Number Theory Constructions

Evan Chen*

DNY-NTCONSTRUCT

§1 Lecture Notes

§1.1 Discussion

This is going to be a lot like the Free class: lots of room for you to make choices (e.g. in constructions). The same two philosophies from the combinatorial counterpart might be helpful here:

- **Bottom-up ("design")**: making conscious design choices, or narrowing the set of things you're considering.
- **Top-down ("desire")**: adding abstract constraints, including any constraints you can prove are necessary (which is especially often the case in number theory).

This time, both of these steps often require number theory skill in order to carry out the correct deductions. (So: globally, it feels like doing a combinatorics problem, but locally, it feels like doing a number theory problem.) This has the weird property that sometimes you'd like to rely on statement that is obviously true ("$n^2 + 1$ is prime infinitely often"), but either hard to prove or open; if you don't know, then you have to make a judgment call. (Whereas in combinatorics, simple true statements are rarely impossible to prove.)

Common tropes in this lecture will include:

- Picking really numbers big with lots of prime factors.
- Chinese Remainder Theorem: add modular conditions with reckless abandon, then let CRT collate them for you.
- Appealing to sledgehammers like Bertrand, Dirichlet, Zsigmondy, Kobayashi, et cetera after having reduced the problems to something.

§1.2 Examples

Problem 1.1 (TSTST 2015/5)**.** Let $\varphi(n)$ denote the number of positive integers less than n that are relatively prime to n. Prove that there exists a positive integer m for which the equation $\varphi(n) = m$ has at least 2015 solutions in n.

Problem 1.2 (IMO 2003/6)**.** Let p be a prime number. Prove that there exists a prime number q such that for every integer n, the number $n^p - p$ is not divisible by q.

Problem 1.3 (December TST 2015/2)**.** Prove that for every positive integer n, there exists a set S of n positive integers such that for any two distinct $a, b \in S$, $a - b$ divides a and b but none of the other elements of S.

*Internal use: Olympiad Training for Individual Study (OTIS). Last update January 17, 2018.

§2 Practice Problems

Solve at least [42♣] (more if you have time), including any problem marked in red.

[2♣] **Problem 1** (USAMO 2017/1)**.** Prove that there exist infinitely many pairs of relatively prime positive integers $a, b > 1$ for which $a + b$ divides $a^b + b^a$.

[5♣] **Problem 2** (JMO 2016/2)**.** Prove that there exists a positive integer $n < 10^6$ such that 5^n has six consecutive zeros in its decimal representation.

[2♣] **Problem 3** (Shortlist 2007 N2)**.** Let $b, n > 1$ be integers. Suppose that for each $k > 1$ there exists an integer a_k such that $b - a_k^n$ is divisible by k. Prove that $b = A^n$ for some integer A.

[2♣] **Problem 4** (IMO 2000/5)**.** Does there exist a positive integer n such that n has exactly 2000 prime divisors and n divides $2^n + 1$?

[3♣] **Problem 5** (BAMO 2011/5)**.** Decide whether there exists a row of Pascal's triangle containing four pairwise distinct numbers a, b, c, d such that $a = 2b$ and $c = 2d$.

[3♣] **Problem 6** (TSTST 2012/5)**.** A rational number x is given. Prove that there exists a sequence $x_0, x_1, x_2, \dots$ of rational numbers with the following properties:

(a) $x_0 = x$;

(b) for every $n \geq 1$, either $x_n = 2x_{n-1}$ or $x_n = 2x_{n-1} + \frac{1}{n}$;

(c) x_n is an integer for some n.

[3♣] **Problem 7** (Shortlist 2014 N4)**.** Let $n > 1$ be an integer. Prove that there are infinitely many integers $k \geq 1$ such that

$$\left\lfloor \frac{n^k}{k} \right\rfloor$$

is odd.

[5♣] **Problem 8** (USAMO 2006/3)**.** For integral m, let $p(m)$ be the greatest prime divisor of m. By convention, we set $p(\pm 1) = 1$ and $p(0) = \infty$. Find all polynomials f with integer coefficients such that the sequence

$$\{p\left(f\left(n^2\right)\right) - 2n\}_{n \geq 0}$$

is bounded above. (In particular, this requires $f(n^2) \neq 0$ for $n \geq 0$.)

[5♣] **Problem 9** (USAMO 2013/5)**.** Let m and n be positive integers. Prove that there exists an integer c such that cm and cn have the same nonzero decimal digits.

[5♣] **Problem 10** (RMM 2012/4)**.** Prove there are infinitely many integers n such that n does not divide $2^n + 1$, but divides $2^{2^n+1} + 1$.

[5♣] **Problem 11** (Shortlist 2013 N4)**.** Determine whether there exists an infinite sequence of nonzero digits a_1, a_2, a_3, … such that the number $\overline{a_k a_{k-1} \dots a_1}$ is a perfect square for all sufficiently large k.

[5♣] **Problem 12** (EGMO 2014/3)**.** We denote the number of positive divisors of a positive integer m by $d(m)$ and the number of distinct prime divisors of m by $\omega(m)$. Let k be a positive integer. Prove that there exist infinitely many positive integers n such that $\omega(n) = k$ and $d(n)$ does not divide $d(a^2 + b^2)$ for any positive integers a, b satisfying $a + b = n$.

[9♣] **Problem 13** (USAMO 2012/3)**.** Determine which integers $n > 1$ have the property that there exists an infinite sequence $a_1, a_2, a_3, \dots$ of nonzero integers such that the equality

$$a_k + 2a_{2k} + \cdots + na_{nk} = 0$$

holds for every positive integer k.

[9♣] **Problem 14** (TSTST 2016/3)**.** Decide whether or not there exists a nonconstant polynomial $Q(x)$ with integer coefficients with the following property: for every positive integer $n > 2$, the numbers

$$Q(0),\ Q(1), Q(2),\ \dots,\ Q(n-1)$$

produce at most $0.499n$ distinct residues when taken modulo n.