OTIS Application Problems

Year V — OTIS 2019-2020

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Due: August 1, 2019 at 11:59PM ET

Thanks for your interest in OTIS! This is the set of application problems for OTIS 2019-2020, for newcomer students. (Returning students do not need to submit these problems.)

Preamble

§1 Philosophy: this is not a test

The application problems are treated differently from other programs. Many other programs have an "entrance exam" or similar in which you have to solve some problems by yourself, and higher scores get in.

This is *not* the intention of the OTIS application problems; I already have your contest scores anyways[1]. Instead, the OTIS application problems are meant to help me with a few things:

- It implicitly serves as summer reading. Part I of my geometry book [3] is basically a pre-requisite for OTIS, so working through the geometry problems in section A will check that you actually know the background.
- It lets me see your writing. If I have an easy time reading and understanding your solutions, that's usually a good sign that OTIS will work well.
- It gives you some practice asking for help (see item 2 in instructions).
- It helps serve as a sanity check that you will have enough time to work on OTIS during the year. This packet consists of actual olympiad-level problems, so you can see what you're getting yourself into.[2]

So, **please treat this like homework rather than a test**. In particular, you can even ask me for help on the problems (see item 2 in instructions). I will not just grade your solutions and sum your scores — in fact, I probably won't even bother assigning scores of any sort. Instead, I am looking to see whether you are someone who is willing and able to solve olympiad problems and take the time to write them up cleanly.

In particular, don't be discouraged if you find the problems challenging! If you start early, work diligently, and are willing to ask for hints, then I think you're likely to do very well.

[1] And, I don't really like the idea that education should be meritocratic. Consequently, I will not merely take the N strongest students.

[2] In the past, I used to have a summer orientation packet for admitted students, which inevitably would cause 20% of the admits to drop out. This is a terrible experience I would rather avoid.

§2 Instructions

1. Solve as many of the problems as you can.
2. **You can ask me for help if you're stuck on something!** Just send me an email telling me what you've tried, and I'll try to push you in the right direction.[3] This is how OTIS works for admitted students, so why not practice now?

   For the geometry problems from [3], you can also use the hints in the back to get you un-stuck.
3. You can also use any other online or print references, e.g. searching the web. However, I ask that you reference any "outside sources" that you used, for each problem, other than those in item 2.
4. You should *not* discuss the problem with humans other than me.
5. **You must write up the solution yourself.** Do not just copy-paste or link someone else's solution.
6. Try to write your solutions well. See [2] for some suggestions on proof-writing.
7. The submission link for the problems, as well as the rest of the application, is available at:

   `https://goo.gl/forms/qJGz5M4Fjsuqzq4k2`
8. Your solutions should be submitted as a **single PDF**, not to exceed 10 MB. Scanned submissions might exceed that limit, so I strongly suggest you use LaTeX instead.
9. The deadline is **August 1, 2019 at 11:59PM ET**. As this packet was posted in early January, this is a strict deadline. No late submissions are accepted.

§3 Math conventions

Here are some conventions and notation that OTIS will use. (For example, they may appear on OTIS practice exams with no further clarification.)

- We let $\mathbb{N} = \mathbb{Z}_{>0} = \{1, 2, \dots\}$ denotes the set of *positive* integers (i.e. 0 is not a natural number). We use $\mathbb{Z}_{\geq 0}$ for nonnegative integers.
- The functions $\lfloor \bullet \rfloor$ and $\lceil \bullet \rceil$ are the floor and ceiling functions.
- The graph-theory terms "graph", "vertex", "edge", "degree", "directed graph", "tournament" will be used freely. Graphs are generally simple graphs unless otherwise specified.
- The function log actually denotes the *natural* logarithm, not the base-10 logarithm.
- We say 0 divides itself, but not any other integer.
- Some problems may refer to chess pieces (e.g. bishops on a chessboard), and a few problems may refer to entire chess games (really!). We take `www-math.bgsu.edu/~zirbel/chess/BasicChessRules.pdf` as the agreed-upon rules for the games of chess for such problems.
- Empty sums are equal to 0 and empty products are equal to 1.

[3] Warning: I travel for MOP/IMO over the summer, so responses then will be slower. Start early.

Problems

§A Geometry

Some pointers for this section:

- **Reading**: please read through Part I of my book ([3]) as the material there is necessary (and sufficient) to solve these problems.
- All problems are themselves from EGMO [3], and you can use the hints there to help you get un-stuck.
- It is not essential that you typeset diagrams for these problems.
- All problems admit synthetic solutions, but computational approaches are okay too. Do whatever you need to.

Problem A.1 (#1.40, Canada 1991)**.** Let P be a point inside circle ω. Consider chords of ω passing through P. Prove that the midpoints of these chords all lie on a fixed circle.

Problem A.2 (#1.41, Russia 1996)**.** Points E and F are given on side BC of convex quadrilateral $ABCD$ (with E closer than F to B). It is known that $\angle BAE = \angle CDF$ and $\angle EAF = \angle FDE$. Prove that $\angle FAC = \angle EDB$.

Problem A.3 (#1.50, IMO 2013)**.** Let ABC be an acute triangle with orthocenter H, and let W be a point on the side $\overline{BC}$, between B and C. The points M and N are the feet of the altitudes drawn from B and C, respectively. Suppose ω_1 is the circumcircle of triangle BWN and X is a point such that $\overline{WX}$ is a diameter of ω_1. Similarly, ω_2 is the circumcircle of triangle CWM and Y is a point such that $\overline{WY}$ is a diameter of ω_2. Show that the points X, Y, and H are collinear.

Problem A.4 (#2.28, JMO 2012)**.** Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic.

Problem A.5 (#2.33, IMO 2000)**.** Two circles G_1 and G_2 intersect at two points M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through the point M, with C on G_1 and D on G_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.

Problem A.6 (#2.35, IMO 2009)**.** Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB respectively. Let K, L M be the midpoints of $\overline{BP}$, $\overline{CQ}$, $\overline{PQ}$, respectively, and let Γ be the circumcircle of $\triangle KLM$. Suppose that $\overline{PQ}$ is tangent to Γ. Prove that $OP = OQ$.

Problem A.7 (#3.21, USAMO 2003)**.** Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Lines AB and DE intersect at F, while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

Problem A.8 (#3.25, USAMO 1993)**.** Let $ABCD$ be a quadrilateral whose diagonals are perpendicular and meet at E. Prove that the reflections of E across the sides of $ABCD$ are concyclic.

Problem A.9 (#3.30, USAMO 2015)**.** Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let X be a variable point on segment $\overline{PQ}$. Line AX meets ω again at S (other than A). Point T lies on arc AQB of ω such that $\overline{XT}$ is perpendicular to $\overline{AX}$. Let M denote the midpoint of chord $\overline{ST}$.

As X varies on segment $\overline{PQ}$, show that M moves along a circle.

Problem A.10 (#4.47, USAMO 2011)**.** Let P be a point inside convex quadrilateral $ABCD$. Points Q_1 and Q_2 are located within $ABCD$ such that

$$\begin{aligned} \angle Q_1BC &= \angle ABP, & \angle Q_1CB &= \angle DCP, \\ \angle Q_2AD &= \angle BAP, & \angle Q_2DA &= \angle CDP. \end{aligned}$$

Prove that $\overline{Q_1Q_2} \parallel \overline{AB}$ if and only if $\overline{Q_1Q_2} \parallel \overline{CD}$.

Problem A.11 (#4.48, Japan 2009)**.** Triangle ABC has circumcircle Γ. A circle with center O is tangent to BC at P and internally to Γ at Q, so that Q lies on arc BC of Γ not containing A. Prove that if $\angle BAO = \angle CAO$ then $\angle PAO = \angle QAO$.

Problem A.12 (#4.53, SL 2002)**.** The incircle Ω of the acute-angled triangle ABC is tangent to its side BC at a point K. Let $\overline{AD}$ be an altitude of triangle ABC, and let M be the midpoint of $\overline{AD}$. If N is the common point of the circle Ω and $\overline{KM}$ (distinct from K), then prove Ω and the circumcircle of triangle BCN are tangent to each other.

§B Inequalities

- **Reading**: You should read *A Brief Introduction to Olympiad Inequalities* [1] as the material there is necessary (and sufficient) to solve these problems.

Problem B.1. Let a, b, c be positive reals. Prove that

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \ge a + b + c.$$

Problem B.2. Suppose that $a^2 + b^2 + c^2 = 1$ for positive real numbers a, b, c. Find the minimum possible value of

$$\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}.$$

Problem B.3. Let a, b, c be positive real numbers such that $a^2+b^2+c^2+(a+b+c)^2 \le 4$. Prove that

$$\frac{ab+1}{(a+b)^2} + \frac{bc+1}{(b+c)^2} + \frac{ca+1}{(c+a)^2} \ge 3.$$

Problem B.4. Let a, b, c, d be positive reals with $(a+c)(b+d) = 1$. Prove that

$$\frac{a^3}{b+c+d} + \frac{b^3}{c+d+a} + \frac{c^3}{d+a+b} + \frac{d^3}{a+b+c} \ge \frac{1}{3}.$$

Problem B.5. For positive real numbers a, b, c satisfying $abc = 1$, prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \ge \frac{3}{2}.$$

Problem B.6. For positive real numbers a, b, c with $abc = 1$ prove that

$$\left(a + \frac{1}{b}\right)^2 + \left(b + \frac{1}{c}\right)^2 + \left(c + \frac{1}{a}\right)^2 \ge 3(a + b + c + 1).$$

§C More

Problem C.1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ for which

$$f(xf(x) + f(y)) = f(x)^2 + y$$

holds for all real numbers x and y.

Problem C.2. Let a, b, c, d be real numbers such that $b - d \geq 5$ and all zeros x_1, x_2, x_3, and x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product $(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$ can take.

Problem C.3. Ana and Banana are playing a game. First Ana picks a word, which is defined to be a nonempty sequence of capital English letters. Then Banana picks a nonnegative integer k and challenges Ana to supply a word with exactly k subsequences which are equal to Ana's word. Ana wins if she is able to supply such a word, otherwise she loses. For example, if Ana picks the word "TST", and Banana chooses $k = 4$, then Ana can supply the word "TSTST" which has 4 subsequences which are equal to Ana's word. Which words can Ana pick so that she can win no matter what value of k Banana chooses?

Problem C.4 (IMO 2017)**.** A hunter and an invisible rabbit play a game in the plane. The rabbit and hunter start at points $A_0 = B_0$. In the nth round of the game ($n \geq 1$), three things occur in order:

(i) The rabbit moves invisibly from A_{n-1} to a point A_n such that $A_{n-1}A_n = 1$.

(ii) The hunter has a tracking device (e.g. dog) which reports an approximate location P_n of the rabbit, such that $P_nA_n \leq 1$.

(iii) The hunter moves visibly from B_{n-1} to a point B_n such that $B_{n-1}B_n = 1$.

Let $N = 10^9$. Can the hunter guarantee that $A_NB_N < 100$?

References

[1] Evan Chen. A brief introduction to olympiad inequalities. http://web.evanchen.cc/handouts/Ineq/en.pdf.

[2] Evan Chen. Remarks on English. http://web.evanchen.cc/handouts/english/english.pdf.

[3] Evan Chen. *Euclidean geometry in mathematical olympiads*. MAA Problem Books Series. Mathematical Association of America, Washington, DC, 2016.