Wait darn this actually has a decent solution.

Let $A = (x_1^2 + 1) \ldots (x_4^2 + 1)$. We claim that $A \geq 16$.

This bound is sharp when $x_1 = x_2 = x_3 = x_4 = 1$, so $16$ is the minimum.

**Compute**

$$A = \frac{(i+x_1)(i+x_2)(i+x_3)(i+x_4)}{(-i+1)}$$

$$A = \prod_{k=1}^{4} (i-x_k)(i+x_k) \cdot (\star) \quad \text{(here } i^2 = -1)$$

$$= \prod_{k=1}^{4} P(i) \cdot P(-i)$$

$$= \left[ 1-b+d + i[c-a] \right]\left[ 1-b+d - i[c-a] \right]$$

$$\geq \left( b-d+1 \right)^2 + (a-c)^2$$

$$\geq (5-1)^2 + 0^2$$

$$= 16.$$

---

1. Stay within the borders when writing your solution.
2. Write DARKLY and LEGIBLY so your faxed copy can be clearly read; points will be taken off if graders can’t read your work!
Problem 2 (Page 1 of 4)

2014 Olympiads

Answer: \( f = 0 \), \( f(x) = x^2 \) when \( x = 2 \), otherwise.

It's easy to check \( f = 0 \) works, and for \( f(x) = x^2 \),

\[
x(2y^2 - x)^2 + y^2(2x - y^2)^2 = x^3 + y^6 = \frac{(x^3)^2}{x} + (y^2)^2
\]

(\text{where} \ x \neq 0)

So \( f(x) = x^2 \) works too. For the last function, see the end of the solution.

Now let's show these are all. Put \( y = 0 \) to obtain

\[
\chi f(2f(0) - x) = \frac{f(x)^2}{x} + f(0). \quad (\ast)
\]

Now we claim \( f(0) = 0 \). If not, pick a prime \( p \mid f(0) \) and choose \( x = p \neq 0 \). Then \( \frac{p f(x)}{x} \) is an integer, so \( p \mid f(p)^2 \Rightarrow p \mid f(p) \Rightarrow p \mid \frac{f(p)^2}{p} \), and hence we obtain (oh also \( p \mid p f(2f(0) - p) \)) that \( p \mid f(0) \), contradiction.

Hence \( f(0) = 0 \). So \( \ast \) rewrites as

\[
\chi f(-x) = \frac{f(x)^2}{x} \quad \forall x \neq 0.
\]

or

\[
\chi^2 f(-x) = f(x)^2
\]

which actually also holds at \( x = 0 \).

1. Stay within the borders when writing your solution.
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Next we show \( f \) is even. For \( x \neq 0 \), note
\[
\begin{align*}
  f(x)^2 &= x^2 f(-x) \\
  f(-x)^2 &= x^2 f(x)
\end{align*}
\]
\[\Rightarrow \quad \left[ f(x) - f(-x) \right] \left[ f(x) + f(-x) + x^2 \right] = 0.\]
In the unlikely event that \( f(x) + f(-x) = -x^2 \), we find
\[ f(x)^2 = x^2 \left[ -x^2 - f(x) \right] \Rightarrow \]
\[ \Rightarrow \quad \left[ f(x) + \frac{1}{2} x^2 \right]^2 = -\frac{3}{4} x^4 < 0 \]
which is absurd. (since \( x \neq 0 \))

So in fact, \( f(x)^2 = x^2 f(x) \). The given is thus
\[ xf(zf(y) - x) + y^2 f(2x - f(y)) = xf(x) + f(yf(y)) \quad \forall x \neq 0 \]

Wait I'm being dumb* the thing above is boxed
\[ (f(x) - [f(x) - x^2]) = 0. \]

LOL. So \( f(x) = 0 \) or \( f(x) = x^2 \) for any particular \( x \).

*Not that I'm not usually being dumb.
Now suppose there exists \( t \in \mathbb{Z} \setminus \{0\} \) such that \( f(t) = 0 \). We show \( f \) is zero everywhere.

Put \( y = t \) in the given to obtain

\[
 t^2 \cdot f(2t) = 0 \quad \forall x \neq 0
\]

Adding on \( f(0) = 0 \), \( f(2k) = 0 \) for all \( k \).

Now, select \( x = 2k + \) in the given. We find

\[
2k \cdot f\left(z[x]_{/(4k)}\right) + y^2 f(4k-f(y))
= \frac{f(2k)^2}{2k} + f(yf(y))
\]

\[
\Rightarrow y^2 \cdot f(4k-f(y)) = f(yf(y))
\]

Assume for contradiction that \( f(y) = y^2 \) now. (Here \( y \neq 0 \)).

Evidently

\[
y^2 \cdot f(4k-y^2) = f(y^3)
\]

If \( f(y^3) \neq 0 \) then \( f(4k-y^2) \neq 0 \), in which case we get \( y^4 = (4k-y^2)^2 \) for all \( k \neq 0 \), impossible. Hence

\[
f(4k-y^2) = f(y^2-4k) = f(y^3) = 0 \quad \forall k \neq 0.
\]

Since \( y \) is odd, \( y^2 = 1 \pmod{4} \), and so

\[
f(n) = 0 \quad \forall n \text{ other than } \pm y^2.
\]

In particular,

1. Stay within the borders when writing your solution.
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However, $f$ is even, so now $f(n) = 0$ for all $n$ except possibly $f(-1)$ and $f(1)$. Now we have

But unless $y = \pm 1$, we can now take $n = y$, contradiction.

Thus we either have $f(x) = 0$ or $f(x) = \frac{1}{x}$, $x = \pm 1$. We now check the last function individually. Fails:

- If $x = \pm 1$ and $y = \pm 1$:
  \[xf(2 - x) + f(2x - 1) = xf(x) + 1\]
  This holds for both $x = 1$ and $x = -1$.

- If $x = \pm 1$ but $y \neq \pm 1$ \(\Rightarrow f(y) = 0\):
  \[xf(x) + y^2f(2x) = xf(x) + 1\]
  \[\Rightarrow 0\]

- If $x \neq \pm 1$, but $y = \pm 1$:
  \[xf(2 - x) + f(2x - 1) = 1\]
  But this is wrong at $x = 5$. LOL!
  Darn, I was pretty scared for a while.

Thus $f(x) = 0$, $f(x) = x^2$ are the only solutions to this functional equation.
Let $X, Y, Z$ denote Lincoln, NE, the North Pole, and any vertex of the Bermuda Triangle. (These are noncollinear since they lie on the spherical Earth.) We use barycentric coordinates with respect to XYZ. For every $n \in \mathbb{Z}$ let us select the point $P_n = \left(1: n - \frac{2014}{3}, (n - \frac{2014}{3})^3\right)$. The coordinate sum is

$$27\left(1 + (n - \frac{2014}{3}) + (n - \frac{2014}{3})^3\right) \equiv 2014^3 \pmod{3} \neq 0 \pmod{3}$$

so this means it is nonzero, and $P_n$ is a Euclidean point (and not a point at infinity).

Let $x = a - \frac{2014}{3}$, $y = b - \frac{2014}{3}$, $z = c - \frac{2014}{3}$.

Define

$$D(a,b,c) = D = \begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix}$$

$$= \sum_{cyc} -x^3(y-z).$$

In that case the points $P_a$, $P_b$, $P_c$ are collinear if and only if the determinant $D$ is zero.

---

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But observe that

\[
(x-y)(y-z)(z-x)(x+y+z) = \sum_{\text{cyc}} x(-x^2y + x^2z - y^2z + y^3x - z^2x + z^3y)
\]

\[
= \sum_{\text{cyc}} (-x^3y - x^3z - xy^3z - x^3y^2 - x^3z^2 + y^3z^2)
\]

\[
= \sum_{\text{cyc}} -x^3(y-z).
\]

So then \(D(a,b,c) = (a-b)(b-c)(c-a)(a+b+c - 2014).\)

Hence \(P_a, P_b, P_c\) are collinear if and only if two of \(a, b, c\) coincide, or \(a+b+c = 2014.\)

\[a+b+c = 2014.\]
The answer is $k=6$. Call the players Ana & Banana, for A and B.

First, we prove that if $k \geq 6$, then Ana cannot always win. Consider the board below, where each space is a circle:

```
  .
  .
  .
  .
  .
  .
  .
  .
  .
  .
```

We have marked several spaces black; extend this pattern indefinitely. We claim that Banana can merely remove the blackened spots. This is always possible since no two such spots are adjacent. Now this implies that at the beginning of Ana’s turn, any consecutive six grid cells have at least two spaces without a counter which are also nonadjacent (in fact, we always have exactly two blackened cells). This makes it impossible to win at $k=6$.

1. Stay within the borders when writing your solution.
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Next we construct a winning strategy for $k = 5$. After Banana's first move, there is exactly one token left, so Ana can construct an "equilateral triangle", whence Banana leaves her with two adjacent counters.

Now, let Ana place two counters at a "gap" two away, as shown:

```
    A1  X1  Y1  B1
   •  •  •  •  •
   O  O  O  O  O  O
 x1  x2  y2  y1
```

Note that this forces Banana to take either $X_1$ or $Y_1$. WLOG he takes $X_1$. Now we consider two cases:

1. Ana adds in $X_1$ and $X_2$.

```
   •  •  •  •  •
   O  O  O  O  O
```

Now, we consider two cases.

We then repeat this procedure until either $X_1$ and $X_2$ are both present, or both $Y_1$ & $Y_2$ are.

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At each step, Banana must take either $X_1$ or $Y_1$, and we simply add it back in, along with a lower-case letter, until one lower-case letter is full. So we have essentially two cases: (Case I

\[ \begin{array}{c}
\text{Case I} \\
(1+2)
\end{array} \]

Let's start with Case II. If $Y_1$ is removed, then add in $\{y, y\}$, reducing to Case I.

If $X_1$ is removed, then we get the following.

Adding on $P_1$ and and $X_1$ forces Banana to take $X_1$ (since $P_1 - X_1 - X_2$ is a threat, as is the same $A_1 - X_1 - Y_1 - B_1$). Similarly we can add $P_2, X_1$ and again force Banana to take $X_1$.

---

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If we now add $X_1$ and $U_1$, then $P_1 - X_1 - X_2$ and $P_2 - X_1 - X_1$ forces Banana to take $X_1$.

Now adding in $V_1$, $W$ leads to a clear win.

Case I is analogous, except the marked cell $L$ is this time not empty, which does not alter the strategy.

Hence Ana can force a win when $k = 5$, as required.
Let $Q$ be the reflection of $P$ across $AC$.

Then, let $B', X', Y'$ denote the reflections of $B, X, Y$ across $AC$.

Directing angles,

\[ \angle AQC = -\angle APC = -\angle AHC = \angle ABC \]

Hence $Q$ lies on $(ABC)$.

well-known lemma here:

reflection of $H$ across $AC$ is on $(ABC)$.

This is just angle chasing.

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Now complex bash with \((ABC)\) the unit circle.

Because \(\angle BAC = 2 \angle CAQ\), we have
\[
\frac{j}{q} \cdot b = \left(\frac{j}{q} \cdot c\right)^3
\]
\[
\Rightarrow b = \frac{c^3}{q^2}.
\]

Also
\[
b' = a + c - ac \overline{b}
\]
\[
= a + c - aq^2 / c^2.
\]

Moreover,
\[
y' = a + q + c.
\]

It remains to compute \(x'\). Note that \(x' - a\) is the circumcenter of the circle with vertices \(0, \quad q-a, \quad b' - a\); hence
\[
x' - a = \begin{vmatrix} q-a & (q-a)^2 \\ b' - a & (b' - a)^2 \end{vmatrix} \quad \text{[Circumcenter Formula]}
\]
\[
\begin{vmatrix} q-a & \overline{q-a} \\ b' - a & \overline{b'-a} \end{vmatrix} \quad \text{[See Addendum 2]}
\]
\[
= \frac{(q-a)(b'-a)[\overline{b'-a} - \overline{q-a}]}{(q-a)(b'-a) - (q-a)(b'-a)}
\]
So now, \[ b' - a = c - \frac{aq^2}{c^2} \]
\[ \Rightarrow b' - a = \frac{1}{c} - \frac{c^2}{aq^2}. \]

Hence

\[
\chi' - a = \frac{(q-a)(c - \frac{aq^2}{c^2})(\frac{1}{c} - \frac{c^2}{aq^2} - \frac{1}{q} + \frac{1}{a})}{(q-a)(\frac{1}{c} - \frac{c^2}{aq^2}) - (\frac{1}{q} - \frac{1}{a})(c - \frac{aq^2}{c^2})}
\]

\[
= \frac{(q-a)(c^3-aq^2)(\frac{aq^2-c^3}{aq^2c} - \frac{1}{q} + \frac{1}{a})}{(q-a)(c - \frac{c^4}{aq^2}) + \frac{q-a}{aq} \cdot (c^3-aq^2)}
\]

\[
= \frac{aq(c^3-aq^2)(\frac{1}{c} - \frac{c^2}{aq^2} - \frac{1}{q} + \frac{1}{a})}{aqc - \frac{c^4}{q} + c^3 - aq^2}
\]

\[
= \frac{aq \left( \frac{aq^2-c^3}{aq^2c} - \frac{1}{q} + \frac{1}{a} \right)(c^3-aq^2)}{-\frac{c}{q} \left( c^3 - aq^2 \right) + (c^3-aq^2)}
\]

\[
= \frac{aq \left( \frac{aq^2-c^3}{aq^2c} + \frac{q-a}{aq} \right)}{1 - \frac{c}{q}}
\]

1. Stay within the borders when writing your solution.
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\[
= \frac{a q^2 - c^3 + q c (q-a)}{q c - c^2} \\
= \frac{a q (q-c) + c(q-c)(q+c)}{q c - c^2} \\
= \frac{a q}{c} + q + c.
\]

Hence,
\[
x' = a + q + c + \frac{a q}{c} \\
y' = a + q + c
\]

\[
\Rightarrow |x' - y'| = \left| \frac{a q}{c} \right| = \frac{1}{1} = 1
\]

as required.

Page 5 mentions addendums, which clarify the theorems I cited.
Addendum 1: It's well-known that if $AB$ is a chord of the unit circle and $Z \in C$ then the foot of $Z$ on $AB$ is

$$p = \frac{1}{2} \left( a + b + z - ab \overline{z} \right)$$

Thus the reflection is $Zp - z = a + b - ab \overline{z}$.

Addendum 2: The arbitrary circumcenter has formula

$$P = \frac{\begin{vmatrix} x & x & x \\ y & y & y \\ z & z & z \end{vmatrix}}{2}.$$  

To see this note that if $p$ is said center then

$$R^2 = |p - x|^2 = p\overline{p} - p\overline{x} + \overline{p}x + xx$$

$$\Rightarrow \quad x\overline{p} + \overline{x}p = -R^2 + p\overline{p} + xx.$$  

and so on. Then Cramer's Rule on $p$, $\overline{p}$ is

$$P = \frac{\begin{vmatrix} x & -R^2 + p\overline{p} + xx \\ y & -R^2 + p\overline{p} + yy \\ z & -R^2 + p\overline{p} + zz \end{vmatrix}}{2} = \frac{\begin{vmatrix} x & xx \\ y & yy \\ z & zz \end{vmatrix}}{2}.$$  

1. Stay within the borders when writing your solution.
2. Write DARKLY and LEGIBLY so your faxed copy can be clearly read; points will be taken off if graders can't read your work!
Let $N = n+1$. Construct a $N \times N$ table as below:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>\ldots</th>
<th>\infty</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(0,0)</td>
<td>(1,0)</td>
<td>(2,0)</td>
<td>\ldots</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>(0,1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>\vdots</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>\vdots</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Here $N=6$. Now in each square, write the smallest prime $p$ dividing $\gcd(a+i, b+j)$.

We claim that for $N$ large enough, at least $0.28N^2$ of the squares have a prime greater than $0.001n^2$ written in them. Indeed, let $p_1, p_2, \ldots, p_r$ denote the set of primes less than $0.001n^2$. 

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Now note that a prime $p$ is written at most
\[ \left( \frac{N}{p} \right)^2 \leq \left( \frac{N}{p} + 1 \right)^2 \]
times. Thus, the number of “bad” squares is less than
\[ \sum_{i=1}^{\infty} \left( \frac{N}{p_i} + 1 \right)^2 = N^2 \left[ \frac{1}{p_1^2} + \ldots + \frac{1}{p_r^2} \right] \]
+ 2N \left[ \frac{1}{p_1} + \ldots + \frac{1}{p_r} \right] + \phi(r) \quad (*)

By the Prime Number Theorem, $r$ grows asymptotically as $N/\ln N$. So we have asymptotic bounds
\[ \frac{1}{p_1} + \ldots + \frac{1}{p_r} < \frac{1}{1} + \ldots + \frac{1}{r} \]
~ $\phi(r)$

Now we have asymptotic bounds
\[ \frac{1}{p_1} + \ldots + \frac{1}{p_r} < \frac{1}{1} + \ldots + \frac{1}{r} \]
\[ \leq 10 \ln r + o(N) \]
\[ < 20 \ln 0.001n \leq A^2 \]

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and of course $r \leq 0.001n^2$. So the last two terms of (*) can be bounded by $0.002N^2$ for large $N$, say.

Moreover,

$$\frac{1}{p_1^2} + \ldots + \frac{1}{p_s^2} < \frac{\pi^2}{6} - \frac{1}{1 - \frac{1}{16}}$$

$$< \frac{\pi^2}{6} - 1 - \frac{1}{16}$$

So the entire sum (*) is bounded above by $0.71n^2$, establishing the claim.

Hence some row/column has at least $2\%$ of its entries as primes exceeding $0.001n^2$. If $N$ is large enough, then $N \geq 0.001n^2$, so all these primes are distinct, and moreover there are at least $0.25n < 0.28n$ such primes.

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Consequently, $a_i$ is $0$ for some $0 \leq i \leq n$,

$$a + \hat{i} \geq (0.26n^2)$$

$$\Rightarrow a \geq (0.001n^2)^{0.26n} - i$$

$$\Rightarrow a > c^n \cdot n^{\frac{1}{2}n}$$

for some constant $c > 0$. As $a,b$ are symmetric, we must therefore have

$$\min \{a,b\} > c^n \cdot n^{\frac{1}{2}n}.$$ 

when $n$ is big enough. Now we simply pick $c$ even smaller so that $c^n \cdot n^{\frac{1}{2}n} < 1$ for the smaller $n$. We’re done.

**Black Problem**

**Addendum:**

(*) as if $p$ divides $a_i$, $a_j$ then $p | i - j$ (i+j here)
so $p \leq |i-j|$. Hence $p \leq N$.

---

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2. Write DARKLY and LEGIBLY so your faxed copy can be clearly read; points will be taken off if graders can’t read your work!