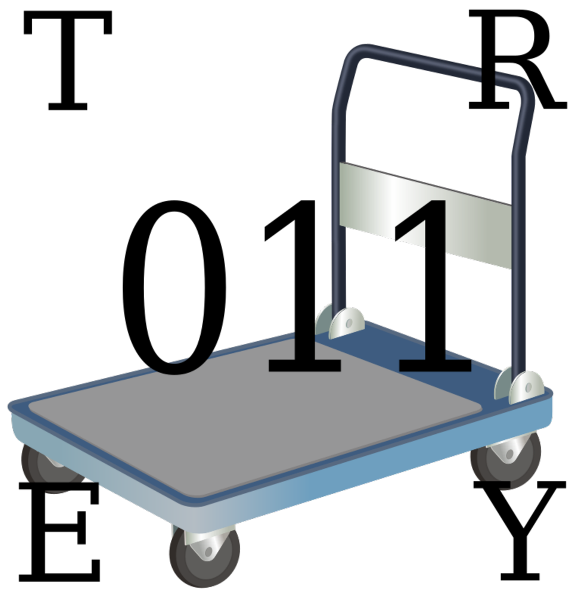
Undergraduate Math 011

a firsT yeaR coursE in geometrY



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Preface

0.1. Synopsis. Among the four olympiad math subjects, geometry has the reputation for being by far the most reliant on specific knowledge to do well in (which haters of geometry often cite when complaining about it). But it occurred to me recently that, despite this reputation, there isn't really *that* much material in the subject.

In the spring of 2016, I published a textbook *Euclidean Geometry in Math Olympiads* (henceforth EGMO) which I think contains every theorem that an IMO gold medalist would be "expected to know". This has as far as I can tell become the American standard for learning the subject, and these notes are based on EGMO.

I want to point out now — EGMO is *pretty thin* as far as things that could be called a "textbook" go. The main matter is about 200 pages. And even then, most of it is not theory — it is a lot of contest problems, worked examples, motivational discussion, and so on.

So it occurred to me one day that **all of olympiad geometry probably fits comfortably in a typical one-semester college course**. In fact, maybe not even that. Half a semester might be more accurate, now that I think about it.

And so to stake my claim, I present to you my latest work:

Undergraduate Math 011: a firsT semesteR coursE in geometrY

Not offered this academic year.

0.2. Goals of this book. The intention of these notes is that they are

- self-contained,
- as short as possible, yet
- technically complete for olympiad purposes.

In particular, most major theorems in EGMO should appear, and the proofs (though terse) should also be included for completeness when it is not unreasonable. It is thus expected that any classical geometry problem appearing at IMO or USAMO will fall entirely in the span of the techniques contained in this document (and perhaps even within the first chapter). The lecture notes are divided into 10 sections, which loosely correspond in order to the 10 chapters of EGMO.

0.3. Casualties in achieving these goals. In achieving these goals, while making the text as short as possible, the following decisions were made:

- No friendly speech or discussion. We are in the fast lane, and everything is correspondingly terse.
- No "worked examples" (okay, there are a few, but not many). The only material presented is the core material; most "applications" are exercises.
- No diagrams. This is a little extreme, but I do remember once trying to read a graph theory textbook that had no figures… so not unheard of.

- Few named theorems. No named problems. (I believe this is the norm.)
- Few contest problems. Most exercises are chosen to be boring "textbook-style" ones.[1]
- In particular, important theorem are often left as exercises.[2] In some cases things which would be theorems in EGMO are downgraded to exercises here, to maximize confusion with inessential results.
- No solutions. We are a math textbook here: "left as an exercise".

0.4. Satirical comments. As a consequence of these design decisions, olympiad contestants will to some extent find these notes useless, and should read EGMO instead. Indeed part of my point in writing this was to draw attention to the contrast between

(i) having a surface-level understanding of an area of math, and
(ii) being able to solve IMO-level problems with it.

In some ways, this is a celebration of just how strong the top high school geometers are. More pessimistically, it can also be considered a tongue-in-cheek jab at how higher math is often taught. For this reason, I also took pride in making it look like the type of writing style and formatting that is common in real mathematics — that is, the usual stuffy, formal, bleed-your-eyes-in-boredom writing that passes for exposition in mediocre textbooks.

In short, here is what olympiad geometry might look like if it was taught in a typical top-tier university.

0.5. Stray unironic comment. I considered not including this preface in order to make this document more of an April Fool's Day gag. But then I realized that there was an uncomfortably real possibility that people might not realize this was a satirical document and try to use it, and that would be very bad.

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[1] Incidentally, there is something qualitatively unappealing about the 101 exercises I chose for these lecture notes. See, a geometry problem which appears on an olympiad seems to me to have a *personality*: it is a concrete, specific problem, that is unlikely to be independently constructed by someone else. Whereas the exercises in these lecture notes are nameless, unmemorable, and somehow all look the same.

[2] For the record: the more I think about it, the more I object to this common practice in certain textbooks. I increasingly suspect it is just due to laziness by authors. The "main matter" is (in my opinion) not the right time to let the reader "work it out for themselves", precisely because they are seeing material for the first time. They should *eventually* be able to work out some proofs for themselves — but not now.

CHAPTER I

Synthetic geometry

Preliminaries

We will generally assume a knowledge of very basic high school geometry, for example the axioms of Euclid.

Let ABC be a triangle. We also take for granted the existence of four remarkable triangle centers:

(1) The *incenter* I is the intersection of the internal angle bisectors, which is also the center of the unique circle inside the triangle tangent to all three sides.
(2) The *centroid* G is the unique point inside the triangle such that ray AG bisects side BC, etc.
(3) The *orthocenter* H is the intersection of the three altitudes of the triangle.
(4) The *circumcenter* O is the center of the unique circle passing through A, B, and C.

Proofs of the existence of these centers can be found in any standard reference, and we will also supply proofs in the exercises. In general, if we state a result involving a triangle ABC then we may refer to any of the names G, H, I, O without unnecessary redefinition.

We also give names to the associated inscribed triangles.

(1) The *intouch triangle* of ABC is the triangle whose vertices are the contact points of the incircle with the sides of ABC.
(2) The *medial triangle* of ABC is the triangle whose vertices are the midpoints of the sides of ABC.
(3) The *orthic triangle* of ABC is the triangle whose vertices are the feet of the altitudes.

We also remind the reader that by (XYZ) we mean the circle through X, Y, Z.

§1. Angles

1.1. Definitions. For most of this book, we are going to be working with so-called directed angles.

1.1.1. Definition. Given any two non-parallel lines ℓ and m, we define the *directed angle*

$$\measuredangle(\ell, m)$$

to be the measure of the angle *starting* from ℓ and *ending* at m, measured *counterclockwise*. If ℓ and m are parallel or coincide, we define the angle to be zero. Moreover, unless specified otherwise, the measures of the angles are always considered modulo 180°.

From here, it is easy to check the following results.

1.1.2. PROPOSITION. Let ℓ, m, n be lines.

(1) $\measuredangle(\ell, m) = -\measuredangle(m, \ell)$.
(2) If ℓ, m, n are concurrent, then $\measuredangle(\ell, m) + \measuredangle(m, n) = \measuredangle(\ell, n)$.
(3) If $m \parallel n$ and ℓ is another line, then $\measuredangle(\ell, m) = \measuredangle(\ell, n)$.

We leave the proof as Exercise 3.
The angle formed by three points then follows as a special case of this.

1.1.3. DEFINITION. Given three points A, O, B we define

$$\measuredangle AOB := \left(\overline{AO}, \overline{BO}\right).$$

Equivalently, if ℓ and m are two lines which intersect at O, then $\measuredangle(\ell, m) = \measuredangle AOB$ for any point A on ℓ and B on m.

One can then translate the earlier proposition to obtain several results in the new notation, which will be used more frequently.

1.1.4. PROPOSITION. For any distinct points A, B, C, P in the plane, we have the following rules.

(1) $\measuredangle APA = 0$.
(2) $\measuredangle ABC = -\measuredangle CBA$.
(3) $\measuredangle APB + \measuredangle BPC = \measuredangle APC$.
(4) $\measuredangle PBA = \measuredangle PBC$ if and only if A, B, C are collinear.
(5) If $\overline{AP} \perp \overline{BP}$, then $\measuredangle APB = \measuredangle BPA = 90°$.
(6) $\measuredangle ABC + \measuredangle BCA + \measuredangle CAB = 0$.
(7) $AB = AC$ if and only if $\measuredangle ACB = \measuredangle CBA$.
(8) If $\overline{AB} \parallel \overline{CD}$, then $\measuredangle ABC + \measuredangle BCD = 0$.

PROOF. (1) $\measuredangle APA = \measuredangle(\overline{AP}, \overline{AP}) = 0$.
(2) $\measuredangle ABC = \measuredangle(\overline{AB}, \overline{BC}) = -\measuredangle(\overline{CB}, \overline{BA}) = -\measuredangle CBA$.
(3) $\measuredangle APB + \measuredangle BPC = \measuredangle(\overline{AP}, \overline{PB}) + \measuredangle(\overline{BP}, \overline{PC}) = \measuredangle(\overline{AP}, \overline{CP}) = \measuredangle APC$.
(4) By using (2) and (3) together, we have $\measuredangle PBA - \measuredangle PBC = \measuredangle PBA + \measuredangle CBP = \measuredangle\left(\overline{CB}, \overline{BP}\right) + \measuredangle\left(\overline{BP}, \measuredangle BA\right) = \measuredangle\left(\overline{CB}, \overline{BA}\right)$. This is equal to zero if and only if lines CB and BA coincide, which is equivalent to A, B, C collinear.
(5) This is obvious.
(6) Introduce a line ℓ through A parallel to $\overline{BC}$ (this is possible by Euclid's fifth postulate). Then

$$\begin{aligned}
&\measuredangle ABC + \measuredangle BCA + \measuredangle CAB \\
&= \measuredangle\left(\overline{AB}, \overline{BC}\right) + \measuredangle\left(\overline{BC}, \overline{CA}\right) + \measuredangle\left(\overline{CA}, \overline{AB}\right) \\
&= \measuredangle\left(\overline{AB}, \ell\right) + \measuredangle\left(\ell, \overline{CA}\right) + \measuredangle\left(\overline{CA}, \overline{AB}\right) \\
&= \measuredangle\left(\overline{AB}, \overline{CA}\right) + \measuredangle\left(\overline{CA}, \overline{AB}\right) \\
&= 0.
\end{aligned}$$

(7) This is equivalent to the fact that an triangle is isosceles if and only if two of the vertex angles are equal.
(8) $\measuredangle(\overline{AB}, \overline{BC}) + \measuredangle(\overline{BC}, \overline{CD}) = \measuredangle\left(\overline{AB}, \overline{CD}\right) = 0$. $\square$

1.2. Cyclic quadrilaterals.

1.2.1. Definition. We say that a set of points is *concyclic* if one can draw a circle through them. Four concyclic points are said to form a *cyclic quadrilateral*.

Since any three non-collinear points are concyclic (as we will prove later on), this definition is only interesting with four or more points.

We state the following important theorem (which will be invoked repeatedly).

1.2.2. Theorem. Four distinct points A, B, X, Y are concyclic if and only if

$$\measuredangle AXB = \measuredangle AYB.$$

Regrettably, its proof is case-dependent and hence rather tiresome, so we omit the proof from this book. However, the utility of this theorem cannot be overstated.

In a degenerate case when two points coincide, we also have the following equivalence.

1.2.3. Theorem. Line ℓ passes through a point P on a circle centered at O. The circle also passes through A and B. Then the following are equivalent:

(1) ℓ is tangent to the circle.
(2) $\ell \perp \overline{OP}$.
(3) $\measuredangle\left(\ell, \overline{PA}\right) = \measuredangle PBA$.

1.3. Applications. We give some first examples of applications of our results now, which can be applied ubiquitously.

1.3.1. Lemma. Let ABC be a triangle with orthic triangle DEF. Then the quadrilaterals $AEHF$, $BFHD$, $CDHE$, $BFEC$, $CFDA$, $ADEB$ are concyclic.

Proof. The quadrilateral $AEHF$ is cyclic since $\measuredangle AEH = \measuredangle AEB = 90^\circ = \measuredangle AFC = \measuredangle AFH$. The cases $BFHD$ and $CDHE$ are similar.

The quadrilateral $BFED$ is cyclic since $\measuredangle BFC = 90^\circ = \measuredangle BEC$, and similarly for the next three. □

1.3.2. Lemma. Let ABC be a triangle and D, E, F points on lines BC, CA, AB. Then (AEF), (BFD), (CDE) are concurrent at a point.

Proof. Let K be the second intersection of the two circles. Then $\measuredangle FKD = \measuredangle FBD = \measuredangle ABC$ and $\measuredangle DKE = \measuredangle DCE = \measuredangle BCA$. As $0 = \measuredangle DKE + \measuredangle EKF + \measuredangle FKD = \measuredangle ABC + \measuredangle BCA + \measuredangle CAB$, it follows $\measuredangle EKF = \measuredangle CAB$, and so $AEKF$ is concyclic. □

1.3.3. Definition. The point of concurrence in the previous lemma is referred to as the *Miquel point* of DEF with respect to ABC.

That being said, because we are taking modulo 180°, we most be careful to not take "half" of a directed angle, since in that situation things may not make sense. Here is an example of a situation in which it is thus necessary to pay some attention to the positions of the points.

1.3.4. Definition. We define the A-*excenter* of triangle ABC as the intersection of the $\angle A$ bisector and the external angle bisectors of $\angle B$ and $\angle C$. It is typically denoted I_a.

1.3.5. Lemma. Let ABC be a triangle.

(i) The midpoint of $\overline{II_a}$ is the arc midpoint of arc $\widehat{BC}$ not containing A.
(ii) The points B, I, C, I_a lie on a circle with diameter $\overline{II_a}$.

Since this is routine, the proof is left as an exercise.

1.4. Phantom points. Moving forward, it will often be helpful to be able to rephrase problems in certain ways. Although this does not need official definition, we shall give an example of a situation in which it might be helpful.

1.4.1. Lemma. Let ABC be a triangle with $AB \neq AC$. The perpendicular bisector of $\overline{BC}$ intersects the $\angle A$ bisector at a point K. Then the points A, B, C, K are concyclic.

Proof. It turns out that there is not an easy way to directly show the angles with the point K was given. However, suppose we define K' as the intersection of the $\angle A$ bisector with the circumcircle. Then (say by Lemma 1.3.5) it follows that K' is the midpoint of arc $\widehat{BC}$, so $BK' = CK'$, and K' lies on the perpendicular bisector of $\overline{BC}$ too. This implies $K = K'$, since two lines can only intersect at a unique point. $\square$

1.5. Similar triangles. We mention now, as we will have need for it later, that similar triangles can also be written in terms of directed angles. In doing so we may also pay attention to their orientations.

1.5.1. Definition. Consider triangles ABC and XYZ. We say they are *directly similar* if

$$\measuredangle ABC = \measuredangle XYZ, \ \measuredangle BCA = \measuredangle YZX, \text{ and } \measuredangle CAB = \measuredangle ZXY.$$

We say they are *oppositely similar* if

$$\measuredangle ABC = -\measuredangle XYZ, \ \measuredangle BCA = -\measuredangle YZX, \text{ and } \measuredangle CAB = -\measuredangle ZXY.$$

If either condition is satisfied we say they are *similar*.

The upshot of this is that we may continue to use directed angles when proving triangles are similar; we just need to be a little more careful. In any case, as you probably already know, similar triangles also produce ratios of lengths.

1.5.2. Proposition. The following are equivalent for triangles ABC and XYZ.
(i) $\triangle ABC \sim \triangle XYZ$.
(ii) (AA) $\angle A = \angle X$ and $\angle B = \angle Y$.
(iii) (SAS) $\angle B = \angle Y$, and $AB : XY = BC : YZ$.
(iv) (SSS) $AB : XY = BC : YZ = CA : ZX$.

Thus, lengths (particularly their ratios) can induce similar triangles and vice versa. However, notice that SAS similarity does not have a directed form; see the exercises.

Exercises.

1. Exercise. If A, B, C are points lying on a circle centered at O, show that $\measuredangle AOC = 2\measuredangle ABC$.

2. Exercise. If A, B, C, D are four distinct points, show that

$$\measuredangle ABC + \measuredangle BCD + \measuredangle CDA + \measuredangle DAB = 0.$$

3. Exercise. Finish the proof of Proposition 1.1.2.

4. Exercise. Find an example of two triangles ABC and XYZ such that $AB : XY = BC : YZ$, $\measuredangle BCA = \measuredangle YZX$, but $\triangle ABC$ and $\triangle XYZ$ are not similar.

5. Exercise. Identify the Miquel point associated to (a) the orthic triangle, (b) the intouch triangle.

6. Exercise. Show that triangle $I_a I_b I_c$ has orthic triangle ABC.

7. Exercise (*). Prove Lemma 1.3.5. Why does it require undirected angles?

8. Exercise. Let $ABCDE$ be a convex pentagon such that $BCDE$ is a square with center O and $\angle A = 90°$. Show that $\overline{AO}$ bisects $\angle BAE$.

9. Exercise. Let P be a point inside circle ω. Consider the set of chords of ω that contain P. Prove that their midpoints all lie on a circle.

10. Exercise. In cyclic quadrilateral $ABCD$, let I_1 and I_2 denote the incenters of $\triangle ABC$ and $\triangle DBC$, respectively. Show that quadrilateral $I_1 I_2 BC$ is cyclic.

11. Exercise. Let ABC be an acute triangle inscribed in circle Ω. Let X be the midpoint of the arc $\widehat{BC}$ not containing A and define Y, Z similarly. Identify the orthocenter of $\triangle XYZ$.

12. Exercise. Let ABC be a triangle with altitudes $\overline{BE}$, $\overline{CF}$ and let M denote the midpoint of $\overline{BC}$. Show that the tangents to (AEF) at E and F meet at M, and the tangent to (AEF) at A is parallel to BC.

13. Exercise (*). Let ABC be a triangle and P a point on its circumcircle. Show that feet of the perpendiculars from P to sides BC, CA, AB are collinear. This line is called the *Simson line* of P.

14. Exercise (*). Let ABC be a triangle with intouch triangle DEF. Let M and N denote the midpoints of $\overline{BC}$ and $\overline{AC}$, and let K denote the intersection of $\overline{BI}$ and $\overline{EF}$. Show that $\angle BKC = 90°$ and moreover K lies on line MN.

§2. Power of a point

In addition to behaving well with respect to angles, circles give rise to length relations which give a powerful tool for our study in geometry.

2.1. The power of a point. The basis of the power of a point theorem starts in the following observation.

2.1.1. Theorem. Let P be a point inside a circle. Two lines through P are drawn. The first line intersects the circle at A and A' (which may be the same point if the line is a tangent). The second line intersects the circle at B and B' (which may be the same point if the line is a tangent). Then

$$PA \times PA' = PB \times PB'.$$

Proof. This is an easy application of similar triangles. Observe that $\measuredangle PB'A = \measuredangle BB'A = \measuredangle BA'A = \measuredangle BA'P$. Similarly, $\measuredangle PA'B = \measuredangle AB'P$. Therefore triangles $PA'B$ and $PB'A$ are similar and oppositely oriented. Consequently, $\frac{PA'}{PB} = \frac{PB'}{PA}$ which rearranges to the given result. $\square$

In light of this, it is convenient to make the following definition.

2.1.2. Definition. Let ω be a circle with radius r and center O. The *power* of P with respect to ω, which we denote as $\mathrm{Pow}_\omega(P)$, is defined as $PO^2 - r^2$.

Note that:

- If P lies on ω, then $\mathrm{Pow}_\omega(P) = 0$.
- If P lies outside ω, and A, A' are as in the previous theorem, then $\mathrm{Pow}_\omega(P) = PA \times PA'$. This follows by taking the corresponding B and B' so that $\overline{BB'}$ is a diameter of the circle, whence $PB \times PB' = (PO + r)(PO - r) = PO^2 - r^2$.
- If P lies inside ω, and A, A' are as in the previous theorem, then $\mathrm{Pow}_\omega(P) = -AP \times PA'$, by a similar argument.

Therefore this definition gives an "intrinsic" way of defining the fixed quantity in Proposition 2.1.1.

There is another way to realize the sign convention above.

2.1.3. Definition. Let A, P, A' be collinear points. We agree that the *signed product* $PA \times PA'$ has negative sign if P lies between A and A', and positive otherwise. We also adopt the convention $AP = -PA$, so that both $AP \times PA' = -PA \times PA'$

2.1.4. Definition. The *signed quotient* $\frac{PA}{PA'} = -\frac{AP}{PA'}$ follows the same sign convention.

Then all three cases above can be succinctly summarized:

2.1.5. Theorem. If a line through a point P intersects a circle ω at two points A and A', then

$$PA \times PA' = \mathrm{Pow}_\omega(P).$$

Note that there is no longer a distinction between whether P lies inside or outside the circle.

With the signed product convention, the power of a point theorem has a converse too.

2.1.6. Theorem. Two lines AA' and BB' meet at a point P such that

$$PA \times PA' = PB \times PB'.$$

Then A, A', B, B' are cyclic.

We leave the proof as Exercise 15.

2.2. Radical axis. Suppose now we consider two circles.

2.2.1. Theorem. Let ω and γ be two non-cocentric circles. Then the locus of points P with $\mathrm{Pow}_\omega(P) = \mathrm{Pow}_\gamma(P)$ is a line perpendicular to the line through their centers.

Proof. We have a rare application of Cartesian coordinates. Let us denote by $(a, 0)$ the center of the first circle, with radius r. Also let us denote by $(b, 0)$ the center of the second circle, with radius s. Then

$$\begin{aligned} \mathrm{Pow}_\omega(P) &= \mathrm{Pow}_\gamma(P) \\ \iff [(x-a)^2 + y^2] - r^2 &= [(x-b)^2 + y^2] - s^2 \\ \iff 2(a-b)x &= b^2 - a^2 + r^2 - s^2 \end{aligned}$$

which describes a vertical line. $\square$

Note that if the two circles have common points, those common points have power 0 and are thus on the locus. In particular, if two circles meet at points A and B, it is more economical to describe their radical axis as line AB.

2.3. Radical center. We are given now *three* circles ω_1, ω_2, ω_3 whose centers are not collinear. We can draw the radical axis of any pair.

2.3.1. THEOREM. In this situation the three radical axes are concurrent.

PROOF. Suppose the radical axis of ω_1 and ω_2 meets the radical axis of ω_2 and ω_3 at P. (Because the centers are not collinear, the radical axes are not parallel.) Then $\operatorname{Pow}_{\omega_1}(P) = \operatorname{Pow}_{\omega_2}(P) = \operatorname{Pow}_{\omega_3}(P)$. So P lies on the third radical axis. $\square$

2.3.2. DEFINITION. The concurrency point is called the *radical center* of the three circles.

This gives a valuable way to show that lines are concurrent which often cannot easily be done with just angle chasing alone

Exercises.

15. EXERCISE. Prove Theorem 2.1.6.

16. EXERCISE. Let $t > 0$. If $\operatorname{Pow}_\omega(P) = t^2 > 0$ give a geometric interpretation for t.

17. EXERCISE. Three circles are drawn, each pair having a common chord. Show that their common chords are concurrent.

18. EXERCISE (*). Let ABC be a triangle with circumradius R and inradius r. Show that $IO^2 = R(R - 2r)$.

19. EXERCISE. Two circles meet at A and B. A line is drawn tangent to the first circle at X, and tangent to the circle at Y. Show that line AB bisects $\overline{XY}$.

20. EXERCISE. Let A, B, C be three noncollinear points, and draw a circle of radius zero at each. Determine their radical center. Use this to give a proof that the circumcenter O exists.

21. EXERCISE. Identify the radical axis of the circles with diameter $\overline{AB}$ and $\overline{AC}$. Use this to prove that the orthocenter H exists.

22. EXERCISE (*). Scalene triangle ABC is given with orthic triangle DEF. Show that circles (AOD), (BOE), (COF) meet at a point other than O.

23. EXERCISE (**). Let $ABCD$ be a quadrilateral and let $E = \overline{AB} \cap \overline{CD}$, $F = \overline{BC} \cap \overline{DA}$. Show that the circles with diameters $\overline{AC}$, $\overline{BD}$ and $\overline{EF}$ have a common radical axis. Deduce that the midpoints of $\overline{AC}$, $\overline{BD}$, $\overline{EF}$ are collinear. (Hint: consider the orthocenters of certain triangles.)

§3. Homothety

3.1. Definition and properties. One way to capture at once a lot of information that normally would form a similar triangles argument is through the notion of homothety.

3.1.1. Definition. A *homothety* h is a transformation defined by a center O and a nonzero real number k (not necessarily positive). It sends a point P to another point $h(P)$, multiplying the distance from O by k.

It is important to note that k can be negative; in that case, O will lie between P and $h(P)$.

It is easy to see that homothety preserves similarity. Homothety also preserves many things, including but not limited to tangency, angles (both vanilla and directed), circles, and so on. They do not preserve length, but they work well enough: the lengths are simply all multiplied by k.

3.1.2. Proposition. Given noncongruent parallel segments $\overline{AB}$ and $\overline{XY}$, there is a unique homothety sending A to X and B to Y.

Proof. Since $AB \neq XY$, the quadrilateral $ABYX$ is not a parallelogram, and we may take O to be the intersection of lines AX and BY. Then $\triangle OAB \sim \triangle OXY$. The common scale factor is then the signed quotient $\frac{OX}{OA} = \frac{OY}{OB}$. $\square$

This is often used with triangles. A consequence of this is the following useful lemma.

3.1.3. Corollary. Let ABC and XYZ be non-congruent triangles such that $\overline{AB} \parallel \overline{XY}$, $\overline{BC} \parallel \overline{YZ}$, and $\overline{CA} \parallel \overline{ZX}$. Then lines AX, BY, CZ concur at some point O, and O is a center of a homothety mapping $\triangle ABC$ to $\triangle XYZ$.

Proof. Exercise 24. $\square$

3.2. Application to the medial triangle. Suppose that $\triangle DEF$ is the medial triangle of $\triangle ABC$, then the corresponding sides are parallel and have ratio $\frac{1}{2}$. Consequently:

3.2.1. Theorem. The centroid G of a triangle is the center of a negative homothety, with ratio $-\frac{1}{2}$, which maps ABC to its medial triangle.

This immediately gives applications.

3.2.2. Corollary. If D is the midpoint of BC, then we have the signed quotient $\frac{GA}{GD} = -2$.

3.2.3. Corollary. Points G, H, O are collinear with signed quotient $\frac{GH}{GO} = -2$.

Proof. Observe that since $\overline{DO} \perp \overline{BC}$, we have $\overline{DO} \perp \overline{EF}$. Therefore O is actually the orthocenter of $\triangle DEF$, so the homothety will map it to H, the orthocenter of $\triangle ABC$. $\square$

3.3. Application to the nine-point circle. Here is yet another application. First, we need the following fact.

3.3.1. Proposition. In triangle ABC,

(a) the reflection of H across side BC lies on (ABC).
(b) the reflection of H across the midpoint of side BC lies on (ABC) diametrically opposite A.

Proof. This is just angle chasing, see Exercise 25. $\square$

3.3.2. Theorem. Let ABC be a triangle. There is a circle passing through the following nine points:

- The midpoints of AH, BH, CH.
- The midpoints of BC, CA, AB.
- The feet of the altitudes.

The center of this circle is the midpoint of $\overline{OH}$, and this circle has half the radius of (ABC).

Proof. Consider a homothety at H with ratio 2. Each of the nine points mentioned goes to a point on (ABC), the first three by definition, the latter six by Proposition 3.3.1. Thus after the homothety, the nine points lie on (ABC). So before the homothety, they must have been on a circle as described. $\square$

3.3.3. Definition. This circle is called the *nine-point circle*. Its center is called the *nine-point center*.

3.3.4. Definition. The *Euler line* is the line through the points G, O, H and the nine-point center.

3.4. An application to tangent circles. Tangents at circles are also nicely handled by homothety.

3.4.1. Proposition. Chord $\overline{AB}$ is given in a circle Ω. Let ω be a circle tangent to chord AB at K and internally tangent to ω at T. Then ray TK passes through the midpoint M of arc $\widehat{AB}$ of Ω, not containing T.

Proof. Since Ω and ω are tangent at T, it follows there is a homothety at T taking ω to Ω. Because the tangent to ω at K (which is line AB) is parallel to the tangent to Ω at M it follows the homothety maps K to M. Hence T, K, M are collinear. $\square$

Exercises.

24. Exercise. Prove Corollary 3.1.3.

25. Exercise. Prove Proposition 3.3.1.

26. Exercise. Given two noncongruent circles, neither contained in the other, show that there is a unique positive homothety sending one to the other, and its center is the intersection of the common external tangents.

27. Exercise. Let ABC be a triangle. The incircle touches $\overline{BC}$ at D, while the A-excircle touches BC at E. Show that $\overline{AE}$ passes through the antipode of D on the incircle.

28. Exercise. In the notation of the previous exercise, show that $\overline{EI}$ bisects the A-altitude.

29. Exercise (*). Consider three circles ω_1, ω_2, ω_3 in the plane no two congruent and with disjoint interiors. For each pair of circles, we construct the intersection of their common external tangents. Prove that these three intersections are collinear.

30. Exercise (*). In triangle ABC with contact triangle DEF, point M is the midpoint of $\overline{BC}$. Prove that the lines AM, EF, DI are concurrent. (Hint: draw a line through $\overline{DI} \cap \overline{EF}$ and let it meet sides $\overline{AB}$ and $\overline{AC}$ at B' and C'. Try to show $AB'IC'$ is cyclic and $B'I = C'I$.)

31. Exercise (**). Let ABC be a triangle and D be a point on $\overline{AB}$. Suppose a circle ω is tangent to $\overline{CD}$ at L, $\overline{AB}$ at K, and also to (ABC). Show that the incenter of ABC lies on line LK.

§4. Trigonometry and lengths

We assume the reader is familiar with the notation sin and cos. These are tools which allow one to relate sides and angles together, and accordingly become quite useful. Since these topics are regularly covered in high school curriculums, we will be brief.

4.1. Notation. Throughout this section (and moving forwards), if $\triangle ABC$ is a given triangle, we will let $a = BC$, $b = CA$, $c = AB$ denote the lengths of the triangle, and $A = \angle BAC$, $B = \angle ABC$, $C = \angle BCA$ denote the measures of the angles. It is also customary to set

$$s = \frac{a+b+c}{2}$$

as the so-called *semiperimeter* of the triangle.

4.2. The extended law of sines. The law of sines is usually given in $\frac{a}{\sin A} = \dots$ but there is in fact an extended form which is more symmetric.

4.2.1. Theorem. If R denotes the circumradius of $\triangle ABC$, then

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

A proof, as well as several applications, is given in the exercises. Note that this means that if we know two angles of a triangle and one side, then we can recover everything else.

4.3. The law of cosines. We would like to also be able to determine a triangle based on two side lengths and the included angle, and the law of cosines does this for us.

4.3.1. Theorem. We have

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Proof. We handle only the case where $\cos C > 0$ since the other cases are analogous. The A-altitude to side BC, which we denote by $\overline{AD}$, has $AD = b\sin C$ and $CD = b\cos C$. Then by the Pythagorean theorem,

$$\begin{aligned} c^2 &= AD^2 + BD^2 \\ &= (b\sin C)^2 + (a - b\cos C)^2 \\ &= b^2(\sin C)^2 + a^2 - 2ab\cos C + b^2(\cos C)^2 \\ &= a^2 + b^2\left[(\sin C)^2 + (\cos C)^2\right] - 2ab\cos C \\ &= a^2 + b^2 - 2ab\cos C. \end{aligned}$$

□

Exercises.

32. Exercise. Deduce the Pythagorean theorem from the law of cosines.

33. Exercise. The $\angle A$ bisector meets side BC at D. Show that $DB/DC = AB/AC$.

34. Exercise. Show that $a = 2R \sin A$ in Theorem 4.2.1, thus establishing the theorem.

35. Exercise. Let ABC be a triangle and D a point on side BC. Show that

$$a \times (DA^2 + DB \times DC) = b^2 \times DB + c^2 \times DC.$$

36. Exercise. Let $A_1A_2A_3A_4A_5$ be a convex pentagon. Assume rays $A_{i+1}A_{i+2}$ and $A_{i+4}A_{i+3}$ meet at point X_i. Show that

$$\prod_{i=1}^{5} X_iA_{i+2} = \prod_{i=1}^{5} X_iA_{i+3}.$$

Here all indices are taken modulo 5.

37. Exercise (*). In a convex cyclic quadrilateral $ABCD$ show that

$$AB \cdot CD + BC \cdot DA = AC \cdot BD.$$

38. Exercise (*). In this problem we derive a formula for the area of a triangle in terms of a, b, c.

(a) Show that the area of ABC is given by $\frac{1}{2}ab \sin C$.
(b) Express $\sin C$ as a function of a, b, c.
(c) Prove that the area of ABC is given by

$$\sqrt{s(s-a)(s-b)(s-c)}.$$

39. Exercise (*). Show that the area of a triangle is given also by $\sqrt{rr_ar_br_c}$, where r is the radius of the incircle and r_a, r_b, r_c are the radii of the excircles.

§5. Ceva and Menelaus

5.1. Signed areas. We saw earlier that we got some convenience from signed quotients of lengths, and we will in what follows want to adopt a similar convention for signed areas. Thus we make the following definition.

5.1.1. Definition. If P, Q, R are three points then we denote by $[PQR]$ the *signed area* of triangle PQR; its absolute value is the same as the usual area, but

- it is zero if P, Q, R are collinear,
- it is positive if P, Q, R are situated in counterclockwise order on its circumcircle, and
- it is positive if P, Q, R are situated in clockwise order on its circumcircle.

Note that unlike with signed quotients and products, we can specify a sign of a single area (rather than, say, a quotient of areas).

5.2. Ceva's theorem. Ceva's theorem is a useful result which allows one to determine concurrence of so-called *cevians* of a triangle: lines joining the vertex of a triangle to a point on the opposite side. The proof will use our convention of directed quotients.

5.2.1. THEOREM. Let ABC be a triangle and D, E, F points on the opposite sides or their extensions, which are distinct from the vertices. Then lines AD, BE, CF are concurrent if and only if

$$\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = 1.$$

PROOF. First let us assume that the three cevians are indeed concurrent at P. Then we have

$$\frac{BD}{DC} = \frac{[PBD]}{[PCD]} = \frac{[ABD]}{[ACD]}$$

where the areas are signed. This means that

$$\frac{BD}{DC} = \frac{[ABD] - [PBD]}{[ACD] - [PCD]} = \frac{[ABP]}{[ACP]}.$$

Multiplying cyclically yields the desired equation.

For the converse, assume that the desired concurrence holds. We let lines BE and CF meet at P, and let line AP meet side BC at D'. Then $\frac{BD'}{D'C} = \frac{BD}{DC}$, which is enough to imply $D = D'$. □

For example, this immediately implies that the centroid of the triangle exists, because the signed ratios above are all equal to 1. We leave the remaining centers as a good exercise.

There is also a "trigonometric" form of the same theorem, in case it is easier to access the angles. For simplicity we only state it with the cevians inside the triangle.

5.2.2. THEOREM. Let ABC be a triangle and D, E, F points on the opposite sides. Then lines AD, BE, CF are concurrent if and only if

$$\frac{\sin\angle BAD}{\sin\angle CAD} \times \frac{\sin\angle CBE}{\sin\angle ABE} \times \frac{\sin\angle ACF}{\sin\angle BCF} = 1.$$

PROOF. We have by the law of sines that

$$\frac{\sin\angle BAD}{\sin\angle CAD} = \frac{BD \times \frac{AD}{\sin B}}{DC \times \frac{AD}{\sin C}} = \frac{BD}{DC} \times \frac{\sin C}{\sin B}$$

and when we multiply this cyclically, we find that this reduces to Ceva's theorem. □

5.3. Menelaus theorem. There is a similar theorem if we are hoping that D, E, F are collinear (rather than lines AD, BE, CF being concurrent).

5.3.1. THEOREM. Let ABC be a triangle and D, E, F points on the opposite sides or their extensions, which are distinct from the vertices. Then points D, E, F are concurrent if and only if

$$\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = -1.$$

Proof. As with Ceva's theorem, there is no hope of the collinearity if the sign is positive, and hence we can focus on the case of negative sign. Moreover, by a similar argument with a point D', it suffices to prove the forwards direction, assuming D, E, F are collinear.

Let $Z = \overline{AD} \cap \overline{BE}$ and define X, Y analogously. Applying Ceva's theorem on $\triangle ACD$ (with cevians meeting at F) and $\triangle ABD$ (with cevians meeting at E) that

$$1 = \frac{AY}{YD} \times \frac{DB}{BC} \times \frac{CE}{EA}$$
$$1 = \frac{AF}{FB} \times \frac{BC}{CD} \times \frac{DZ}{ZA}.$$

If we write down the four analogous expressions and multiply them all together, we will find that

$$1 = \left(\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB}\right)^2$$

which, since we are only considering the case of negative sign, implies the result. □

Exercises.

40. Exercise. Use Ceva's theorem to establish the existence of the incenter I.

41. Exercise. Use Ceva's theorem to establish the existence of the orthocenter H.

42. Exercise (*). Let $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ be cevians meeting in the interior of $\triangle ABC$. Let $\overline{DX}$, $\overline{EY}$, $\overline{FZ}$ be cevians meeting in the interior of $\triangle DEF$. Show that lines AX, BY, CZ are concurrent.

43. Exercise. Let $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ be cevians meeting at a point P in the interior of $\triangle ABC$. Denote by D' the reflection of D across the midpoint of $\overline{BC}$ and define E', F' similarly. Show that cevians $\overline{AD'}$, $\overline{BE'}$, $\overline{CF'}$ are concurrent. (The concurrency point is called the *isotomic conjugate* of P.)

44. Exercise (*). Let P be a point in the interior of $\triangle ABC$. Show that there exists a point Q inside $\triangle ABC$ such that $\measuredangle BAP = \measuredangle QAC$, $\measuredangle CBP = \measuredangle QBA$, $\measuredangle ACP = \measuredangle QCB$. (The point Q is called the *isogonal conjugate* of P.)

45. Exercise. What is the isogonal conjugate of the orthocenter?

46. Exercise (*). Let ABC be a triangle and let $s = \frac{1}{2}(a+b+c)$ denote its semiperimeter. We also let D, E, F be the contact points of the incircle on the opposite sides.

(a) Prove that $BD = s - b$ and $CD = s - c$.
(b) Show that lines AD, BE, CF are concurrent. The concurrency point is called the *Gergonne point*.

47. Exercise (**). In triangle ABC the A-excircle touches side BC at D, and points E, F are defined similarly. Show that lines AD, BE, CF are concurrent. The concurrency point is called the *Nagel point*.

CHAPTER II

Coordinate systems

§6. Complex numbers

6.1. The complex plane. Let $\mathbb{C}$ and $\mathbb{R}$ denote the set of complex and real numbers, respectively. Our intention is to assign a *complex number* $z \in \mathbb{C}$ to each point Z in the plane.

Each $z \in \mathbb{C}$ can be expressed as

$$z = a + bi = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

where $a, b, r, \theta \in \mathbb{R}$ and $0 \le \theta < 2\pi$. We write $|z| = r = \sqrt{a^2+b^2}$ and $\arg z = \theta$. Recall that

$$|z_1 z_2| = |z_1|\,|z_2| \text{ and } \arg z_1 z_2 = \arg z_1 + \arg z_2.$$

More importantly, each z is associated with a *complex conjugate* $\overline{z} = a - bi$, which commutes with addition and multiplication We also have the following important identity.

6.1.1. Proposition. For any complex number z, $|z|^2 = z\overline{z}$.

Proof. This is obvious by expanding. $\square$

When doing geometry problems, it is customary that the coordinates of a point denoted by a capital letter are represented by the lowercase letter; for example the complex coordinate of A is represented a. We adopt this convention without further comment in what follows.

6.2. First results.

6.2.1. Proposition. Let A, B, C, D be pairwise distinct points. Then $\overline{AB} \perp \overline{CD}$ if and only if $\frac{d-c}{b-a} \in i\mathbb{R}$; i.e.

$$\frac{d-c}{b-a} + \overline{\left(\frac{d-c}{b-a}\right)} = 0.$$

Proof. It's equivalent to $\frac{d-c}{b-a} \in i\mathbb{R} \iff \arg\left(\frac{d-c}{b-a}\right) \equiv \pm 90^\circ \iff \overline{AB} \perp \overline{CD}$. $\square$

6.2.2. Proposition. Let A, B, C be pairwise distinct points. Then A, B, C are collinear if and only if $\frac{c-a}{c-b} \in \mathbb{R}$; i.e.

$$\frac{c-a}{c-b} = \overline{\left(\frac{c-a}{c-b}\right)}.$$

Proof. Similar to the previous one. $\square$

6.2.3. PROPOSITION. Let A, B, C, D be pairwise distinct points. Then A, B, C, D are concyclic if and only if

$$\frac{c-a}{c-b} : \frac{d-a}{d-b} \in \mathbb{R}.$$

PROOF. It's not hard to see that $\arg\left(\frac{c-a}{c-b}\right) = \angle ACB$ and $\arg\left(\frac{d-a}{d-b}\right) = \angle ADB$. (Here angles are directed). □

6.3. The unit circle, and triangle centers. On the complex plane, the *unit circle* is of critical importance. Indeed if $|z| = 1$ we have

$$\overline{z} = \frac{1}{z}.$$

This means that we can get the following useful formulas.

6.3.1. LEMMA. If A and B lie on the unit circle, then then the foot from Z to $\overline{AB}$ is

$$\frac{1}{2}\left(z + a + b - ab\overline{z}\right).$$

PROOF. By earlier propositions, we want a point w such that

$$\frac{w-z}{a-b} = -\frac{\overline{w}-\overline{z}}{1/a - 1/b}$$
$$\frac{w-a}{w-b} = \frac{\overline{w}-\overline{a}}{\overline{w}-\overline{b}}$$

and solving the system for w gives the result above. □

6.3.2. LEMMA. If A, B, C, D lie on the unit circle then the intersection of $\overline{AB}$ and $\overline{CD}$ is given by

$$\frac{ab(c+d) - cd(a+b)}{ab - cd}.$$

PROOF. By earlier propositions, we want a point w such that

$$\frac{w-a}{w-b} = \frac{\overline{w}-\overline{a}}{\overline{w}-\overline{b}}$$
$$\frac{w-c}{w-d} = \frac{\overline{w}-\overline{c}}{\overline{w}-\overline{d}}$$

and again solving gives the result. □

These are much easier to work with than the corresponding formulas in general.

6.4. Triangle centers. We can also obtain the triangle centers immediately:

6.4.1. THEOREM. Assume points A, B, C lie on the unit circle. Then the circumcenter, centroid, and orthocenter of ABC are given by 0, $\frac{1}{3}(a+b+c)$, $a+b+c$, respectively.

Observe that the Euler line follows from this.

PROOF. The results for the circumcenter and centroid are immediate. Let $h = a + b + c$. By symmetry it suffices to prove $\overline{AH} \perp \overline{BC}$. We may set

$$z = \frac{h-a}{b-c} = \frac{b+c}{b-c}.$$

Then

$$\overline{z} = \overline{\left(\frac{b+c}{b-c}\right)} = \frac{\overline{b}+\overline{c}}{\overline{b}-\overline{c}} = \frac{\frac{1}{b}+\frac{1}{c}}{\frac{1}{b}-\frac{1}{c}} = \frac{c+b}{c-b} = -z$$

so $z \in i\mathbb{R}$ as desired. □

There is also a result for the incenter. It is more complicated and we will not use it, so we do not prove it here.

6.4.2. Theorem. Points A, B, C lie on the unit circle. Then one can assign a choice of complex numbers x, y, z on the unit circle such that $a = x^2$, $b = y^2$, $c = z^2$ and

(a) the arc midpoints opposite the vertices of the triangle are $-yz$, $-zx$, $-xy$, and
(b) the incenter has coordinate $-(xy + yz + zx)$.

Exercises.

48. Exercise. Let W be the reflection of Z across $\overline{AB}$. Show that

$$w = \frac{(a-b)\overline{z} + \overline{a}b - a\overline{b}}{\overline{a} - \overline{b}}.$$

49. Exercise (*). Show that the signed area of $\triangle ABC$ is given by the determinant

$$\frac{i}{4}\begin{vmatrix} a & \overline{a} & 1 \\ b & \overline{b} & 1 \\ c & \overline{c} & 1 \end{vmatrix}.$$

50. Exercise. Find the coordinates of the nine-point center of a triangle ABC inscribed in the unit circle, in terms of a, b, c.

51. Exercise. Points A and B lie on the unit circle which are not diametrically opposite. Show that the tangents at A and B meet at a point with coordinates $\frac{2ab}{a+b}$.

52. Exercise. Show that if A and B lie on the unit circle, then a point P lies on line AB if and only if

$$p + ab\overline{p} = a + b.$$

53. Exercise. In Theorem 6.4.2, show that (b) follows from (a).

54. Exercise (*). Let $ABCD$ be a cyclic quadrilateral. Let H_A, H_B, H_C, H_D denote the orthocenters of BCD, CDA, DAB, ABC. Show that $\overline{AH_A}$, $\overline{BH_B}$, $\overline{CH_C}$, $\overline{DH_D}$ are concurrent.

55. Exercise (**). Let ABC be a triangle and P a point on its circumcircle. Show that the Simson line of P bisects $\overline{PH}$.

56. Exercise. Let A, B, C, D be points. Show that lines AB and CD intersect at

$$\frac{(\bar{a}b - a\bar{b})(c-d) - (a-b)(\bar{c}d - c\bar{d})}{(\bar{a}-\bar{b})(c-d) - (a-b)(\bar{c}-\bar{d})}.$$

57. Exercise (*). Let ABC be a triangle and erect equilateral triangles on sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ outside of ABC with centers O_A, O_B, O_C. Prove that $\triangle O_AO_BO_C$ is equilateral and that its center coincides with the centroid of triangle ABC.

58. Exercise (**). Prove that the Euler lines of triangles IAB, IBC, ICA, ABC are concurrent.

§7. Barycentric coordinates

We present a second coordinate system. In this chapter, $\triangle ABC$ is a fixed non-degenerate reference triangle with vertices in counterclockwise order. The lengths will be abbreviated $a = BC$, $b = CA$, $c = AB$. These correspond with points in the vector plane $\vec{A}$, $\vec{B}$, $\vec{C}$.

7.1. Normalized coordinates. Each point in the plane is assigned an ordered triple of real numbers $P = (x, y, z)$ such that

$$\vec{P} = x\vec{A} + y\vec{B} + z\vec{C} \quad \text{and} \quad x + y + z = 1.$$

These are called the *barycentric coordinates* of the point.

The most important result about these coordinates is that one can recover signed areas using them. (For this reason, they are sometimes called *areal coordinates* instead.) We state this in the following theorem. To avoid technicalities about how to define the notion of "area" rigorously, we do not include the proof.

7.1.1. Theorem. Let $P = (x_P, y_P, z_P)$, $Q = (x_Q, y_Q, z_Q)$, $R = (x_R, y_R, z_R)$. The signed area of a triangle $[PQR]$ is given by

$$[PQR] = \begin{vmatrix} x_P & y_P & z_P \\ x_Q & y_Q & z_Q \\ x_R & y_R & z_R \end{vmatrix}.$$

This theorem right away gives an equivalent, useful definition for the coordinates of point P.

7.1.2. Corollary. For each point P,

$$P = \left(\frac{[BPA]}{[ABC]}, \frac{[CPB]}{[ABC]}, \frac{[APC]}{[ABC]} \right).$$

Another important corollary is that equations of a line take the form of linear equations.

7.1.3. Theorem. Let u, v, w be real numbers not all equal. Then the locus of points (x, y, z) satisfying

$$ux + vy + wz = 0$$

is a straight line, and moreover all lines are of this form.

Proof. One direction follows by fixing two points Q and R in Theorem 7.1.1 noting that the locus of points lying on line QR is precisely those points with $[PQR] = 0$.

Conversely, suppose WLOG that $u \neq v$ and $u \neq w$. Then one may take $Q = (\frac{-v}{u-v}, \frac{u}{u-v}, 0)$ and $R = (\frac{-w}{u-w}, 0, \frac{u}{u-w})$ and line QR will have the desired form in Theorem 7.1.1. $\square$

7.2. Coordinates of triangle centers. We now work out the coordinates of some common triangle centers. The remaining ones can be found in the exercises.

7.2.1. PROPOSITION. The barycentric coordinates of the centroid are $G = (1/3, 1/3, 1/3)$.

PROOF. The centroid is given by $\frac{1}{3}(\vec{A} + \vec{B} + \vec{C})$. ☐

7.2.2. PROPOSITION. The barycentric coordinates of the incenter are $I = (\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c})$.

FIRST PROOF. Note that $[BIC] = ar$, where r is the inradius, whilst $[CIA] = br$ and $[AIB] = cr$. So it follows from Corollary 7.1.2. ☐

SECOND PROOF. By the angle bisector, point I is collinear with $A = (1, 0, 0)$ and the point $D = (0, \frac{b}{b+c}, \frac{c}{b+c})$.

The equation of line AD is then seen to be $cy - bz = 0$. Indeed, it fits the format of Theorem 7.1.3 and it passes through both of the desired points.

So the second and third coordinate of I are in a $b : c$ ratio. Proceeding in a cyclic fashion gives the result. ☐

7.3. Homogenized coordinates. One may by now notice that the denominators are quite cumbersome. So, the following convention will be convenient.

7.3.1. DEFINITION. Let u, v, w be real numbers with nonzero sum. Then $(u : v : w)$ is shorthand for $\left(\frac{u}{u+v+w}, \frac{v}{u+v+w}, \frac{w}{u+v+w}\right)$.

We distinguish these conventions with colons and commas respectively. The former will be called *homogeneous barycentric coordinates* while the latter (original) is *normalized barycentric coordinates*. For example, we may now conveniently write $I = (a : b : c)$, which is shorter and more informative.

7.3.2. REMARK. It's nice to note that, because in Theorem 7.1.3 the terms are all degree one in x, y, z these homogeneous coordinates work out of the box, as well.

As an example, let's consider the incenter

$$I = (a : b : c) = \left(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c}\right).$$

Moreover let $\ell\colon ux + vy + wz = 0$ be a line. If we want to see whether I lies on ℓ, the condition is

$$\begin{aligned} & u \cdot a + v \cdot b + w \cdot c = 0 \\ \iff & u \cdot \frac{a}{a+b+c} + v \cdot \frac{b}{a+b+c} + w \cdot \frac{c}{a+b+c} = 0. \end{aligned}$$

This is nearly the same as saying we can use the coordinates $(a : b : c)$ directly in calculations.

7.4. The vector plane and dot product. To make further progress we adopt the following conventions. We have been working with vectors in the plane of ABC; at this point, we agree that the circumcenter of the triangle ABC will be designated as the origin This means that $\vec{A}$, $\vec{B}$, $\vec{C}$ each have magnitude equal to the circumradius R of $\triangle ABC$; in symbols,

$$\left\|\vec{A}\right\| = \left\|\vec{B}\right\| = \left\|\vec{C}\right\| = R.$$

7.4.1. Definition. Recall that the *dot product* $\cdot$ is defined as follows: if $\vec{v}$ and $\vec{w}$ are vectors and the angle between them is θ then the dot product is defined as the real number

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\theta.$$

The dot product is commutative and distributes over addition.

The dot product has the following properties.

7.4.2. Proposition. (i) We have $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$.
(ii) We have $\vec{v} \cdot \vec{w} = 0$ if and only if the vectors are perpendicular.

We can also compute the dot products of our basis vectors.

7.4.3. Proposition. We have $\vec{A} \cdot \vec{A} = R^2$ and $\vec{A} \cdot \vec{B} = R^2 - c^2/2$, and the cyclic variations.

Proof. The first part is clear. The second calculation can be realized as:

$$\begin{aligned} \vec{A} \cdot \vec{B} &= R^2 \cos \angle AOB \\ &= R^2 \cos 2C \\ &= R^2 \left(1 - 2\sin^2 C\right) \\ &= R^2 - \frac{1}{2}\left(2R \sin C\right)^2 \\ &= R^2 - c^2/2. \end{aligned}$$ □

Thus we can compute the dot products of any vectors expressed as sums of $\vec{A}$, $\vec{B}$, $\vec{C}$.

7.5. Displacement vectors. We may now compute the distance between any two points. For convenience, given two points P and Q we can consider their *displacement vector*, thought of as the vector from P to Q.

7.5.1. Definition. A *displacement vector* of two (normalized) points $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$ is denoted by $\overrightarrow{PQ}$ and is equal to $(q_1 - p_1, q_2 - p_2, q_3 - p_3)$.

Note that the sum of the coordinates of a displacement vector is 0.

7.5.2. Theorem. Let P and Q be two arbitrary points and consider a displacement vector $\overrightarrow{PQ} = (x, y, z)$. Then the distance from P to Q is given by

$$|PQ|^2 = -a^2 yz - b^2 zx - c^2 xy.$$

Proof. Observe that

$$|PQ|^2 = \left(x\vec{A} + y\vec{B} + z\vec{C}\right) \cdot \left(x\vec{A} + y\vec{B} + z\vec{C}\right).$$

Applying the properties of the dot product

$$\begin{aligned} |PQ|^2 &= \sum_{\text{cyc}} x^2 \vec{A} \cdot \vec{A} + 2 \sum_{\text{cyc}} xy \vec{A} \cdot \vec{B} \\ &= R^2 (x^2 + y^2 + z^2) + 2 \sum_{\text{cyc}} xy \left(R^2 - \frac{1}{2}c^2\right). \end{aligned}$$

Collecting the R^2 terms,

$$\begin{aligned}|PQ|^2 &= R^2(x^2+y^2+z^2+2xy+2yz+2zx) - (c^2xy+a^2yz+b^2zx) \\ &= R^2(x+y+z)^2 - a^2yz - b^2zx - c^2xy \\ &= -a^2yz - b^2zx - c^2xy. \qquad \square\end{aligned}$$

7.6. The equation of a circle. Since a circle is the locus of points equidistant from a point, we ought to be able to work out its equation. In fact, we can say slightly more.

7.6.1. THEOREM. Let ω be a circle. Then there exist real numbers u, v, w such that for any point $P = (x, y, z)$ we have

$$\operatorname{Pow}_\omega(P) = -a^2yz - b^2zx - c^2xy + (ux+vy+wz)(x+y+z).$$

In particular, the points of ω are those for which $\operatorname{Pow}_\omega(P) = 0$.

We choose to present the equation in this form so that the right-hand side is entirely degree 2 as a polynomial in x, y, z, even though we have $x + y + z = 1$. This is so that homogeneous coordinates can be used to determine whether a point lies on the circle; the denominators factor immediately (cf. Remark 7.3.2).

PROOF. Assume the circle has center $O = (x', y', z')$ and radius r. Then

$$\begin{aligned}\operatorname{Pow}_\omega(P) &= PO^2 - r^2 \\ &= -a^2(y-y')(z-z') - b^2(z-z')(x-x') \\ &\qquad - c^2(x-x')(y-y') - r^2 \\ &= -a^2yz - b^2zx - c^2xy + \lambda_1 x + \lambda_2 y + \lambda_3 z + \lambda_4\end{aligned}$$

where λ_1, λ_2, λ_3, λ_4 are constants. We may then take $u = \lambda_1 + \lambda_4$, $v = \lambda_2 + \lambda_4$, $w = \lambda_3 + \lambda_4$. $\square$

As with Theorem 7.1.3, one can determine the equation of a circle by selecting any particular points. See Exercise 61.

Exercises.

59. EXERCISE. Use barycentric coordinates to give alternate proofs of Ceva and Menelaus.

60. EXERCISE (*). Let ABC be a triangle and let P be a point in its interior. Lines PA, PB, PC intersect sides BC, CA, AB at D, E, F, respectively. Prove that

$$[PAF] + [PBD] + [PCE] = \frac{1}{2}[ABC]$$

if and only if P lies on at least one of the medians of triangle ABC.

61. EXERCISE. Show that the equation of the circumcircle of ABC is given by $-a^2yz - b^2zx - c^2xy = 0$.

62. EXERCISE. Find the coordinates of the A-excenter.

63. EXERCISE. Use Theorem 7.6.1 to give another proof that the radical axis of two circles is a straight line.

64. EXERCISE. Show:

(a) The barycentric coordinates of the circumcenter are given by

$$O = (\sin 2A : \sin 2B : \sin 2C)\,.$$

(b) The barycentric coordinates of the orthocenter are given by

$$H = (\tan A : \tan B : \tan C)\,.$$

65. Exercise. Use the previous exercise to give another proof that the orthocenter, circumcenter, and centroid are collinear.

66. Exercise (*). Find the equation of the tangent to the circumcircle of triangle ABC at A.

67. Exercise. Find the coordinates of the Gergonne and Nagel points (defined in Exercises 46 and 47). Show that the Nagel point is collinear with the incenter and centroid.

68. Exercise (*). Prove that if $P = (x : y : z)$ is a point inside triangle ABC, then its isogonal conjugate (defined in Exercise 44) is given by $P^* = (a^2/x : b^2/y : c^2/z)$.

69. Exercise. Show that the perpendicular bisector of BC has equation

$$0 = a^2(z - y) + x(c^2 - b^2).$$

70. Exercise. Let $\overrightarrow{MN} = (x_1, y_1, z_1)$ and $\overrightarrow{PQ} = (x_2, y_2, z_2)$ be displacement vectors. Show that $\overline{MN} \perp \overline{PQ}$ if and only if

$$0 = a^2(z_1y_2 + y_1z_2) + b^2(x_1z_2 + z_1x_2) + c^2(y_1x_2 + x_1y_2).$$

71. Exercise (**). Let $A_1A_2A_3A_4$ be a non-cyclic quadrilateral. For $1 \le i \le 4$, let O_i and r_i be the circumcenter and the circumradius of triangle $A_{i+1}A_{i+2}A_{i+3}$ (where $A_{i+4} = A_i$). Prove that

$$\frac{1}{O_1A_1^2 - r_1^2} + \frac{1}{O_2A_2^2 - r_2^2} + \frac{1}{O_3A_3^2 - r_3^2} + \frac{1}{O_4A_4^2 - r_4^2} = 0.$$

CHAPTER III

Advanced techniques

§8. Inversion in the plane

In what follows, we consider the usual Euclidean plane $\mathbb{R}^2$ with an additional "point at infinity", which we denote as ∞. We consider every line to pass through this point ∞.

8.1. Definition and first properties.

8.1.1. Definition. For a circle ω with center O and radius $r > 0$, an *inversion* around ω is a map that sends each point P to be a point P^* as follows.

- If $P = \infty$, then $P^* = O$.
- If $P = O$, then $P^* = \infty$.
- For any other point P, we choose P^* to be the unique point satisfying $OP \cdot OP^* = r^2$.

We immediately identify some properties of the inversion.

8.1.2. Proposition. Inversion is an involution: $(P^*)^* = P$.

8.1.3. Proposition. The point P lies on ω if and only if P^* lies on ω.

In the case where P lies outside the circle, there is also a geometric interpretation.

8.1.4. Theorem. Let P be a point outside ω and suppose $\overline{PA}$, $\overline{PB}$ are tangents to the circle. Then P^* coincides with the midpoint of $\overline{AB}$.

The proof of this theorem is left as Exercise 72.

8.2. Generalized lines and circles. We continue to fix a circle ω with center O and radius r, through which we will perform inversions.

If ℓ is a line, by its inverse ℓ^* we mean the set

$$\ell^* = \{P^* \mid P \in \ell\}.$$

Similarly for a circle γ its inverse is the set

$$\gamma^* = \{P^* \mid P \in \gamma\}.$$

The main result of this section is that inverses of lines and circles are themselves lines and circles.

We check this using the following propositions.

8.2.1. Proposition. If ℓ is a line passing through O, then $\ell^* = \ell$.

Proof. This is immediate by definition. $\square$

8.2.2. PROPOSITION. If γ is a circle passing through O, its inverse γ^* is a line not passing through O. Dually, if ℓ is a line not passing through O, its inverse ℓ^* is a circle passing through O.

PROOF. By an appropriate homothety, we may reduce to the case where γ intersects the circle ω at two points A and B, and thus $r = OA = OB$. Let C be any point on γ now and let $D = \overline{AB} \cap \overline{CO}$. Then we have $\measuredangle ACO = \measuredangle ABO = \measuredangle OAB = \measuredangle OAD$, and thus $\triangle ACO \sim \triangle OAD$, so $OC \cdot OD = OA^2$. Thus $D = C^*$, ergo C^* lies on line AB. In other words γ^* is contained in line AB. It is not hard to see that every point on line AB can be obtained this way, and also γ^* should contain ∞ since γ contained O, so we are done. $\square$

8.2.3. PROPOSITION. If γ is a circle not passing through O, its inverse γ^* is a circle not passing through O.

PROOF. Suppose a line from O meets γ at A and B, and a second line meets O at C and D. Then letting $t^2 = OA \cdot OB = OC \cdot OD$ we quickly deduce $\frac{r^4}{t^2} = OA^* \cdot OB^* = OC^* \cdot OD^*$, and so by power of a point from O the four points A^*, B^*, C^*, D^* are cyclic as well. If we treat AB as fixed and vary C, D, (noting hat t does not change either) we obtain the result. Note the circle γ^* does not contain O since γ does not pass through ∞. $\square$

This can be succinctly summarized as follows. We say a *generalized circle* is either a circle or a line: thus we may think of a usual line as a "circle passing through ∞".

8.2.4. THEOREM. The inverse of a generalized circle is itself a generalized circle.

Then, all the qualifiers about "passing through O or not" and "passing through ∞ or not" can be captured in this statement.

8.3. Inversion distance formula. We have the following additional result, sometimes known as the "distance formula".

8.3.1. THEOREM. In the usual notation, we have

$$A^*B^* = \frac{r^2}{OA \cdot OB} \cdot AB.$$

Here is one application, generalizing Exercise 37 from before.

8.3.2. THEOREM. For any four points in the plane A, B, C, D we have

$$AB \cdot CD + BC \cdot DA \geq AC \cdot BD.$$

Equality holds if $ABCD$ is cyclic, in that order.

PROOF. Consider an inversion about the circle centered at D with radius r. On the one hand we have by the triangle inequality

$$A^*B^* + B^*C^* \geq A^*C^*$$

and the inequality is sharp if $ABCD$ was cyclic in that order. However, we have that

$$A^*B^* = \frac{r^2}{DA \cdot DB} \cdot AB$$
$$B^*C^* = \frac{r^2}{DB \cdot DC} \cdot BC$$
$$A^*C^* = \frac{r^2}{DA \cdot DC} \cdot AC$$

which rearranges to the inequality. □

We give many more applications of inversion in the exercises. Typically, one can get interesting results if one inverts around a point with many circles passing through it, and carefully analyzing the results.

Exercises.

72. Exercise. Prove Theorem 8.1.4.

73. Exercise. Show that we always have $\measuredangle OAB = -\measuredangle OB^*A^*$.

74. Exercise (*). Let ABC be a triangle with intouch triangle DEF. Show that the inverse of the circumcircle of $\triangle ABC$ with respect to the incircle coincides with the nine-point circle of $\triangle DEF$. Conclude that the Euler line of $\triangle DEF$ passes through O if $\triangle ABC$ is not equilateral.

75. Exercise. Show that if $\triangle ABC$ has circumcenter O and we invert around the circle centered at C with radius r, then O^* is the reflection of C across $\overline{A^*B^*}$.

76. Exercise. Give another proof of Theorem 8.1.4 by inversion around the circle centered at M with radius $MB = MC$. Deduce additionally that $MK \cdot MT = MB^2 = MC^2$.

77. Exercise. We say two circles γ and ω are *orthogonal* if they intersect at two points A and B, and the tangents to γ and ω at A are perpendicular. Show that in that case, if we invert around ω, then $\gamma^* = \gamma$.

78. Exercise. Let ABC be a right triangle with $\angle C = 90^\circ$ and let X and Y be points in the interiors of $\overline{CA}$ and $\overline{CB}$, respectively. Construct four circles passing through C, centered at A, B, X, Y. Prove that the four points lying on at exactly two of these four circles are concyclic.

79. Exercise (*). Let $ABCD$ be a quadrilateral whose diagonals are perpendicular and meet at E. Prove that the reflections of E across the sides of $ABCD$ are concyclic.

80. Exercise (*). Let ω_1, ω_2, ω_3, ω_4 be circles with consecutive pairs tangent at A, B, C, D. Prove that quadrilateral $ABCD$ is cyclic.

81. Exercise (**). Let ABC be triangle whose A-excircle touches $\overline{BC}$ at E. Suppose γ is a circle tangent to $\overline{AB}$, to $\overline{AC}$, and internally tangent to the circumcircle of $\triangle ABC$ at T. Prove that $\measuredangle BAT = \measuredangle EAC$.

§9. Projective geometry

This time, we consider the usual Euclidean plane $\mathbb{R}^2$ with an additional "point at infinity" along *each* set of parallel lines, rather than just one point at infinity. Thus, any two lines intersect at a unique point; the point is a usual Euclidean point if they are not parallel, but it is considered to intersect at an infinity point otherwise.

9.1. Cross ratios. We begin with the following definition.

9.1.1. Definition. Given four distinct collinear points A, B, X, Y (which may be at infinity) we define the cross ratio as:

$$(AB; XY) = \frac{XA}{XB} \div \frac{YA}{YB}.$$

Note that the cross ratio has the following uniqueness property.

9.1.2. Proposition. Given four distinct collinear points A, B, X, Y (which may be at infinity) and let Y' be a fifth point different from A, B, X. Assume that $(AB; XY) = (AB; XY')$. Then $Y = Y'$.

It is possible also to define the cross ratio of four concurrent *lines*, in the following way.

9.1.3. Definition. Suppose four distinct lines a, b, x, y are concurrent at some point P. Their cross ratio is then defined as

$$(ab; xy) = \frac{\sin\angle(x, a)}{\sin\angle(x, b)} \div \frac{\sin\angle(y, a)}{\sin\angle(y, b)}.$$

Some care is required to describe the signs in the above definition; it is more economical to declare that the sign of the entire cross ratio is positive if one of the four angles formed by line a and b contains both x and y, and negative otherwise.

The key result is the following equivalence.

9.1.4. Theorem. Suppose four distinct lines a, b, x, y are concurrent at some point P. A line ℓ distinct from P meets a, b, x, y at four points A, B, X, Y. Then

$$(AB; XY) = (ab; xy).$$

This is shown using the law of sines, see Exercise 85. It has widespread applications. For example, the immediate corollary is itself very useful.

9.1.5. Corollary. Let ℓ and ℓ' be two lines and P a point not on either line. Let A, B, X, Y be four points on ℓ. Then let A' be the intersection of line PA and ℓ' and define B', X', Y' similarly. Then

$$(AB; XY) = (A'B'; X'Y').$$

When applying this corollary it is customary to use the notation $(AB; XY) \stackrel{P}{=} (A'B'; X'Y')$ to indicate the point through which we are projecting.

We now state without proof one more result that will allow us to analyze conics: we define the cross ratio of four points relative to a circumconic of those four points.

9.1.6. Definition. Let γ be a nondegenerate conic. Then four points A, B, X, Y on γ we define $(AB; XY)_\gamma = (PA, PB; PX, PY)$ where P is any fifth point on the conic; this doesn't depend on the choice of P.

Here is a nice application of the results so far, the so-called Pascal theorem.

9.1.7. Theorem. Let A, B, C, D, E, F be six points inscribed in a conic γ. Define $X = \overline{AB} \cap \overline{DE}$, $Y = \overline{CD} \cap \overline{FA}$, $Z = \overline{BC} \cap \overline{EF}$. Then X, Y, Z are collinear.

Proof. Let $P = \overline{AB} \cap \overline{EF}$ and $Q = \overline{DE} \cap \overline{AF}$ and let $Z' = \overline{XY} \cap \overline{EF}$. Then

$$(EP; ZF) \overset{B}{=} (EA; CF)_\gamma \overset{D}{=} (QA; YF) \overset{X}{=} (EZ'; PF)$$

so $Z' = Z$. □

9.2. The case of cross ratio -1. The most important special case is the situation of four points with cross ratio -1, which is sometimes known as a *harmonic* cross ratio. To make progress on these we use state the following useful lemma, whose proof is trivial, hence left as an exercise.

9.2.1. Lemma. If A, B, X, Y are distinct collinear points such that $(AB; XY) = (AB; YX)$ then $(AB; XY) = -1$.

We now give two examples of situations in which the cross ratio of -1 occurs.

9.2.2. Proposition. Let ABC be a triangle with concurrent cevians $\overline{AD}$, $\overline{BE}$, $\overline{CF}$. Line EF meets BC at X. Then $(XD; BC) = -1$.

Proof. Let $Y = \overline{AD} \cap \overline{EF}$. Then $(BC; XD) \overset{A}{=} (FE; XY) \overset{P}{=} (BC; DX)$. □

9.2.3. Proposition. Let γ be a nondegenerate conic and X and Y two points on the conic. Suppose the tangents to γ at X and Y meet at P. Consider another line through P meeting γ at two points A and B. Let $Q = \overline{AB} \cap \overline{XY}$. Then

(a) $(AB; PQ) = -1$ and
(b) $(AX; BY)_\gamma = -1$.

Proof. We have $(AB; PQ) \overset{X}{=} (AB; XY) \overset{Y}{=} (AB; QP)$. □

One nice application of Lemma 9.2.3 is that we can make the following definition.

9.2.4. Definition. Let γ be a nondegenerate conic and P a point. Consider lines through P intersecting γ at X, Y and let Q be the unique point on line XY with $(XY; PQ) = -1$. Then the locus of Q lies on a line called the *polar* of P (with respect to γ). Point P is the *pole* of Q.

9.2.5. Remark. If PX, PY are tangents from P to γ, then the polar of P coincides with line XY.

Two more examples of this situation are given in Exercise 82 and 89.

9.3. An aside on projective transformations. This is a rich topic which is largely beyond the scope of these notes, so we merely mention some famous results.

9.3.1. Definition. A *projective collineation* is a bijection of the projective plane which preserves collinearity of points.

The following result is sometimes called the "fundamental theorem of projective geometry".

9.3.2. Theorem. Let τ be a projective collineation. Then the image of any conic under τ is a conic. Moreover cross ratios are preserved in the following senses:

- If A, B, C, D are four collinear points and τ maps them to W, X, Y, Z then $(AB;CD) = (WX;YZ)$.
- If a, b, c, d are four concurrent lines and τ maps them to w, x, y, Z then $(ab;cd) = (wx;yz)$.
- If A, B, C, D are four points on a conic γ and τ maps these points to W, X, Y, Z and the conic γ to γ' then $(AB;CD)_\gamma = (WX;YZ)_{\gamma'}$.

The following existence theorem guarantees the existence of homographies satisfying certain conditions.

9.3.3. Theorem. (a) There exists a unique homography taking any four points, no three collinear, to any other quadruple of four points with no three collinear.
(b) Let γ be a circle or ellipse and P a point inside it. One can find a unique homography mapping every point of γ to a (possibly different) point of γ, while mapping P to the center of γ.

This means many results which can be stated only in projective terms can be proved in special cases by taking a suitable homography. For example, one can prove Proposition 9.2.2 in the following way: consider a homography mapping A, B, C, and $\overline{BE} \cap \overline{CF}$ to an equilateral triangle and its center. If we let B', C', D', X' be the images of B, C, D, X then we have $(B'C';D'X') = -1$ since D' is the midpoint of $B'C'$ while X' is the point at infinity along line $B'C'$. Therefore we actually have $(BC;DX) = (B'C';D'X') = -1$ as needed.

We won't use this technique at all, but it is worth noting that such proofs are possible.

9.4. Exercises.

82. Exercise. Let M be the midpoint of a segment AB and ∞ the point at infinity along line AB. Show that $(AB;M\infty) = -1$.

83. Exercise. Verify Lemma 9.2.1.

84. Exercise. Show that if γ is a circle and convex, quadrilateral $AXBY$ is inscribed in it, then in fact

$$(AB;XY)_\gamma = -\left|\frac{XA}{XB}\right| \div \left|\frac{YA}{YB}\right|.$$

85. Exercise. Prove Theorem 9.1.4 using the law of sines.

86. Exercise. Give another proof of Lemma 9.2.2 by using Ceva and Menelaus theorems.

87. Exercise. Give another proof of Lemma 9.2.3 in the case where γ is a circle by using similar triangles and Exercise 84.

88. Exercise (*). Let $ABXC$ be a quadrilateral inscribed in circle ω with $(AX;BC)_\omega = -1$. Let M be the midpoint of $\overline{AC}$.
(a) Show that $\measuredangle BAM = \measuredangle XAC$.
(b) Prove that line AX is concurrent with the tangents to ω at B and C, possibly at infinity.

(The line $\overline{AX}$ is a *symmedian* of $\triangle ABC$.)

89. Exercise (*). Let X, A, Y, B be collinear points in that order and let C be any point not on this line. Show that any two of the following conditions implies the third condition.

(i) $(AB;XY) - 1$;
(ii) $\angle XCY = 90°$; and
(iii) $\overline{CY}$ bisects $\angle ACB$.

90. Exercise. Prove that if γ is a circle with center O and P is a point outside it, then the polar of P is the line through the inverse P^* which is perpendicular to line OP.

91. Exercise. Let γ be a nondegenerate conic. Show that point P lies on the polar of Q if and only if Q lies on the polar of P.

92. Exercise. Let AB, CD, PQ be chords of a conic γ concurrent at M. Let $X = \overline{PQ} \cap \overline{AD}$, $Y = \overline{PQ} \cap \overline{BC}$, then prove that $(PM;XQ) = (PY;MQ)$. Conclude that if $\frac{MP}{MQ} = -1$ then $\frac{MX}{MY} = -1$.

93. Exercise (**). Let $ABCD$ be a quadrilateral inscribed in nondegenerate conic γ. Set $P = \overline{AB} \cap \overline{CD}$, $Q = \overline{BC} \cap \overline{DA}$, and $R = \overline{AC} \cap \overline{BD}$. Prove that P, Q, R are the poles of QR, RP, PQ, respectively. (Hint: use Proposition 9.2.2 and Proposition 9.2.3 repeatedly.)

94. Exercise. Let $ABCD$ be a quadrilateral and γ any circumconic. Let $XYZW$ be a quadrilateral and ω any circumconic. Assuming that $(AB;CD)_\gamma = (XY;ZW)_\omega$, there exists a (unique) homography sending $ABCD$ to $XYZW$ and which maps each point of γ to a point of ω.

95. Exercise (*). Let A, B, C, D be four points, no three collinear. Define the points $P = \overline{AD} \cap \overline{BC}$, $Q = \overline{AB} \cap \overline{CD}$, and $R = \overline{AC} \cap \overline{BD}$. Let X_1, X_2, Y_1, Y_2 denote $\overline{PR} \cap \overline{AD}$, $\overline{PR} \cap \overline{BC}$, $\overline{QR} \cap \overline{AB}$, $\overline{QR} \cap \overline{CD}$. Prove that lines X_1Y_1, X_2Y_2, and PQ are concurrent.

§10. Complete quadrilaterals

If we say $ABCDEF$ is a *complete quadrilateral*, we mean that A, B, C, D are four points, no three collinear, and the points $E = \overline{AB} \cap \overline{CD}$, $F = \overline{BC} \cap \overline{DA}$ exist. The complete quadrilateral is cyclic if A, B, C, D lie on a circle; we may then refer to the circumcircle and center of this circle.

10.1. Spiral similarity.

10.1.1. Definition. A *spiral similarity* with a center O combines a rotation about O with a dilation.

The most commonly occurring case of a spiral similarity is between two segments, in which case we have direct similarity. We prove the relevant existence result now:

10.1.2. Theorem. Let $\overline{AB}$ and $\overline{CD}$ be segments such that $ABDC$ is not a parallelogram. There exists a unique spiral similarity taking A to C and B to D.

PROOF. We determine O in terms of A, B, C, D via complex numbers. The similarity we want is equivalent to:

$$\frac{c-o}{a-o} = \frac{d-o}{b-o}.$$

Solving gives

$$o = \frac{ad-bc}{a+d-b-c}.$$

The denominator is nonzero so we are done. □

We now give a synthetic description of when such spiral similarities exist in nature.

10.1.3. LEMMA. Let $\overline{AB}$ and $\overline{CD}$ be segments, and suppose $X = \overline{AC} \cap \overline{BD}$ exists. If (ABX) and (CDX) intersect again at O, then O is the center of the unique spiral similarity taking $\overline{AB}$ into $\overline{CD}$.

PROOF. This is actually just a matter of angle chasing. We have

$$\measuredangle OAB = \measuredangle OXB = \measuredangle OXD = \measuredangle OCD$$

and similarly

$$\measuredangle OBA = \measuredangle ODC.$$

That implies $\triangle OAB \sim \triangle OCD$ directly, which is sufficient. □

10.2. Miquel point of a cyclic quadrilateral. We begin with the following surprising observation. Suppose O is the center of the spiral similarity sending $\overline{AB}$ to $\overline{CD}$ as in the previous sections, meaning

$$\triangle OAB \sim \triangle OCD.$$

Then we have $\measuredangle AOB = \measuredangle COD$. This implies $\measuredangle AOC = \measuredangle BOD$. Moreover, we have $\frac{AO}{BO} = \frac{CO}{DO}$; so in fact we have

$$\triangle OAC \sim \triangle OBD.$$

In other words, *spiral similarities come in pairs.*

We can apply this observation to now get a famous theorem, which does not refer to spiral similarity at all.

10.2.1. THEOREM. Let $ABCDEF$ be a complete quadrilateral. The four circles (FAB), (FDC), (EAD), (EBC) concur at a point M.

PROOF. Let M be the center of the spiral similarity sending $\overline{AB}$ to $\overline{DC}$, hence also sending $\overline{AD}$ to $\overline{BC}$. Then Lemma 10.1.3 gives M lying on $(FAB) \cap (FDC)$ as well as M lying on $(EAD) \cap (EBC)$, as needed. □

10.2.2. DEFINITION. The point M is called the *Miquel point* of quadrilateral $ABCD$.

In fact, we can get the following animated motion of the result.

10.2.3. PROPOSITION. Let $ABCD$ be a quadrilateral and let $E = \overline{AB} \cap \overline{DC}$. Consider variable points X and Y on lines AB and DC satisfying $\frac{AX}{BX} = \frac{DY}{CY}$. Then the circumcircle of $\triangle EXY$ passes through the Miquel point M.

PROOF. The spiral similarity mapping A to D and B to C should also map X to Y, and we apply Lemma 10.1.3 again. □

10.3. The Miquel point of a cyclic quadrilateral. In the case where $ABCD$ is cyclic the point M has an explicit description.

10.3.1. THEOREM. If $ABCDEF$ is a cyclic complete quadrilateral with Miquel point M, then M coincides with the inverse of $\overline{AC} \cap \overline{BD}$ with respect to the circumcircle of A, B, C, D.

PROOF. Let O denote the circumcenter. If we apply Proposition 10.2.3 with X and Y as the midpoints of $\overline{AB}$ and $\overline{CD}$ we conclude that M lies on the circumcicrle of $\triangle EXY$. This is the circle with diameter $\overline{EO}$. Thus $\angle EMO = 90°$. Similarly $\angle FMO = 90°$. Thus M is the foot from O to line EF. Then we use Exercise 93. $\square$

Exercises.

96. EXERCISE. Give a direct proof of Theorem 10.2.1 by angle chasing, without referring to spiral similarity.

97. EXERCISE. Let $ABCD$ be a quadrilateral, and let E and F be points on sides AD and BC, respectively, such that $\frac{AE}{ED} = \frac{BF}{FC}$. Ray FE meets rays BA and CD at S and T, respectively. Prove that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through a common point.

98. EXERCISE (*). If $ABCDEF$ is a complete quadrilateral with Miquel point M, show that M is concyclic with the circumcenters of $\triangle EAD$, $\triangle EBC$, $\triangle FAB$, $\triangle FCD$.

99. EXERCISE (*). If $ABCDEF$ is a complete quadrilateral with Miquel point M, show that the feet of the altitudes from M to AB, BC, CD, DA are collinear.

100. EXERCISE. Let M be the Miquel point of cyclic complete quadrilateral $ABCDEF$ with circumcenter O. Show that the M is the second intersection of circles (OAC) and (OBD).

101. EXERCISE. Let M be the Miquel point of cyclic complete quadrilateral $ABCDEF$ with circumcenter O. Show that $\overline{MO}$ bisects $\angle AMC$ and $\angle BMD$.

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