

EGMO 2013 Solution Notes

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This is a compilation of solutions for the 2013 EGMO. The ideas of the solution are a mix of my own work, the solutions provided by the competition organizers, and solutions found by the community. However, all the writing is maintained by me.

These notes will tend to be a bit more advanced and terse than the “official” solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered “standard”, then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like “let \mathbb{R} denote the set of real numbers” are typically omitted entirely.

Corrections and comments are welcome!

Contents

0 Problems	2
1 Solutions to Day 1	3
1.1 EGMO 2013/1, proposed by David Monk (UNK)	3
1.2 EGMO 2013/2, proposed by Matti Lehtinen (FIN)	4
1.3 EGMO 2013/3, proposed by Dan Schwarz (ROU)	7
2 Solutions to Day 2	8
2.1 EGMO 2013/4, proposed by Vesna Iršič (SLV)	8
2.2 EGMO 2013/5, proposed by Waldemar Pompe (POL)	9
2.3 EGMO 2013/6, proposed by Emil Kolev (BGR)	10

§0 Problems

1. The side BC of the triangle ABC is extended beyond C to D so that $CD = BC$. The side CA is extended beyond A to E so that $AE = 2CA$. Prove that if $AD = BE$ then the triangle ABC is right-angled.
2. Determine all integers m for which the $m \times m$ square can be dissected into five rectangles, the side lengths of which are the integers $1, 2, \dots, 10$ in some order.
3. Let n be a positive integer.
 - (a) Prove that there exists a set S of $6n$ positive integers such that the least common multiple of any two is at most $32n^2$.
 - (b) Show that every set T of $6n$ positive integers contains two elements with least common multiple exceeding $9n^2$.
4. Find all positive integers a and b for which there are three consecutive integers at which the polynomial $P(n) = \frac{1}{b}(n^5 + a)$ takes integer values.
5. Let Ω be the circumcircle of the triangle ABC . The circle ω is tangent to the sides AC and BC , and it is internally tangent to the circle Ω at the point P . A line parallel to AB intersecting the interior of triangle ABC is tangent to ω at Q . Prove that $\angle ACP = \angle QCB$.
6. Snow White and the Seven Dwarves are living in their house in the forest. On each of 16 consecutive days, some of the dwarves worked in the diamond mine while the remaining dwarves collected berries in the forest. No dwarf performed both types of work on the same day. On any two different (not necessarily consecutive) days, at least three dwarves each performed both types of work. Further, on the first day, all seven dwarves worked in the diamond mine. Prove that, on one of these 16 days, all seven dwarves were collecting berries.

§1 Solutions to Day 1

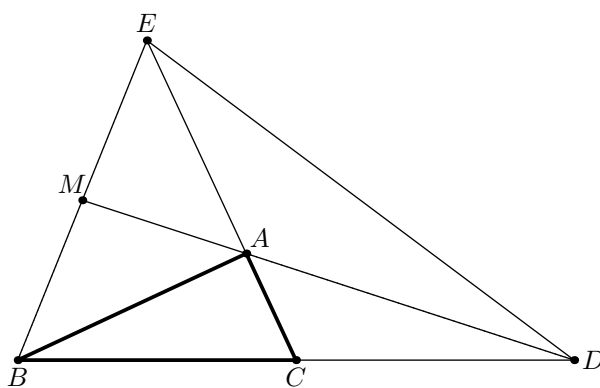
§1.1 EGMO 2013/1, proposed by David Monk (UNK)

Available online at <https://aops.com/community/p3013167>.

Problem statement

The side BC of the triangle ABC is extended beyond C to D so that $CD = BC$. The side CA is extended beyond A to E so that $AE = 2CA$. Prove that if $AD = BE$ then the triangle ABC is right-angled.

Let ray DA meet \overline{BE} at M . Consider the triangle EBD . Since the point lies on median \overline{EC} , and $EA = 2AC$, it follows that A is the centroid of $\triangle EBD$.



So M is the midpoint of \overline{BE} . Moreover $MA = \frac{1}{2}AD = \frac{1}{2}BE$; so $MA = MB = ME$ and hence $\triangle ABE$ is inscribed in a circle with diameter \overline{BE} . Thus $\angle BAE = 90^\circ$, so $\angle BAC = 90^\circ$.

§1.2 EGMO 2013/2, proposed by Matti Lehtinen (FIN)

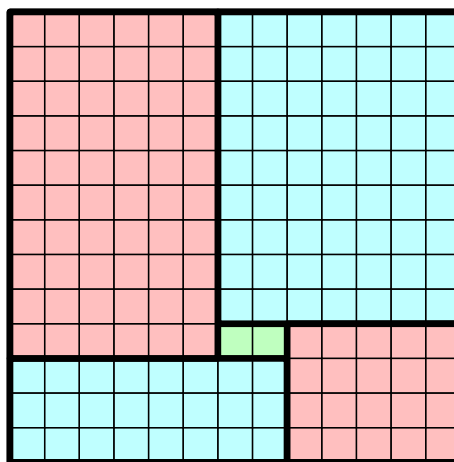
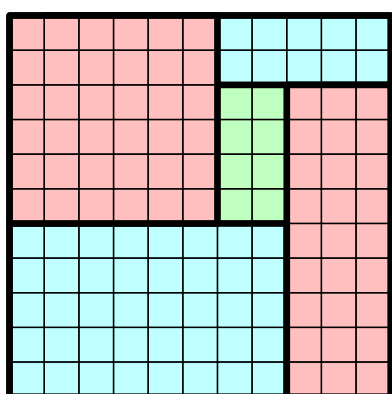
Available online at <https://aops.com/community/p3013170>.

Problem statement

Determine all integers m for which the $m \times m$ square can be dissected into five rectangles, the side lengths of which are the integers $1, 2, \dots, 10$ in some order.

The answer is that this is possible if and only if $m = 11$ or $m = 13$.

¶ **Constructions for $m = 11$ and $m = 13$.** See the figures below.



¶ **Proof the task is impossible unless $11 \leq m \leq 13$.** The total area of the rectangles is of the form $A = a_1b_1 + \dots + a_5b_5$ for (a_1, \dots, b_5) a permutation of $(1, \dots, 10)$. By applying the rearrangement inequality repeatedly one can prove that

$$A \leq 1 \cdot 2 + 3 \cdot 4 + 5 \cdot 6 + 7 \cdot 8 + 9 \cdot 10 = 190.$$

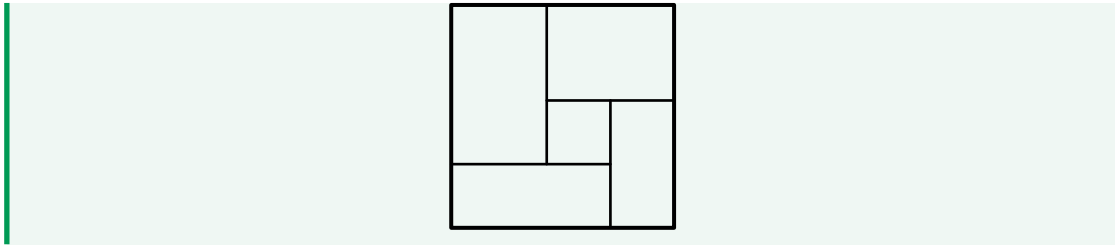
Analogously, we can get a matching lower bound

$$A \geq 1 \cdot 10 + 2 \cdot 9 + 3 \cdot 8 + 4 \cdot 7 + 5 \cdot 6 = 110.$$

Since $A = m^2$ for integer m , it follows $11 \leq m \leq 13$.

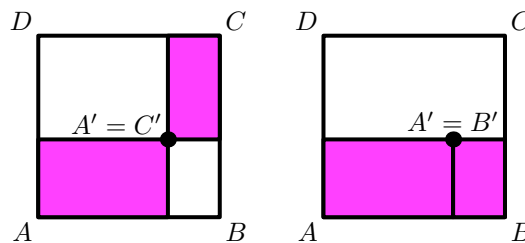
¶ **Proof the task is impossible for $m = 12$.** It remains to rule out such a tiling for a 12×12 square. We start with the following structure claim:

Claim — Any tiling for $m > 10$ must either be the spiral pattern below or a reflection of it.



Proof. Label the square $ABCD$. Because $m > 10$, each corner of the square must be part of a different rectangle. Consider the rectangle covering A , and let the vertex opposite A be A' . Define B', C', D' similarly.

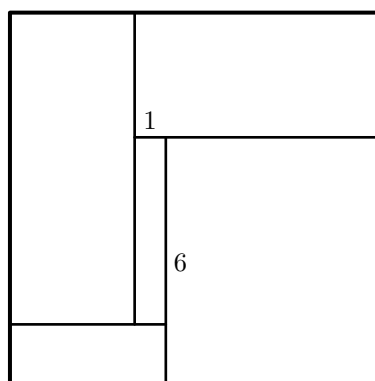
We contend that no two of A', B', C', D' coincide. Indeed, if $A' = C'$, then we get the figure on the left where the two purple rectangles are part of the tiling. The remaining two “quadrants” must be dissected into three rectangles, but there is general a quadrant cannot be divided into either 2 or 3 rectangles in a way that doesn’t create two equal lengths.



Meanwhile, if $A' = B'$, the two purple rectangles already obviously have a common dimension (right figure).

Hence A', B', C', D' are all different points. They must each be part of some rectangle not using the corner. Moreover, every rectangle in the dissection has at least one vertex not on the boundary of the square. So the fifth rectangle must be $A'B'C'D'$ exactly. \square

Now on to the proof that $m = 12$ is impossible. Assume for contradiction a construction exists and refer to the spiral figure above. Note that because $12 > 1 + 10$, the side length 1 cannot belong to any edge of the outer 12×12 square. This means that 1 must be a side length of the center rectangle. Moreover, because the outer square has perimeter 48, and we know $1 + 2 + \dots + 10 = 55$, the center rectangle has perimeter 7. In other words, the inner rectangle must be 1×6 exactly.



Hence, we would need to partition the remaining numbers $\{2, 3, 4, 5, 7, 8, 9, 10\}$ into eight pairs such that two pairs differ by 1 and differ by 6 (in the picture, the paired numbers are on opposite sides of the square). Note that 7 must be paired with 8 (since 7 ± 6 and $7 - 1$ are not available) and 5 must be paired with 4 (since 5 ± 6 and $5 + 1$ are not

available). But then the remaining pairs need to have difference 6, and there is no way to form even a single such pair in $\{2, 3, 9, 10\}$. This produces the desired contradiction.

§1.3 EGMO 2013/3, proposed by Dan Schwarz (ROU)

Available online at <https://aops.com/community/p3013178>.

Problem statement

Let n be a positive integer.

- (a) Prove that there exists a set S of $6n$ positive integers such that the least common multiple of any two is at most $32n^2$.
- (b) Show that every set T of $6n$ positive integers contains two elements with least common multiple exceeding $9n^2$.

For part (a), let

$$S = \{1, 2, \dots, 4n\} \cup \{4n + 2, 4n + 4, \dots, 8n\}$$

which works.

For (b), the main idea is as follows. Let $x_0 < x_1 < \dots < x_m$ be these integers. Assume for contradiction the LCM's are less than L . The main idea is the estimate

$$L \geq \frac{x_i x_{i+1}}{\gcd(x_i, x_{i+1})} \geq \frac{x_i x_{i+1}}{x_{i+1} - x_i}$$

whence

$$\frac{1}{L} \leq \frac{1}{x_i} - \frac{1}{x_{i+1}}.$$

so taking the sum starting from any k gives

$$\frac{m+1-k}{L} \leq \frac{1}{x_k} - \frac{1}{x_m} < \frac{1}{x_k}.$$

For $(k, m) = (3n, 6n)$ and $L = 9n^2$ this is a contradiction (since $x_k \geq 3n$).

Remark. China TST 2007/6 follows the same idea.

§2 Solutions to Day 2

§2.1 EGMO 2013/4, proposed by Vesna Iršič (SLV)

Available online at <https://aops.com/community/p3014762>.

Problem statement

Find all positive integers a and b for which there are three consecutive integers at which the polynomial $P(n) = \frac{1}{b}(n^5 + a)$ takes integer values.

The answer is

- Either $b = 1$; or
- $a \equiv \pm 1 \pmod{11}$ and $b = 11$ (and $a > 0$).

We consider only the case $b > 1$. Suppose that P is a prime power dividing b . Evidently P is not even. Then there exists an n for which

$$n^5 \equiv (n+1)^5 \equiv (n-1)^5 \pmod{P}.$$

Therefore,

$$0 \equiv (n+1)^5 + (n-1)^5 - 2n^5 = 20n^3 + 10n \pmod{P}.$$

Since $n \not\equiv 0 \pmod{p}$ for obvious reasons, we obtain that

$$n^2 \equiv -\frac{1}{2} \pmod{p}.$$

Then,

$$\begin{aligned} 0 &\equiv 2((n+1)^5 - (n-1)^5) \pmod{P} \\ &= 20n^4 + 40n^2 + 4 \\ &\equiv 20 \cdot -\frac{1}{4} + 40 \cdot -\frac{1}{2} + 4 \pmod{P} \\ &= -11. \end{aligned}$$

This implies $P = 11$. Since P can be any prime power dividing b , this forces $b = 11$.

In other words, $b \in \{1, 11\}$.

Finally, observe that

$$3^5 \equiv 4^5 \equiv 5^5 \equiv 1 \pmod{11}$$

and

$$(-3)^5 \equiv (-4)^5 \equiv (-5)^5 \equiv -1 \pmod{11}$$

are the only such triples modulo 11. Therefore the solution set is the one claimed above.

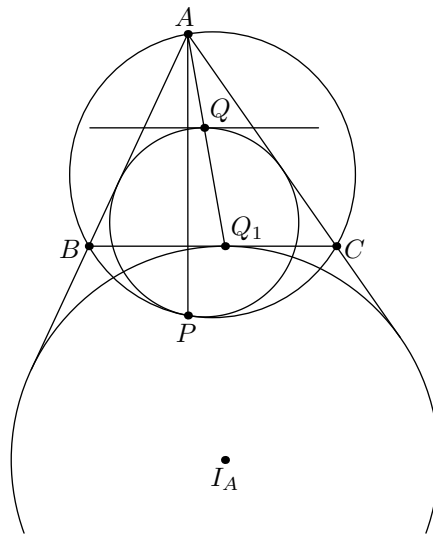
§2.2 EGMO 2013/5, proposed by Waldemar Pompe (POL)

Available online at <https://aops.com/community/p3014767>.

Problem statement

Let Ω be the circumcircle of the triangle ABC . The circle ω is tangent to the sides AC and BC , and it is internally tangent to the circle Ω at the point P . A line parallel to AB intersecting the interior of triangle ABC is tangent to ω at Q . Prove that $\angle ACP = \angle QCB$.

First, let us extend \overline{AQ} to meet \overline{BC} at Q_1 . By homothety, we see that Q_1 is just the contact point of the A -excircle with \overline{BC} .



Now let us perform an inversion around A with radius $\sqrt{AB \cdot AC}$ followed by a reflection around the angle bisector; call this map Ψ . Note that Ψ fixes B and C . Moreover it swaps \overline{BC} and (ABC) . Hence, this map swaps the A -excircle with the A -mixtilinear incircle ω . Hence Ψ swaps P and Q_1 . It follows that \overline{AP} and $\overline{AQ_1}$ are isogonal with respect to $\angle BAC$, meaning $\angle BAP = \angle CAQ_1$. Since $\angle CAQ = \angle CAQ_1$ we are done.

§2.3 EGMO 2013/6, proposed by Emil Kolev (BGR)

Available online at <https://aops.com/community/p3014769>.

Problem statement

Snow White and the Seven Dwarves are living in their house in the forest. On each of 16 consecutive days, some of the dwarves worked in the diamond mine while the remaining dwarves collected berries in the forest. No dwarf performed both types of work on the same day. On any two different (not necessarily consecutive) days, at least three dwarves each performed both types of work. Further, on the first day, all seven dwarves worked in the diamond mine. Prove that, on one of these 16 days, all seven dwarves were collecting berries.

Let Q_n denote the vector space $\{0, 1\}^n$. For a vector v , let $v[i]$ denote the i th component. We may identify each day with a vector v_k , where $v_k[i] = 0$ if dwarf i worked in the diamond mine, and $v_k[i] = 1$ otherwise. Let $V = \{v_1, v_2, \dots, v_{16}\}$ be the subset of Q_7 in the problem, and assume $v_{16} = 0000000$.

We first prove the following:

Lemma

Exactly two vectors start with each of 000, 001, \dots , 111. Similar statements hold for any choice of three indices.

Proof. If three vectors start with 000 (say) then we run into problems. \square

Now we know that v_{16} is the all-null vector. Ignoring that vector (and hence considering just the first 15 vectors), let n_i be the number of 1's in v_i , where $i = 1, 2, \dots, 15$.

Claim — We have

$$\begin{aligned}\sum_{i=1}^{15} \binom{n_i}{1} &= \frac{16}{2} \binom{7}{1} = 56 \\ \sum_{i=1}^{15} \binom{n_i}{2} &= \frac{16}{2^2} \binom{7}{2} = 84 \\ \sum_{i=1}^{15} \binom{n_i}{3} &= \frac{16}{2^3} \binom{7}{3} = 70.\end{aligned}$$

Proof. This follows using the lemma and double counting. For example, the third equation is counting the number of quadruples (i, j_1, j_2, j_3) such that the i th string has a 1 in the j_1 th, j_2 th, j_3 th position, where $j_1 < j_2 < j_3$. The left-hand side counts the number when summed by i ; the right-hand side counts the number according to the $\binom{7}{3}$ choices of (j_1, j_2, j_3) . \square

We may then rewrite the lemma as

$$\sum_{i=1}^{15} n_i = 56$$

$$\sum_{i=1}^{15} n_i^2 = 2 \cdot 84 + 56 = 224$$

$$\sum_{i=1}^{15} n_i^3 = 6 \cdot 70 + 3 \cdot 224 - 2 \cdot 56 = 980.$$

Now remark that $(n_i - 3)(n_i - 4)(n_i - 7) \leq 0$ for each integer $1 \leq n_i \leq 7$. We compute

$$\begin{aligned} 0 &\geq \sum_{i=1}^{15} (n_i - 3)(n_i - 4)(n_i - 7) \\ &= \sum_{i=1}^{15} n_i^3 - 14n_i^2 + 61n_i - 84 \\ &= 1 \cdot 980 - 14 \cdot 224 + 61 \cdot 56 - 15 \cdot 84 \\ &= 980 - 3136 + 3416 - 1260 \\ &= 0. \end{aligned}$$

This implies that $n_i \in \{3, 4, 7\}$ for each i . If $n_i \neq 7$ for any i then we have

$$224 = \sum n_i^2 \equiv 15 \cdot 2 \not\equiv 0 \pmod{7}$$

which is impossible. This means that $n_i = 7$ for some i , and therefore, $1111111 \in V$, as desired.

Remark. Up to permutation there turns out to be a unique set of 16 vectors. Xinyang Chen gives the following compact description of the unique solution:

- 0000000 and 1111111, of course
- 1101000 and its six *cyclic shifts* 0110100, 0011010, 0001101, 1000110, 0100011, 1010001.
- The complements of the above seven strings (i.e. 0010111, 1001011, etc.).