OTIS Mock Olympiad

Exam Sample 02 USAMO

Mock USAMO (4 hours)

USAMO 4. Let $ABCD$ be a convex quadrilateral. Assume that the incircle of triangle ABD is tangent to $\overline{AB}$, $\overline{AD}$, $\overline{BD}$ at points W, Z, K. Also assume that the incircle of triangle CBD is tangent to $\overline{CB}$, $\overline{CD}$, $\overline{BD}$ at points X, Y, K. Prove that quadrilateral $WXYZ$ is cyclic.

USAMO 5. Positive integers x_1, x_2, $\dots$, x_n ($n \ge 4$) are arranged in a circle such that each x_i divides the sum of the neighbors; that is,

$$\frac{x_{i-1} + x_{i+1}}{x_i} = k_i$$

is an integer for each i, where $x_0 = x_n$, $x_{n+1} = x_1$. Prove that

$$2 \le \frac{k_1 + \dots + k_n}{n} < 3.$$

USAMO 6. Let m and s be positive integers with $2 \le s \le 3m^2$. Define a sequence $a_1, a_2, \dots$ recursively by $a_1 = s$ and

$$a_{n+1} = 2n + a_n \qquad \text{(for } n = 1, 2, \dots \text{)}.$$

Prove that if the numbers $a_1, a_2, \dots, a_m$ are prime, then a_{s-1} is also prime.

Time limit: 4 hours
Each problem is worth 7 points