p3460731.

Problem statement

Determine all positive integers $n \geq 2$ for which there exist integers $x_1, x_2, \dots, x_{n-1}$ satisfying the condition that if $0 < i < n$, $0 < j < n$, $i \neq j$ and n divides $2i + j$, then $x_i < x_j$.

The answer is $n = 2^k$ and $n = 3 \cdot 2^k$, for each $k \geq 0$ (excluding $n = 1$).

We work with the set $S = \{1, 2, \dots, n-1\} \bmod n$ of nonzero residues modulo n instead. We define the relation $\prec$ on S to mean that $2i + j \equiv 0 \pmod n$ and $i \neq j$, for $i, j \in S$. Then the problem satisfies the conditions if and only if $\prec$ has no cycles, i.e. $\prec$ imposes a partial order on S.

The existence of a cycle for $\prec$ is equivalent to some choice of $t_1 \in S$ and an integer $m \geq 2$ such that

$$t_1 \prec t_2 \prec \dots \prec t_m \prec t_1.$$

Unwinding the definition, this is equivalent to two conditions:

- We need $t_i \not\equiv t_{i+1} \pmod n$ for $i = 1, \dots, m$ (where $t_{m+1} = t_1$). This is equivalent to
$$3 \cdot 2^{i-1} \cdot t_1 \equiv 0 \pmod n \qquad (\heartsuit).$$
- For $t_m \prec t_1$ to be true, we need
$$(-2)^m t_1 \equiv t_1 \pmod n \iff \left((-2)^m - 1\right) t_1 \equiv 0 \pmod n. \qquad (\spadesuit)$$

We now analyze three cases:

- Let $n = 2^k$. Suppose for contradiction some cycle exists. Then $(-2)^m - 1$ is coprime to n, so $(\spadesuit)$ would imply $t_1 \equiv 0 \pmod n$, contradiction.
- Let $n = 3 \cdot 2^k$. Suppose for contradiction some cycle exists. If $(\spadesuit)$ holds for some m, then $2^k \mid t_1$, so the only possibility is that $t_1 \equiv \pm 2^k \pmod n$ and $3 \mid (-2)^m - 1$. However, in that case $(\heartsuit)$ is violated for $i = 1$, contradiction.
- Suppose n had an n has an odd divisor $d \mid n$ and $d \geq 5$. Then taking $t_1 = n/d$ and $m = \varphi(d)$, the equation $(\spadesuit)$ is true. Moreover, $(\heartsuit)$ is true because there is at least one odd prime p with $\nu_p(n) > \nu_p(3t_1) = \nu_p(3n/d)$ (since $d \geq 5$ is odd). So indeed it's possible to construct a cycle.

Thus these are all the answers and the only answers.

§2.2 EGMO 2014/5, proposed by Romania

Available online at https://aops.com/community/p3460733.

Problem statement

Let n be a positive integer. We have n boxes where each box contains a non-negative number of pebbles. In each move we are allowed to take two pebbles from a box we choose, throw away one of the pebbles and put the other pebble in another box we choose. An initial configuration of pebbles is called *solvable* if it is possible to reach a configuration with no empty box, in a finite (possibly zero) number of moves. Determine all initial configurations of pebbles which are not solvable, but become solvable when an additional pebble is added to a box, no matter which box is chosen.

The point of the problem is to characterize all the solvable configurations. We claim that it is given by the following:

Claim — A configuration $(a_1, \dots, a_n)$ is solvable if and only if

$$\sum_1^n \left\lceil \frac{a_i}{2} \right\rceil \ge n.$$

Proof. The proof is by induction on the number of stones. If there are fewer than n stones there is nothing to prove. Now assume there are at least n stones, and let $S = \sum \lceil a_i/2 \rceil$. Then:

- If $S < n$, this remains true after any operation, so by induction the configuration is not solvable.
- Suppose $S \ge n$, and also that there is an empty box (else we are already done). Then there must be some box with at least two stones. In that case, using those two stones to fill the empty box does not change the value of S, but decreases the total number of stones by one, as desired. □

From here we may then extract the answer to the original problem: we require all a_i to be even and $\sum a_i = 2n - 2$.

Remark. It should be unsurprising that a criteria of this form exists, since (1) intuitively, one loses nothing by filling empty boxes as soon as possible, and then ignoring boxes with one pebble in them, (2) the set of configurations is a graded partially ordered set, so one can inductively look at small cases.

§2.3 EGMO 2014/6, proposed by Netherlands

Available online at https://aops.com/community/p3460735.

Problem statement

Solve over $\mathbb{R}$ the functional equation

$$f(y^2 + 2xf(y) + f(x)^2) = (y + f(x))(x + f(y)).$$

A key motivation throughout the problem is that the left-hand side is asymmetric while the right-hand side is symmetric. Thus any time we plug in two values for x and y we will also plug in the opposite pair. Once f is injective this will basically kill the problem.

First, we prove the following.

Lemma

There is a unique $z \in \mathbb{R}$ such that $f(z) = 0$.

Proof. Clearly by putting $y = -f(x)$ such z exists. Now, suppose $f(u) = f(v) = 0$. Then:

- Plug $(x, y) = (u, v)$ gives $f(v^2) = uv$.
- Plug $(x, y) = (v, u)$ gives $f(u^2) = uv$.
- Plug $(x, y) = (u, u)$ gives $f(u^2) = u^2$.
- Plug $(x, y) = (v, v)$ gives $f(v^2) = v^2$.

Consequently $u^2 = uv = v^2$ which yields $u = v$. □

Next let $(x, y) = (z, 0)$ and $(x, y) = (0, z)$ to get

$$\begin{aligned} & f\left(2zf(0)\right) = f\left(z^2 + f(0)^2\right) = 0 \\ \implies & 2zf(0) = z^2 + f(0)^2 = z \\ \implies & \boxed{f(0) = z \in \left\{0, \frac{1}{2}\right\}}. \end{aligned}$$

We now set to prove:

Lemma

The function f is injective.

Proof. By putting $(x, y) = (x, z)$ and $(x, y) = (z, x)$ we get

$$f\left(f(x)^2 + z^2\right) = f\left(2zf(x) + x^2\right) = x(z + f(x)).$$

Now suppose $f(x_1) = f(x_2)$ but $x_1 \neq x_2$. This can only happen if $f(x_1) = f(x_2) = -z$. And now

$$f(x_i)^2 + z^2 = 2zf(x_i) + x_i^2 = z \qquad i = 1, 2.$$

Solving, we have $x_i = \pm 1$, $z = \frac{1}{2}$, (since $z = 0$ is not permissible). Thus we have "almost injectivity".

Now plug in $(x, y) = (-1, 0)$ and $(x, y) = (0, -1)$ in the original and equate in order to obtain $f(-\frac{3}{4}) = f(\frac{5}{4})$, which contradicts the work above. $\square$

Finally we may use the symmetry trick in full to obtain

$$y^2 + 2xf(y) + f(x)^2 = x^2 + 2yf(x) + f(y)^2. \qquad (\heartsuit)$$

In particular, by setting $y = 0$ we obtain

$$f(x)^2 = (z - x)^2 .$$

Two easy cases remain:

- In the $z = 0$ case simply note that ($\heartsuit$) gives $2xf(y) = 2yf(x)$, so for $x \neq 0$ the value $f(x)/x$ is constant and hence $f(x) \equiv \pm x$ follows.
- In the $z = \frac{1}{2}$ case ($\heartsuit$) becomes $(2f(y) + 1)\, x = (2f(x) + 1)\, y$ and hence we're done again by the same reasoning.