Math 249: Algebraic Combinatorics

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Notes for the course MATH 249, instructed by Lauren K Williams.

3 September 5, 2013

Okay so there's no Putnam class today, so these are notes for MATH 249.

Problem set 1 is due two weeks from today.

3.1 Incidence Algebras

Let P be a locally finite[1] poset, and $Int(P)$ is the set of intervals in P. Let K be a field. If $f : Int(P) \to K$, we abbreviate $f(x, y)$ as $f([x, y])$.

Now we define the incidence algebra $I(P)$ is a K-algebra of functions $f : Int(P) \to K$ where

$$fg(x, y) \stackrel{\text{def}}{=} \sum_{x \le z \le y} f(x, z)g(z, y).$$

The identity δ or 1 is the identity, given by

$$\delta(x, y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}.$$

Proposition 3.1. *Let $f \in I(P)$, then the following are equivalent.*

- *f has a left inverse.*
- *f has a right inverse.*
- *f has a two-sided inverse.*
- *$f(x, x) \neq 0 \quad \forall x \in P$.*

If f^{-1} exists, then $f^{-1}(x, y)$ depends only on $[x, y]$

Proof. First we write down the definition of inverse. Now the statement that $fg = \delta$ is equivalent to

$$f(x, x)g(x, x) = 1 \text{ and } \sum_{x \le z \le y} f(x, z)g(z, y) = 0 \quad \forall x < y.$$

The last sum may be rewritten as

$$f(x, x)g(x, y) + \sum_{x < z \le y} f(x, z)g(z, y).$$

So we require $g(x, y) = -f(x, x)^{-1} \sum_{x<z\le y} f(x, z)g(z, y)$ if we would like g to be the right inverse. This is okay as long as $f(x, x) \neq 0$ for all $x \in P$. Furthermore, the value of g depends only on $x < z \le y$; i.e. only on the interval $[x, y]$.

The same reasoning applies with $hf = \delta$. Now finally, if we have both $fg = \delta$ and $hf = \delta$, then $hfg = h = g$ so this is a two-sided inverse. □

For finite sets, there's something about upper triangular matrices and being invertible iff all the diagonal entries are nonzero, LOOK THIS UP.

[1] Every interval is finite

3.2 Zeta Function

Definition 3.2. The *zeta function* ζ is defined by $\zeta(x,y) = 1$ for all $x \le y$ in P.

We begin convoluting it.

$$\zeta^2(x,y) = \sum_{x \le z \le y} 1 = \#[x,y].$$

Now, if k is a positive integer,

$$\zeta^k(x,y) = \sum_{x \le z_1 \le z_2 \le \dots \le z_k = y} 1 = \#\text{multichains of length } k \text{ from } x \text{ to } y.$$

Note that multichains differ from chains in that chains have a strict inequality.
Similarly,

$$(\zeta - 1)(x,y) = \begin{cases} 1 & x < y \\ 0 & \text{otherwise} \end{cases}$$

whence $(\zeta - 1)^k(x,y)$ reports the number of chains of length k from x to y.

Remark 3.3. When $L = J(P)$ the number of chains of length $|P|$ equals $e(P)$, the number of linear extensions of P.

Here,

Definition 3.4. $J(P)$ is the set of all order ideals of P, with $I_1 \le I_2$ if $I_1 \subseteq I_2$.

and

Definition 3.5. A lattice is a poset L such that any $x, y \in L$ has a least upper bound and a greatest lower bound.

Now consider $2 - \zeta \in I(P)$. Evidently,

$$(2 - \zeta)(x,y) = \begin{cases} 1 & \text{if } x = y \\ -1 & \text{if } x < y \end{cases}.$$

By the previous proposition, $2 - \zeta$ is invertible.

Proposition 3.6. $(2-\zeta)^{-1}(x,y)$ *returns the total number of chains* $x = x_0 < x_1 < \dots < x_k = y$ *from* x *to* y*, of any length.*

Proof. There exists ℓ the length of the longest chain from x to y.
Evidently $(\zeta - 1)^{\ell+1}(u,v) = 0$ for all $x \le u \le v \le y$. Now consider

$$(2-\zeta)\left[1 + (\zeta - 1) + (\zeta - 1)^2 + \dots + (\zeta - 1)^\ell\right](u,v)$$

If this is the identity function then we win! But $2 - \zeta = 1 - (\zeta - 1)$ and we win by expansion: it evaluates to $1 - (\zeta - 1)^{\ell+1} = 1$. ☐

Exercise 3.7. Let

$$\eta(x,y) = \begin{cases} 1 & \text{if } y \text{ covers } x \\ 0 & \text{otherwise.} \end{cases}$$

Show that $(1-\eta)^{-1}(x,y)$ is the total number of maximal[2] chains in $[x,y]$.

[2] Maximal by inclusion. That means it can't be extended.

3.3 Möbius Inversion

Definition 3.8. The Möbius function μ is defined by $\mu = \zeta^{-1}$.

That means $\mu\zeta = \delta$, so bashing

1. $\mu(x,x) = 1$ for all $x \in P$.
2. $\mu(x,y) = -\sum_{x \le z < y} \mu(x,z)$ for all $x < y$ in P.

Example 3.9. Let P and Q be the posets in the above figure. Compute $\mu(\hat{0}, x)$, where $\hat{0}$ is the smallest element of each poset.

Solution. Shown. □

Now we have the following.

Proposition 3.10 (Möbius Inversion)**.** *Let P be a poset in which every principal order ideal*[3] *is finite. Let $f, g : P \to \mathbb{C}$. Then*

$$g(x) = \sum_{y \le x} f(y) \Leftrightarrow f(x) = \sum_{y \le x} g(y)\mu(y,x)$$

for all $x \in P$.

Proof. Let $\mathbb{C}^P$ be the vector space of all functions $P \to \mathbb{C}$. Now $I(P)$ acts on $\mathbb{C}^P$ as follows: for any $f \in \mathbb{C}^P$ and all $\alpha \in I(P)$, we have

$$(f\alpha)(x) \overset{\text{def}}{=} \sum_{y \le x} f(y)\alpha(y,x)$$

It's easy to check this is an action.

Then Möbius inversion says that $g = f\zeta \Leftrightarrow f = g\mu$, but $\mu = \zeta^{-1}$. □

Example 3.11 (Inclusion-Exclusion)**.** Recall that given finite sets A, B and C, we have

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|.$$

Consider finite sets $S_1, S_2, \dots, S_n$, let P be the poset of all their intersections ordered by inclusion (including $\varnothing$ and let $\hat{1} = \cup_{i \ge 1} S_i$ be the maximal element).

If $T \in P$ let $f(T) = \#\{x \in T : x \notin T' \forall T > T' \in P\}$. Then, let $g(T) = |T|$.

We want an expression for $|\cup S_i| = g(\hat{1}) = \sum_{T \le \hat{1}} f(T)$.

But $g(T) = \sum_{T' \le T} f(T)$. Also, $f(\hat{1}) = 0$ because any element in $\hat{1}$ belong to some S_i. So by Möbius Inversion, we have

$$0 = f(\hat{1}) = \sum_{T \le \hat{1}} g(T)\mu(T,\hat{1}) = g(\hat{1})\mu(\hat{1},\hat{1}) + \sum_{T < \hat{1}} g(T)\mu(T,\hat{1})$$

In particular, $g(\hat{1}) = -\sum_{T < \hat{1}} \#T\mu(T,\hat{1})$. But we can check that $\mu(T,\hat{1})$ does what we want (see figure).

[3] The principal order ideal generated by x is $\{y \in P : y \le x\}$.

3.4 Computing the Möbius Function

Let us have some lemmas!

Definition 3.12 (Direct Product)**.** If P and Q are locally finite posets, define $P \times Q$ as poset with elements $\{(x,y) \mid x \in P, y \in Q\}$ and $(x_1,y_1) \le (x_2,y_2)$ if and only if $x_1 \le x_2$ and $y_1 \le y_2$.

Proposition 3.13. *Let P and Q be locally finite posets. If $(x,y) \le (x',y')$ are elements of $P \times Q$, then*

$$\mu_{P\times Q}\left((x,y),(x',y')\right) = \mu_P(x,x')\mu_Q(y,y').$$

Proof. Compute

$$\begin{aligned}\sum_{(x,y)\le(u,v)\le(x',y')} \mu_P(x,u)\mu_Q(y,v) &= \left(\sum_{x\le u\le x'} \mu_P(x,u)\right)\left(\sum_{y \le v \le y'} \mu_Q(y,v)\right) \\ &= \delta_{xx'}\delta_{yy'} \\ &= \delta(x,y)(x'y')\end{aligned}$$

as desired. (Note that μ's are unique, so this is indeed our desired μ.) ☐

Example 3.14. Let $P = B_n$, the Boolean algebra of subsets of $\{1,2,\dots,n\}$ ordered by inclusion. Let $\underline{2}$ be the poset with two elements (it has two elements $\hat{0} \le \hat{1}$). It is obvious that

$$B_n \cong \underbrace{\underline{2} \times \underline{2} \times \cdots \times \underline{2}}_{n \text{ times}}$$

because we can view B_n as a zero-one vector. Let us compute μ for B_n.

μ for $\underline{2}$ is given by $\mu(0,0) = \mu(1,1) = 1$ and $\mu(0,1) = -1$. With the lemma, we find that $\mu(T,S) = (-1)^{|T-S|}$ i.e. $\mu(T,S) = (-1)^{\#\text{ times } T \text{ and } S \text{ differ}}$.

Example 3.15. What is the Möbius function for chain $\underline{n}$? Evidently $\mu(i,i) = 1$, $\mu(i,+1) = -1$ and then we find $\mu(i,j) = 0$ for $j \ge i+2$. That is,

$$\mu(i,j) = \begin{cases} 1 & \text{if } i = j \\ -1 & \text{if } i+1 = j \\ 0 & \text{otherwise} \end{cases}.$$

Example 3.16. Let $\mu(n)$ be the Möbius function from number theory. Let's try and get it from this perspective.

Let $n_1, n_2, \dots, n_k$ be nonnegative integers and define $P = \underline{n_1+1} \times \cdots \times \underline{n_k+1}$.

If $N = \prod_{i=1}^k p_i^{n_i}$ where the p_i are distinct primes, then P is isomorphic to the poset of divisors of N ordered by divisibility. So,

$$\mu\left((a_1,\dots,a_k),(b_1,\dots,b_k)\right) = \begin{cases} (-1)^{\sum b_i - a_i} & \text{if } b_i - a_i \le 1 \forall i \\ 0 & \text{otherwise} \end{cases}.$$

Now this is equal to the number-theoretic μ applied to $\prod_{i=1} p_i^{b_i - a_i}$. In particular, if all the a_i are zero, then it's equal to $\mu(N)$.

3.5 Topology

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Proposition 3.17. *P is a finite poset and $\hat{P}$ is P with an extra $\hat{0}$ and $\hat{1}$ (add a minimal and maximal element). Let $c_i = \#$chains of length i between $\hat{0}$ and $\hat{1}$. Then*

$$\mu_{\hat{P}}(\hat{0}, \hat{1}) = c_0 - c_1 + c_2 - c_3 + \dots.$$

Proof.

$$\begin{aligned} \mu_{\hat{P}}(\hat{0}, \hat{1}) &= (1 + (\zeta - 1))^{-1} (\hat{0}, \hat{1}) \\ &= \left(1 - (\zeta - 1) + (\zeta - 1)^2 - (\zeta - 1)^3 + \dots\right) (\hat{0}, \hat{1}) \\ &= c_0 - c_1 + c_2 - c_3 + \dots \end{aligned}$$

$\square$

Consider a Δ-simplicial complex. An element $S \in \Delta$ is called a face if $\dim(S) = |S| - 1$.

If Δ finite, let f_i be the number of i-dimenisonal faces of Δ. The *reduced Euler characteristic* is defined by

$$\tilde{\chi}(\Delta) = \sum_i (-1)^i f_i$$

i.e. one less than the ordinary Euler characteristic.

Recall that the order complex $\Delta(P)$ associated to P is the simplicial complex with vertex set P whose faces are chains of P. Now $\mu_{\hat{P}}(\hat{0}, \hat{1}) = \tilde{\chi}(\Delta(P))$.

4 September 10, 2013

4.1 Simplicial Complexes

Definition 4.1. A maximal (by inclusion) face of a simplicial complex is called a *facet*.

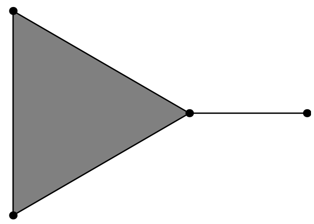

Figure 1: A simplicial complex with two facets.

Definition 4.2. A simplicial complex is *pure* if all facets have the same dimension.

Definition 4.3. For each face F of Δ, let $\overline{F}$ denote the subcomplex generated by F, i.e.

$$\overline{F} = \{G \in \Delta \mid G \subseteq F\}.$$

Key definition for today.

Definition 4.4. A simplicial complex Δ, not necessarily pure, is *shellable* if its facets can be arranged in linear order $F_1, F_2, \dots, F_t$ such that the subcomplex

$$\left(\bigcup_{i=1}^{k-1} \overline{F_i} \right) \cap \overline{F_k}$$

is pure, of dimension $\dim F_k - 1$,[4] for $k = 2, \dots, t$.

Intuitively, that means we can add on

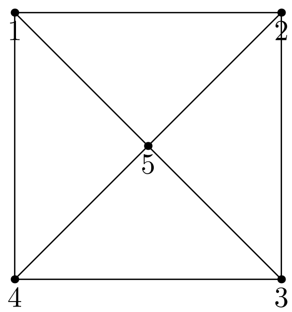

Figure 2: A complex with facets $\{125, 235, 345, 145\}$.

For example, in the above simplex, we add the facets in clockwise order starting from 125; at each step, the previous intersection is merely a segment. A non-example of a shelling is $125, 345, \dots$. In the latter case, we get a 0-dimensional intersection at $k = 2$.

In particular, a shellable simplicial complex must be connected (except for student counterexamples by pairs of points).

Example 4.5. In the pentagon example, $456, 15, 12, 23, 34$ is a shelling.

[4]The empty set is pure of dimension -1 here.

Definition 4.6. Let $F_1, \dots, F_t$ be shelling of Δ. Let $\Delta_j = \bigcup_{i=1}^{j} \overline{F_i}$. We say F_k is a *homology facet* if $\forall x \in F_k$

$$F_k \setminus \{x\} \in \Delta_{k-1},$$

i.e. the entire boundary of F_k lies in Δ_{k-1}. Let h_j denote the number of homology facets of dimension j.

Example 4.7. 34 is a homology facet in the pentagon example, because the entire boundary is already placed.

4.2 Topology

Recall that

Definition 4.8. A *wedge* $\bigvee_{i=1}^{n} X_i$ of n mutually disjoint connected topological spaces X_i means the space obtained by selecting a base point for each X_i and identifying the spaces at those base points.

So we glue the spaces together at the specified base point.

Theorem 4.9 (Bjoner and Wacks). *A shellable simplicial complex is homotopy equivalent to a wedge of spheres, where for each i, the number of i-spheres h_i.*

Recall that

Definition 4.10. Two (continuous) mappings of topological spaces $f_0, f_1 : T_1 \to T_2$ are *homotopic* (written $f_0 \sim f_1$) if there exists a (continuous[5]) mapping (called a homotopy) $F : T_1 \times [0, 1] \to T_2$ such that $F(t, 0) = f_0(t)$ and $F(t, 1) = f_1(t)$ for all $t \in T_1$.

We're kind of abusing notation by associating a simplicial complex with its geometric realization.

Definition 4.11. Two spaces T_1 and T_2 are homotopy equivalent if $\exists f_1 : T_1 \to T_2$ and $f_2 : T_2 \to T_1$ such that $f_2 \circ f_1 \sim \mathrm{id}_{T_1}$ and $f_1 \circ f_2 \sim \mathrm{id}_{T_2}$. We denote this as $T_1 \simeq T_2$.

Definition 4.12. A space which is homotopy-equivalent to a point is called *contractable*.

Lemma 4.13. *Let $F_1, F_2, \dots, F_t$ be a shelling of Δ. Let $F_{i_1}, \dots, F_{i_t}$ be a rearrangement of $F_1, \dots, F_t$ obtained by first taking all non-homology facets in induced order and then the homology facets in arbitrary order. Then this is a shelling.*

Let us say that $F_{i_1}, F_{i_2}, \dots, F_{i_r}$ are the non-homology facets and $F_{i_{r+1}}, \dots, F_{i_t}$ are the homology facets. Then for each F_{i_s} with $r + 1 \le s \le t$, the body of F_{i_s} is contained in $\bigcup_{k=1}^{n} \overline{F}_{i_k}$

Proof. Since the entire boundary of each homology facet F_k is contained in Δ_{k-1}, removing them doesn't change the fact that $F_{i_1}, \dots, F_{i_r}$ is a shelling of $\bigcup_{j=1}^{r} \overline{F_{i_j}}$.

The second claim follows for the same reason. $\square$

Reading the proof is probably more confusing than actually doing it.

Now to prove the main theorem.

[5] In this class, mappings between topologies are by definition continuous.

Proof. By the lemma, we may assume that the non-homology facets come first. So, suppose the shelling is $F_1, F_2, \dots, F_r$ of non-homology facets followed by $F_{r+1}, \dots, F_t$ of homology facets.

We claim that the subcomplex $\bigcup_{i=1}^{r} \overline{F_i}$ is contractable. We proceed by induction. Clearly $\overline{F_1}$ is contractable. For each $i = 2, \dots, r$, F_i has a free face[6] because it is not a homology facet. So we can contract F_i onto $\bigcup_{j=1}^{i-1} \overline{F_j}$ into what we've seen so far. This proves the claim.

The rest of the claim is obvious after we contract $\bigcup_{i=1}^{r} \overline{F_i}$ to a point. $\square$

4.3 Shellability for Order Complexes of Posets

Recall that

Definition 4.14. The *order complex* $\Delta(P)$ of a poset P is the simplicial complex with vertex set P whose faces are the chains of P.

Example 4.15. See paper.

Example 4.16. Given simplicial complex Γ, let $F(\Gamma)$ be its face poset[7]. What is $\Delta(F(\Gamma))$?

See the diagram. This is called the *barycentric subdivision* of Γ. Note that $\Delta(F(\Gamma)) \cong \Gamma$.

Part of this remark is that order complexes can be arbitrarily general; topologically we can get every simplicial complex as an order complex (in particular, the order complex of its face poset).

Remark 4.17. How uniquely does $\Delta(P)$ determine P? If P is a poset and Q is its dual[8], then they have the same order complex. So certainly not uniquely.

Now we wish to find properties of P that imply $\Delta(P)$ is shellable.

Definition 4.18. An *edge labeling* of a finite poset P is a map $\lambda : \mathcal{E}(P) \to \Lambda$ where $\mathcal{E}(P)$ is the edges of the Hasse diagram of P, and Λ is a poset.

Remark 4.19. Very often we will take $\Lambda = \mathbb{Z}$.

To each maximal chain $\mathcal{C}$ in P, we associate the word obtained by reading the edge labels from bottom to top.

Let $\mathcal{C}$ be a maximal chain and $\lambda(\mathcal{C}) = (\lambda_1, \lambda_2, \dots, \lambda_r)$.

Definition 4.20. We say $\mathcal{C}$ is increasing if $\lambda_1 < \lambda_2 < \cdots < \lambda_r$, and decreasing if $\forall i$, $\lambda_i \not< \lambda_{i+1}$.

Now we can order the maximal chains of $\Delta(P)$ by using the lexographic order on the associated words (having chosen a linear extension).

Definition 4.21. Let P be a finite poset. We say that an *EL-labeling* of P is an edge labeling such that in each interval $[x, y]$ of P there exists a unique increasing maximal chain, which moreover, lexicographically precedes all other maximal chains (not necessarily increasing) in that interval.

[6] A proper face contained in no other facet we've seen so far.

[7] Faces are order with $a \le b$ if a is contained in the closure of P

[8] Reverse all orderings

Okay, so why do we care?

Theorem 4.22 (Bjorner)**.** *Let P be a finite poset such that $\hat{P}$ has*[9] *an EL-labeling. Then the lexical order of the maximal chains of P is a shelling of $\Delta(P)$.*

In particular, $\Delta(P)$ is homotopy equivalent to a wedge of spheres, where the number i-spheres of dimension i is equal to the number of decreasing chains of length $i+2$.

Example 4.23. Let us try to construct an EL-labelling for the Boolean algebra B_n. Take the Hasse diagram and on each edge, label the element which is being added.

In particular, by the theorem, $\Delta(B_n - \hat{0} - \hat{1})$ is homotopy equivalent to the sphere of dimension $n-2$.

[9] $\hat{P} = P \cup \hat{0} \cup \hat{1}$.

5 September 12, 2013

So we prove the theorem we had from yesterday. Some reminders:

Definition. A simplicial complex Δ, not necessarily pure, is *shellable* if its facets can be arranged in linear order $F_1, F_2, \dots, F_t$ such that the subcomplex

$$\left(\bigcup_{i=1}^{k-1} \overline{F_i} \right) \cap \overline{F_k}$$

is pure, of dimension $\dim F_k - 1$,[10] for $k = 2, \dots, t$.

Definition. Let $F_1, \dots, F_t$ be shelling of Δ. Let $\Delta_j = \bigcup_{i=1}^{j} \overline{F_i}$. We say F_k is a *homology facet* if $\forall x \in F_k$

$$F_k \setminus \{x\} \in \Delta_{k-1},$$

i.e. the entire boundary of F_k lies in Δ_{k-1}. Let h_j denote the number of homology facets of dimension j.

Recall that $\Delta(P)$ is the simplicial complex whose facets are maximal chains. Recall also the definition of edge-labeling and our assignment of a word $\lambda(\mathcal{C}) = (\lambda_1, \dots, \lambda_r)$ for each maximal chain $\mathcal{C}$ in P. $\mathcal{C}$ is increasing if $\lambda_i < \lambda_{i+1}$ and decreasing if $\lambda_i \not< \lambda_{i+1}$ for all i. Also recall the definition of EL-labelling.

5.1 Bjorner's Theorem

We will now prove Bjorner's theorem, copied again below.

Theorem (Bjorner)**.** *Let P be a finite poset such that $\hat{P}$ has an EL-labeling. Then the lexical order of maximal chains in $\lambda(\mathcal{C})$ is a shelling of $\Delta(P)$. In particular, $\Delta(P)$ is homotopy equivalent to a wedge of spheres, where the number i-spheres of dimension i is equal to the number of decreasing chains of length $i+2$.*

The prove invokes the following lemma, which gives a sufficient condition for shellability.

Lemma 5.1. *Let Δ be a simplicial complex and let $F_1, \dots, F_t$ be an ordering of the facets such that*

$$\forall 1 \le i < j \le t, \exists 1 \le k < j \text{ and } z \in F_j : F_i \cap F_j \subseteq F_k \cap F_j = F_j \setminus \{z\}. \qquad (1)$$

Then $F_1, \dots, F_t$ is a shelling.

Proof. Let $D_j \overset{\text{def}}{=} \overline{F_j} \cap \bigcup_{i=1}^{j-1} \overline{F_i}$; it is a simplicial complex and we need to show that it is closed. Take any point $x \in D_j$. Supposing (1) holds, we need to show x is contained in a $(\dim F_j - 1)$-dimensional face of D_j. Clearly x cannot be contained in a face of dimension exceeding $\dim F_j$.

BY definition, $x \in F_j \cap F_i$ for some i, and (1) implies $\exists 1 \le k < j$ and $z \in F_j$ such that $x \in F_i \cap F_j \subseteq F_k \cap F_j = F_j \setminus \{z\}$. But x lies in $F_k \cap F_j$, which has the correct dimension because $F_j \ \{z\}$ has dimension $F_j - 1$. $\square$

Now for the proof of the main theorem.

[10] The empty set is pure of dimension -1 here.

Proof. First we will show we have a shelling, and then identify the homology facets.

Suppose $\hat{P}$ has an EL-labeling. First we will show the lexical order of maximal chains is a shelling of $\Delta(P)$.

Consider two maximal chains C_i and C_j of $\Delta(P)$ and WLOG assume $\lambda(C_i) < \lambda(C_j)$ lexographically. We want to find a third chain C_k and $z \in P$ such that (1) holds; that is, $\lambda(C_k) \le \lambda(C_j)$ and $C_i \cap C_j \subseteq C_k \cap C_j = C_j \setminus \{z\}$.

Let $C_j = \left\{\hat{0} = p_0 < p_1 < \cdots < p_s = \hat{1}\right\}$. Suppose C_i and C_j in their first q elements $p_0, \dots, p_q$ and then differ at the next position $q+1$. Let r be minimal such that $r \ge q+1$ and $p_r \in C_i$; the point which they meet up at again (indexed by the C_j chain). This r exists because both chains contain $\hat{1}$.

Consider the interval $[p_q, p_r]$ in $\hat{P}$. Since $\lambda C_i < \lambda C_j$, and the chains agree in the first q positions, $\lambda(p_q < p_{q+1} < \cdots < p_r)$ is NOT increasing, since any increasing chain must be lex earlier than C_i by definition of EL-labeling. That means we can find u such that $q < u < r$ and $\lambda(p_{u-1} < u < p_{u+1})$ is not increasing. So now, by EL-labeling, we can find an increasing chain $\tilde{C}$ in $[p_{u-1}, p_{u+1}]$ which is lex earlier than the chain $p_{u-1} < p_u < p_{u+1}$.

Now (1) is immediate, so we get a shelling.

Finally, we claim that the homology facets correspond to decreasing chains. Here's the proof. A maximal chain $C = p_1 < \cdots < p_{s-1}$ is a homology facet iff $\forall p_i$, $C - \{p_i\}$ lies in a maximal chain C' with $\lambda(C') < \lambda(C)$. This is equivalent to $\lambda(p_{i-1}, p_i, p_{i+1})$ is not increasing for each i, which is equivalent to $\hat{0} = p_0 < p_1 < \cdots < p_s = \hat{1}$ being decreasing. $\square$

5.2 Examples

Corollary 5.2. *If $\hat{P} = P \cup \hat{0} \cup \hat{1}$ is a finite poset and it has an EL-labeling, then*

$$\mu(\hat{0}, \hat{1}) = \sum_{C \text{ decreasing from } \hat{0} \text{ to } \hat{1}} (-1)^{|C|}.$$

Proof. We know that $\mu(\hat{0}, \hat{1}) = \tilde{\lambda}(\Delta(P))$, and $\Delta(P)$ is homotopy equivalent to a wedge of spheres, one for each decreasing chain from $\hat{0}$ to $\hat{1}$. Also, $\tilde{\chi}(S^d) = -1 + f_0 - f_1 + f_2 - \cdots = (-1)^d$ (can be proved by induction with the base cases $d = 0$ and $d = 1$). And yet $(\tilde{\chi}(T \vee T') = \tilde{\chi}(T) + \tilde{\chi}(T')$ yields the conclusion. $\square$

Theorem 5.3. *Let L be a finite distributive lattice. Then L has an EL-labelling (so $\Delta(L - \hat{0} - \hat{1})$ is shellable). Moreover, $\Delta(L - \hat{0} - \hat{1})$ is contractible or homotopy equivalent to a sphere.*

Proof. By the fundamental theorem of finite distributive lattices[11], $L = J(P)$ for some poset P.

Choose some linear extension $\lambda : P \to \{1, 2, \dots, |P|\}$. The cover relations for $J(P)$ have the form I covers $I' \Leftrightarrow I' = I \cup \{p\}$ and $I' \ne I$ (where $I, I' \in J$). This encourages us to label that edge by $\lambda(p)$.

We need to show in every interval $[I, I,]$ we need a unique increasing chain that is lex-earlier than all other maximal chains. Let $I' - I = \{p_1, p_2, \dots, p_k\}$ where $\lambda(p_1) < \lambda(p_2) < \cdots < \lambda(p_k)$. Then

$$I < I \cup \{p_1\} < I \cup \{p_1, p_2\} < \cdots < I'$$

is the unique increasing maximal chain, which is certainly lex least.

[11] What is this?

We do need to check that $I \cup \{p_1, p_2, \dots, p_j\}$ is indeed an order ideal. This is obvious because if $q < p_i$ for some $1 \le i \le j$, then $\lambda(q) < \lambda(p_i)$ because λ is a linear extension.

Therefore we have an EL-labeling.

To show that L is either contractable, uh, do magic. $\square$

6 September 17, 2013

Recall that if P is a finite poset, an EL-labeling of P is an edge-labeling such that for every interval $[x, y]$, the lexicographically earliest maximal chain in $[x, y]$ is the unique increasing maximal chain.

6.1 Loose Ends

Recall the theorem from earlier. Let us complete the proof about homotopy stuffs.

For the first case, suppose L is a Boolean algebra; i.e. P is an antichain with no two comparable elements. Then we have one decreasing chain, so it is a sphere.

In the second case, suppose P is not an antichain; i.e. $\exists p < p'$ in P. We claim that in this case there are no decreasing chains. To build maximal chains of order ideals in $J(P)$, we need to add p before p', breaking the decreasing. Hence no chain can be decreasing.

6.2 Some Examples

Example 6.1. For each $k \le n$ define the truncated Boolean algebra by

$$B_n^k = \{A \subseteq \{1, 2, \dots, n\} = mid\, |A| \ge k\}$$

ordered by inclusion.

Find an EL-labeling and then the homotopy type of $B_n^k - \hat{1}$.

Solution. Define an edge-labeling λ of $B_n \cup \left\{\hat{0}\right\}$ by

$$\lambda(A_1, A_2) = \begin{cases} \max A_2 & \text{if } A_1 = \hat{0} \text{ and } |A_2| = k \\ a & \text{if } A_2 - A_1 = \{a\}. \end{cases}$$

It is easy to see these are the only types of edges. To check this is indeed an EL-labeling. We only need to check intervals of the form $[\hat{0}, A_2]$ since the first case has already been checked. But we can check that the unique increasing chain is

$$\hat{0} = I_k < I_{k+1} < \dots < I_\ell = A_2$$

where I_k is the smallest k elements of A_2, and $I_{k+t+1} = I_{k+t} \cup \min(A_2 - I_{k+t})$

So now we want to check the decreasing chains.

Indeed, consider a decreasing chain

$$\hat{0} < I_k < I_{k+1} < \dots < I_n = \{1, \dots, n\}.$$

The label n had better appear on the edge from $\hat{0}$ to I_k, so it is necessary I_k to pick I_k containing n. Then we get $I_{k+1}, \dots$ by adding elements of $\{1, \dots, n\} - I_k$ in decreasing order.

Therefore, there are $\binom{n-1}{k-1}$ decreasing chains. So $\Delta(B_n^k - [n])$ is homotopy equivalent to a wedge of $\binom{n-1}{k-1}$ spheres of dimension $1 + (n-k) - 2 = n - k - 1$. □

6.3 CW Complexes

Recall the definition of homeomorphism; we need a continuous bijection $f : X \to Y$ with a continuous inverse.

Definition 6.2. A *cell* is a topological space homeomorphic to an open ball[12].

We will denote a cell of dimension n by e^n. A closed ball of dimension n will be denoted D^n (for disk), and the sphere by S^n. Note that e^0 is a single point.

A CW-complex is, roughly, a space that can be formed by gluing together cells. One can check that S^n and B^n are all CW-complexes.

Here is the formal definition.

Definition 6.3. A finite-dimensional CW complex is a space built inductively as follows:

(i) Start with a discrete set X^0 whose points are regarded as 0-cells.

(ii) Inductively form the n-skeleton X^n from X^{n-1} by attaching n-cells e^n_α (where α is an index) via attaching maps

$$\Psi_\alpha : S^{n-1} \to X^{n-1}.$$

Here the S^{n-1} is the boundary of e^n_α.

(iii) Then X^n is the quotient space of

$$X^{n-1} \bigsqcup_\alpha D^n_\alpha$$

of $X^n - 1$ with collection of n-disks D^n_α under identifications $x \sim \Psi_\alpha(x)$ for $x = \partial D^n_\alpha = S^{n-1}$.

Definition 6.4. If a topological space X can be decomposed into a CW complex, this is called a CW decomposition of X.

Remark 6.5. These decompositions are in general not unique. Consider a circle in Figure 3.

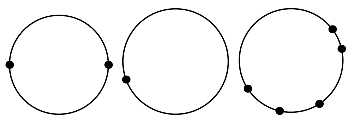

Figure 3: Several CW-decompositions of the 1-sphere. The left and right are regular; the middle one is not.

Definition 6.6. A CW complex is *regular* if each attaching map Ψ_α is a homeomorphism.

Definition 6.7. Given a CW-complex X with set of cells $\mathcal{C}$, define the *face poset* $\mathcal{F}(x)$ to be a poset on $\mathcal{C} \cup \hat{0}$ where $C_1 \le C_2$ iff $C_1 \subseteq \overline{C_2}$.

This is going in the opposite direction of previous lectures, where we started with a poset P and analyzed the topological object $\Delta(P)$.

Question 6.8. To what extent does $\mathcal{F}(X)$ determine the topology of X?

[12]A ball minus its boundary.

Definition 6.9. Given a CW decomposition of X, let f_i be the number of i-dimensional cells. Then the Euler characteristic is

$$\chi(X) = f_0 - f_1 + f_2 - f_3 + \dots .$$

As a standard theorem of algebraic topology,

Theorem 6.10. *The Euler characteristic of X depends only on the homotopy type of X; in particular, it is independent of the CW decomposition.*

This is not trivial; check this in each of the 1-sphere in Figure 3.

However, two non-homotopic spaces could have the same face poset. (Exercise)

Theorem 6.11 (Lindell and Weingram)**.** *If X is a regular CW-complex, then*

$$\Delta(\mathcal{F}(X) - \hat{0})$$

is homeomorphic to X.

So regular CW complexes are "combinatorial objects".

The idea of the proof is the order complex $\Delta(\mathcal{F}(X) - \hat{0})$ is the barycentric subdivision of X. See images.

Question 6.12. When is a poset the face poset of a regular CW complex?

In fact, there is a complete characterization.

Definition 6.13. A poset is a *CW poset* if

(a) P has a least element $\hat{0}$

(b) P is nontrivial, i.e. it has more than one element

(c) For every $x \in P$ with $x \neq \hat{0}$, the open interval $(\hat{0}, x)$ has an order complex $\Delta\left((\hat{0}, x)\right)$ which is homeomorphic to a sphere.

Note the interval is open.

Proposition 6.14. *A poset P is a CW-poset if and only if it is a face poset of a regular CW-complex.*

Proof. Let X be a regular CW complex and let $\overline{\mathcal{C}_\alpha}$ be the closed n-dimensional cell in X. Then its boundary $\partial\mathcal{C}_\alpha$ is homeomorphic to a sphere. Also, the open interval $(\hat{0}, \overline{\mathcal{C}_\alpha})$ in $\mathcal{F}(X)$ consists of cells having a regular CW decomposition of $\partial\mathcal{C}_\alpha$.

Then the theorem earlier implies that $\Delta(\hat{0}, \overline{\mathcal{C}_\alpha})$ is homeomorphic to a sphere.

The other direction is constructive; we glue in cells one by one. Let P be a CW poset and let $P_k = \{x \in P \mid r(x) = k\}$.[13] We will assume a CW complex K_{n-1} has been constructed such that $\mathcal{F}(K_{n-1}) = \bigcup_{k \le n} P_k$.

Each $x \in P_{n+1}$ corresponds to $(\hat{0}, x)$ which is a sphere in the regular CW subcomplex $K_x \subseteq K_{n-1}$ such that

$$\Delta(\mathcal{F}(K_x) - \hat{0}) \cong S^{n-1}.$$

So we can attach an n-cell to K_{n-1} for each $x \in P_{n+1}$ by maps $S^{n-1} \cong K_x \to K_{n-1}$, so we obtain a CW complex K_n such that $\mathcal{F}(K_n) \cong \cup_{k \le n+1} P_k$. This is regular since we just use identifying maps to map the boundary of each e^n_α to $S^{n-1} \cong K_x$

In other words, put things where you have to go. ☐

[13] It's not hard to check that in a CW poset all the saturated chains are graded.

7 September 19

Guest lecture by Michelle Wachs.

7.1 Definitions

First, let us describe the homology type of posets.

Definition 7.1. Let P be a finite poset and let $C_r(P)$ be the vector space generated by r-chains of P; i.e. linear combinations of the chains

$$x_0 < x_1 < \cdots < x_r.$$

Definition 7.2. The boundary map $\partial_r : C_r(P) \to C_{r-1}(P)$ is defined by taking $x_0 < x_1 < \cdots < x_r$ to

$$\sum_{i=0}^{r} (-1)^i \left(x_0 < x_1 < \cdots < x_{i-1} < x_{i+1} < \cdots < x_r\right).$$

One can check that $\partial_{r-1}\partial_r$ is the zero map.

Now (C_r, ∂_r) is an algebraic complex.

Definition 7.3. The homology $\tilde{H}_r(P) = \ker \partial_r / \text{im } \partial_{r+1}$.

This is equivalent to the reduced simplicial homology of the order complex $\Delta(P)$.

Often, we will only consider when r is the length of the poset P, so that

$$\tilde{H}(P) = \ker \partial_r \subseteq C_R(P).$$

7.2 Homotopy and Homology

Homotopy type determines homology. Here is an example.

Proposition 7.4. *If P has length ℓ and $\Delta(P)$ has homotopy type of a wedge of m ℓ-spheres, then*

$$\tilde{H}_i(P) = \begin{cases} 0 & \text{if } i < \ell \\ k^m & \text{if } i = \ell. \end{cases}$$

where k is the underlying field of $C_r(P)$.

Example 7.5. Consider $B_3 = \underline{2} \times \underline{2} \times \underline{2}$; i.e. subsets of $\{1, 2, 3\}$ ordered by inclusion.

Let us compute $\Delta(\overline{B_3})$.[14] It is easy to see that it is homeomorphic to S^1.

Set $\rho = (1 < 12) - (2 < 12) + (2 < 23) - (3 < 23) + (3 < 13) - (1 < 13)$. One can check $\partial_1(\rho) = 0$, so $\rho \in H_1(\overline{B_3})$, and hence $H_1(\overline{B_3}) = \langle p \rangle$.

Example 7.6. In general, $\Delta(\overline{B_n})$ is the barycentric subdivision of the boundary of the $(n-1)$-simplex.

So, $\dim\left(\tilde{H}_{n-2}\left(\overline{B_n}\right)\right) = 1$. Then $\tilde{H}_{n-2}\left(\overline{B_n}\right) = \langle \rho_n \rangle$, where

$$\rho_n = \sum_{\sigma \in S_n} \left(\sigma(1) < \sigma(1)\sigma(2) < \cdots < \sigma(1)\sigma(2)\dots\sigma(n-1)\right).$$

The cycle is called a *fundamental cycle.*

Fact 7.7. Every spherical simplicial complex has a unique fundamental cycle, up to sign, which generates the homology.

[14] Here $\overline{P} = P - \{\hat{0}, \hat{1}\}$.

7.3 The Partition Lattice

In the homework, we check that Π_n is EL-shellable, and is a wedge of $(n-1)!$ spheres of dimension $n-3$.

By the proposition, $\dim \tilde{H}_{n-3}\left(\overline{\Pi_n}\right) = (n-1)!$. We wish to find a basis for this homology. Let $\mathcal{T}_n$ be the set of rooted trees on $\{1, 2, \dots, n\}$. For any $T \in \mathcal{T}_n$, let Π_T be the induced subposet of Π_n, where the partitions are those which may be obtained by removing edges from T.

An example for $n = 4$ is drawn. Note that the "rooted" condition isn't relevant yet; it will come up in a few minutes.

We see that Π_T is isomorphic to the Boolean algebra B_{n-1}. Let ρ_T be the fundamental cycle of $\Delta(\overline{\Pi_T})$.

Theorem 7.8 (Bjorner's NBC basis). *The set*

$$\{\rho_T \mid T \in \mathcal{T}_n \text{ and } T \text{ is increasing.}\}$$

is a basis for $\mathcal{H}_{n-3}\left(\overline{\Pi_n}\right)$.

Here, we have

Definition 7.9. A tree is *increasing* if all the children of a vertex exceed their parent. Here 1 is the root.

Remark 7.10. Note that the dimensions check out. The number of rooted increasing trees is $(n-1)!$; this is an extremely easy combinatorics exercise.

Let us sketch the proof of the theorem.

Proof. Prove that for each falling chain C of a certain EL-labeling of Π_n, there is a unique increasing $T \in \mathcal{T}_n$ such that c is one of the maximal chains of Π_T. Once we have a correspondence between trees and falling chains, use this to show the relevant set is independent.

Here's the EL-labelling we use, due to Stanley and Bjorner. Let π be covered by τ and let

$$\lambda(\pi, \tau) = (\min A, \min B)$$

where A and B are the blocks of π that are merged to get τ, and $\min A \le \min B$. Here λ maps into $\{(a, b) \mid 1 \le a < b \le n\}$, sorted lexicographically. □

7.4 Free Lie Algebra

$$\tilde{H}_{n-3}\left(\overline{\Pi_n}\right) \cong_{S_n} \mathrm{Lie}(n) \otimes \mathrm{sgn}_n.$$

...what is this. Homology isomorphic to free Lie algebra somehow...

Double bracketed free Lie algebra.

Things to Google: Lindon basis.

7.5 Weighted Partitions

Definition 7.11. A weighted partition of $\{1, 2, \dots, n\}$ is a set

$$\{B_1^{v_1}, B_2^{v_2}, \dots, B_t^{v_t}\}$$

where $(B_1, \dots, B_t)$ is a partition of $\{1, 2, \dots, n\}$ and $0 \le v_i \le |B_i| - 1$.

Definition 7.12. Let Π_n^w be the poset of weighted partitions of $\{1, 2, \dots, n\}$ with covering relation given by

$$\left\{A_1^{u_1}, \dots, A_S^{u_s}\right\} \lessdot \left\{B_1^{v_1}, \dots, B_t^{v_t}\right\}$$

if

(i) The A_i is covered by B_i in Π_n, and

(ii) If $B_k = A_i \cup A_j$, then either $v_k = u_i + u_j$ or $v_k = u_i + u_j + 1$.

(iii) All other weights remain the same.

For example, $17^0 \mid 235^1 \mid 468^2$ is covered by either $17^0 \mid 234568^3$ or $17^0 \mid 234568^4$. Π_3^w is drawn. Note that Π_3^w has a unique minimum $1^0 \mid 2^0 \mid 3^0$ but not unique maximum but three maximal elements 123^0, 123^1, 123^2. The obvious generelizations to n instead of 3 are true and easy to verify.

Theorem 7.13 (González D'León and Michelle Wachs)**.** $\Pi_n^w \cup \left\{\hat{1}\right\}$ *is EL-shellable.*

Two prior published proofs gave CL-shellability, which is weaker. The second proof found a mistake in the first proof, and a mistake was found in the second proof by Wachs. Hopefully induction doesn't continue.[15] We really only need to show that $[\hat{0}, [n]^i]$ is EL-shellable for each $i = 0, \dots, n-1$.

Consider $\alpha \lessdot \beta < \hat{1}$. We set

$$\lambda(\alpha, \beta) = (\min A, \min B)^u$$

where A and B are blocks of α that are merged to get β, $\min A \le \min B$ and $u \in \{0, 1\}$ according to whether the weight increases by one or not in β.

The order on the label set is not as natural. Define

$$\Gamma_a = \{(a, b)^u : a < b \le n + 1, u \in \{0, 1\}\}$$

and declare $(a, b)^u \le (a, c)^v$ if $b \le c$ and $u \le v$. Let $\Lambda_n = \Gamma_1 \oplus \Gamma_2 \oplus \cdots \oplus \Gamma_n$, where $\oplus$ is teh ordinal sum; that means all elements of Γ_i are greater than those in Γ_j if $j > i$. Γ_1 is drawn.

Note that in this case, the order on the label set is NOT a total order! Hence, a better name for decreasing/falling chain is "ascent-free chains".

Remark 7.14. This is a generalization of Bjorner's labeling, since the intervals $[\hat{0}, [n]^0]$ and $[\hat{0}, [n]^{n-1}]$ are identical copies of Π_n with Bjorner's labeling.

From a computation with the Moëbius function, we get the fololwing.

Theorem 7.15 (González D'Leon, Michelle Wachs)**.** *For each* $i = 0, \dots, n-1$*,* $\Delta(\hat{0}, [n]^i)$ *has the homotopy type of a wedge of* $|\mathcal{T}_{n,i}|$ *spheres of dimension* $n - 3$*.*

Here, $\mathcal{T}_{ni}$ is the set of rooted trees on n with i descents. Note that when $i = 0$ this just becomes the case Π_n, and in that event $\mathcal{T}_{n,0} = \mathcal{T}_n$ which is the set of increasing trees (i.e. no descents).

Remark 7.16. $|\mathcal{T}_{ni}|$ has no closed form, but $\sum_{i \ge 0} |\mathcal{T}_{ni}| = n^{n-1}$.

The basis for the homology of the interval $\left(\hat{0}, [n]^i\right)$ is as follows. Let $T \in \mathcal{T}_{ni}$, and form Π_T^w an induced sub-poset of Π_n^w in a similar manner as before: we remove edges form T to get the underlying partition. The weight on each block will be the number of descents. Note that $\Pi_T^w \cong B_{n-1}$ again, so we have a fundamental cycle ρ_T^w. In that case, the basis for $\tilde{H}_{n-3}\left(\hat{0}, [n]^i\right)$ is $\{\rho_T^w \mid T \in \mathcal{T}_{ni}\}$. This is harder to prove.

[15] Michelle Wachs comment: it is so easy to get EL-shellability wrong. Where it's wrong is always where they say "it's obvious that".

8 September 24, 2013

8.1 Review

Review of last week: a CW complex is a space built inductively by

- Choosing some 0-cells, the set of which is denoted by X^0,
- Attaching some 1-cells by gluing their boundaries to X^0. The result is called X^1.
- Rinse and repeat.

We terminate this process at some finite dimension.

We say that a CW complex is *regular* if all the attaching maps are injective.

Given the CW complex X with set of cells $\mathcal{C}$, the *face poset* $\mathcal{F}(X)$ is the poset on $\mathcal{C} \cup \hat{0}$ where $C_1 \le C_2$ if and only if $C_1 \subseteq \overline{C_2}$.

Theorem 8.1. *If X is a regular CW-complex, then $\Delta(\mathcal{F}(X) - \hat{0}) \cong X$.*

In other words, if X is a regular CW-complex, then its topology is determined entirely by its combinatorial properties; i.e. the face poset $\mathcal{F}(X)$.

Björner already answered "when is a poset the face poset of a regular CW complex"; see previous lectures.

The forward direction is to construct the X^i by the rank.

8.2 What happened to Coxeter groups?

Theorem 8.2 (Danarja-Klee)**.** *Let Γ be a pure shellable d-dimensional simplicial complex such that each codimenison-one face is contained in at most two facets. Then Γ is homeomorphic to either a d-sphere ora d-ball. Moreover, Γ is homeomorphic to a d-sphere if and only if each codimension-one face is contained inexactly two facets.*

We will not discuss the proof, but as an example,

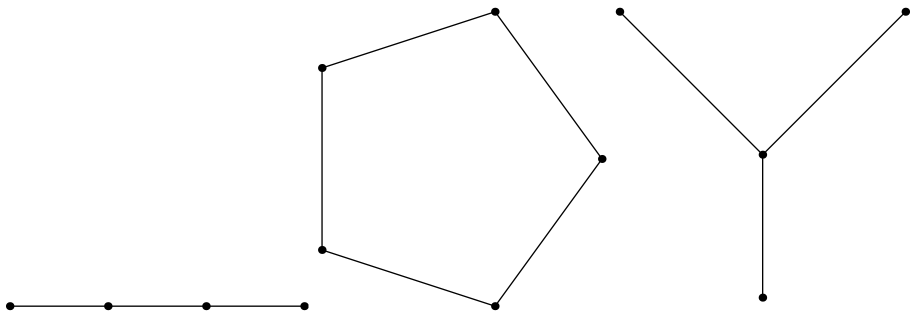

Figure 4: A 1-ball, and a 1-sphere. The theorem applies in both cases. On the far right is a non-example of the theorem

Question 8.3. When does $\Delta(P)$ satisfy the hypothesis of the theorem, where P is a graded [16] poset?

Answer 8.4. All rank 2 intervals of P are either $\diamond$ or a chain of length 2.

Combing the above result with our theorem, we obtain that

[16] The graded condition is required for $\Delta(P)$ to be pure

Proposition 8.5. *Let P be a nontrivial poset such that*

(a) P has $\hat{0}$,

(b) Every interval of rank 2 is a diamond, and

(c) Every interval $[\hat{0}, x]$, where $x \in P$, is finite and shellable.

Then P is a CW poset.

8.3 Bruhat Order

Let $\pi = (a_1, a_2, \dots, a_n)$ or $a_1 a_2 \dots a_n$ denote the permutation of $[n]$ sending $i \mapsto a_i$.

Let $s_i = (i\ i+1)$ denote a transposition of i and $i+1$, and in general set $(i\ j)$ is the permutation which swaps i and j.

Then a presentation for the symmetric group can be given by $\langle s_1, s_2, \dots, s_{n-1} \rangle$ subject to $s_i^2 = 1$, $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, and $s_i s_j = s_j s_i$ if $|i - j| \geq 2$.

Now we add a notion of length. Write $w = s_{i_1} s_{i_2} \dots s_{i_m}$ as a minimal product of s_i's. Then this is called a *reduced expression for* w, and m is called the length $\ell(w)$ of w.

Exercise 8.6. The length $w \in S_n$ is equal to the number of inversion. That is,

$$\ell(w) = \# \{(i, j) \mid 1 \leq i < j \leq n, w(i) > w(j)\}.$$

Definition 8.7. The *(strong) Bruhat order* on S_n is the poset whose cover relations are as follows: $v \lessdot w$ if $\ell(w) = \ell(v) + 1$ and $\exists i < j$ such that $w = (i\ j)v$.

If we add the condition $j = i + 1$ then this is called the weak left Bruhat order.

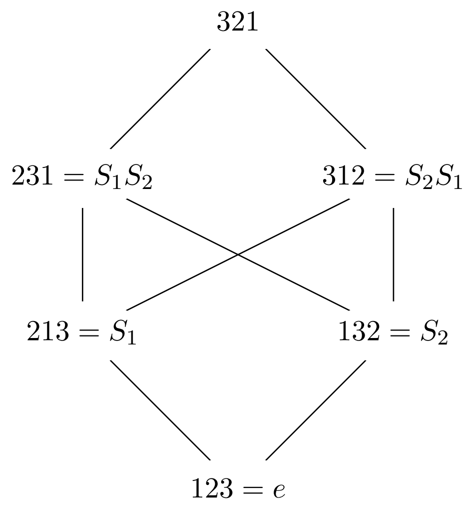

Figure 5: The Bruhat order when $n = 3$.

Remark 8.8. 3412 does not cover 4213, even though the former has 4 inversions and the latter has three.

8.4 Results

Theorem 8.9 (Edelman 1981). *The Bruhat order on S_n is shellable.*

Theorem 8.10 (Björner, Wachs 1982). *The Bruhat order on any Coxeter group is CL-shellable.*

Theorem 8.11 (Dyer, 1991). *The Bruhat order on any Coxeter group is EL-shellable.*

Definition 8.12. A poset is *thin* if every rank 2 interval is a diamond.

Exercise 8.13. The Bruhat order on S_n is thin.

Corollary 8.14 (Björner). *Bruhat order on S_n is a CW psoet, i.e. it is the face poset of regular CW complex. Moreover, since it has a single maximal element, it is a regular CW decomposition of a ball.*

Question 8.15. Is there a regular CW complex "in nature" whose face poset is the Bruhat order?

Fomin-Shaprio had a conjectured an answer, and Patricia Hersh proved that the construction was indeed a CW complex. The constructions arise from total positivity.

8.5 Proving the Bruhat Order is EL-Shellable

First we need a labelling. For the label of $v \lessdot w$, where $w = (i\ j)v$, we'll just use the label (i, j) and sort lexicographically; here $i < j$. For example, when $n = 3$, our label set is simply $(1, 2) < (1, 3) < (2, 3)$.

We claim this is an EL-labeling.

First, we claim that in any interval, the lex least chain is increasing. Let $\mathcal{C}$ be the lex least chain in $[x, y]$, and denote it by

$$x = \pi_0 \lessdot \pi_1 \lessdot \cdots \lessdot \pi_m = y$$

and suppose for contradiction it has a decrease. Then for some r, we have $\pi_{r-1} \lessdot \pi_r$.

One case is where $\lambda(\pi_{r-1}, \pi_r) = (i, j)$ and $\lambda(\pi_r, \pi_{r+1}) = (i, k)$ with $j > k$, and $k > i$. The other case (i, j) and (k, ℓ) is similar. Now suppose that

$$\pi_{r-1} = a_1 a_2 \dots i \dots j \dots k \dots a_n$$

You can check that i, j, k must appear in this order, otherwise we run into a contradiction. So then

$$\pi_r = a_1 a_2 \dots j \dots i \dots k \dots a_n,$$

$$\pi_{r+1} = a_1 a_2 \dots j \dots k \dots i \dots a_n.$$

Define $\pi'_r = a_1 a_2 \dots k \dots j \dots i \dots a_n$. Then $\pi_{r-1} \lessdot \pi'_r \lessdot \pi_{r+1}$. But then teh chain obtained by swapping π_r with π'_r is lexicographically smaller.

Finally we show that there is at most one increasing chain between any interval from x to y.

Let $x = a_1 a_2 \dots a_n$ and $y = b_1 b_2 \dots b_n$. Let i be the smallest number such that $x^{-1}(i) = y^{-1}(i)$. Note that no number less than i will appear in any label in a chain from x to y, because of the way the Bruhat order is constructed. Furthermore, $x^{-1}(i) < y^{-1}(i)$; otherwise i appears later in x than in y, and any chain from x to y swaps i with a larger number, but then this swap decreases inversions. Now note$i$ must appear in some label; because the chain increases, this means that it must be in the first label. Finally, if $\lambda(x, \pi_1) = (i, k)$, then we must have $x^{-1}(k) \le y^{-1}(i)$ (check this).

Now let j be the smallest number $j \ge i+1$ such that $x^{-1}(i) < x^{-1}(j) \le y^{-1}(i)$.

Finally we show that the first label on the increasing chain is (i, j). This last remark will complete the solution, because this will force at most one choice of an increasing chain.

Oops time.

9 September 26, 2013

9.1 Wrapping Up Loose Ends

Let us complete the proof that the Bruhat order is EL-shellable. Recall that we label the edges of the Hasse diagram by (i, j), where $(i\ j)$ is the transposition between between x and y; i.e. if $x \lessdot y$ then $\lambda(x, y) = (i, j)$ if $y = (i\ j)x$. (Here $i < j$.) We place the lexographic order on the label set.

To prove that this as an EL-labeling, first show that an increasing chain exists by showing that the lex-earliest chain is increasing. Now consider $x \lessdot y$. The following remarks yield a proof.

Remark 9.1. No number less than i appears in any label in $[x, y]$.

Remark 9.2. $x^{-1}(i) < y^{-1}(i)$.

Remark 9.3. In an increasing chain, the first label contains i.

Remark 9.4. If $\lambda(x, \pi_1) = (i, k)$, where π_1 is the second permutation in an increasing chin from x to y, then $x^{-1}(k) \le y^{-1}(i)$.

Remark 9.5. Let j be the smallest number such that $i < j$ and $x^{-1}(i) < x^{-1}(j) \le y^{-1}(i)$. The first label on an increasing chain is (i, j).

Let's prove the last remark. Proceed by contradiction. Then the label is of the form (i, k) for some $k \ne j$. Then $x^{-1}(i) < x^{-1}(k)$ (because $i < k$, and going upwards in a chain increases inversions). Furthermore, $x^{-1}(k) \le y^{-1}(i)$ by the fourth remark. Thus $j < k$.

Then it is not possible that $x^{-1}(j) < x^{-1}(k)$, because $\pi_1 = (i\ k)x$ has at least two more inversions than x. Therefore, $x^{-1}(k) < x^{-1}(j)$. So $\pi_1 = (i\ k)x$ has k, i, j in that order. Then at some point we must swap i and j, contradicting the fact that the chain is increasing, as (i, j) now appears after (i, k) in our chain.

These remarks imply that there is at most one increasing chain, as desird.

9.2 Grassmanian

Definition 9.6. The *Grassmanian* $\mathrm{Gr}_{k,n}$ is the set of k-dimensional subspaces of an n-dimensional vector space ($\mathbb{R}^n$ or $\mathbb{C}^n$.)

We can represent each $A \in \mathrm{Gr}_{k,n}$ by full rank $k \times n$ matrices. Two such matrices M_1 and M_2 are equivalent if they span some subspace, so we can think of $\mathrm{Gr}_{k,n}$ as

$$\mathrm{Gr}_{k,n} = \{\text{full rank } k \times n \text{ matrices}\} / \sim$$

where $\sim$ identifies these equivalent matrices.

What kind of observations preserve $\sim$?

1. Scale a given row
2. Permute rows
3. Add a row to another.

Hence each element of $\mathrm{Gr}_{k,n}$ is equivalent to a unique matrix in reduced row echelon form.

The subset of $\mathrm{Gr}_{k,n}$ represented by row-echelon matrices with $*$'s in exactly those positions is homeomorphic is $\mathbb{R}^{\#\text{ of stars}}$, which is a cell. This cell is called the Schubert cell.

$$\begin{pmatrix} 0 & 1 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{pmatrix}$$

AS a consequence, $\mathrm{Gr}_{k,n}$ is a disjoint union of its Schubert cells. Each Schubert cell corresponds to a Young diagram Ω formed by consider the $*$'s, which is bounded in a $k \times (n-k)$ box.

More explicitly, let e_j be the vector with a 1 in the jth position and 0's everywhere else. Then fix the "complete flag"

$$0 = V_0 \subset V_{\subset} \cdots \subset V_n = \mathbb{R}^n$$

where $V_i = \langle e_n, e_{n-1}, \dots, e_{n-i+1} \rangle$.

Example 9.7. In the above array, $k = 4$, $n = 9$, $v_1 = \langle e_9 \rangle$, $v_2 = \langle e_9, e_8 \rangle$, and so on. Our partition is $\lambda = (4, 3, 3, 1)$. Let $W \in \mathrm{Gr}_{k,n}$ be represented by a such a matrix. It is easy to check that

$$\begin{aligned} \dim(W \cap V_1) &= 0 \\ \dim(W \cap V_2) &= 1 \\ \dim(W \cap V_3) &= 1 \\ \dim(W \cap V_4) &= 1 \\ \dim(W \cap V_5) &= 2 \\ \dim(W \cap V_6) &= 3 \\ \dim(W \cap V_7) &= 3 \\ \dim(W \cap V_8) &= 4 \\ \dim(W \cap V_9) &= 4 \end{aligned}$$

Since W is a four-dimensional space, four is the maximum possible dimension.

The important point is that rank jumps can be expressed in terms of λ. Specifically, the rank jumps occur at $\lambda n - k - i + i$.

We can use this to write down an alternative definition of the Schubert cell in terms of λ. We can encode this in symbols by

$$\Omega_\lambda = \{W \in \mathrm{Gr}_{k,n} \mid \dim(W \cap V_j) = i \text{ if } \lambda_{n-k-i} + i \le j \le \lambda_{n-k-i-1} + i\}.$$

Definition 9.8. The *Schubert variety* is the set

$$X_\lambda = \left\{W \in \mathrm{Gr}_{k,n} \mid \dim(W \cap V_{\lambda_{n-k-i}+i}) \ge i \text{ for } 1 \le i \le k\right\}.$$

Proposition 9.9. *For all partitions λ contained in an $k \times (n-k)$ rectangle,*

(i) $\Omega_\lambda \cong \mathbb{R}^{|\lambda|}$

(ii) $X_\lambda = \overline{\Omega_\lambda} = \bigsqcup_{\mu \subseteq \lambda} \Omega_\mu$

(iii) $X_\mu \subset X_\lambda$ *if and only if* $\mu \subset \lambda$.

So the face poset of this cell decomposition of $\mathrm{Gr}_{k,n}$ is the poset of partitions of a $k \times (n-k)$ rectangle, ordered by containment of Young diagrams.

Proof. The first statement is trivial / already proved.

For the second statement, fix some $W \in \Omega_\mu \subset \mathrm{Gr}_{k,n}$. We first prove $X_\mu = \bigsqcup_{\mu \subseteq \lambda} \Omega_\mu$. Then the sequence of dimensions of $W\ capV_i$ runs from 0 to k, increasing at each step by 0 or 1. There exist k jumps at steps $\mu_{n-k-i}+i$. For $W \in X_\lambda \Leftrightarrow \dim(W \cap V_{\lambda_{n-k-i}+i}) \ge i$, then the first i jumps have taken place before $\lambda_{n-k-i}+i$, That is, $\lambda_{n-k-i}+i \ge \mu_{n-k-i}+i$. This implies $\mu \subseteq \lambda$ and so we derive $X_\lambda = \bigsqcup_{\mu \subseteq \lambda} \Omega_\mu$ immediately.

To see $X_\lambda = \overline{\Omega_\lambda}$, we use the row echelon form for Ω_λ. We write elements of Ω_λ non-uniquely as

$$\begin{pmatrix} 0 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \end{pmatrix}$$

so the leftmost $*$ in each row must be nonzero. Now, we just vary the $*$'s continuously! When we vary $*$'s to be zero, we can obtain any matrix of type Ω_μ where $\mu \subseteq \lambda$. Thus, $\Omega_\mu \subset \overline{\Omega_\lambda} \forall \mu \subseteq \lambda$.

As a result we derive $\Omega_\lambda \subseteq X_\lambda \subseteq \Omega_\lambda$, but X_λ is a closed for some reason about minors. That forces $X_\lambda = \overline{\Omega_\lambda}$.

The third part was already done above. □

9.3 Complete Flag Varieties

Let F_n be the set of complete flags in $\mathbb{R}^n$. Here,

Definition 9.10. A *flag* is a chain of subspaces

$$0 = W_0 \subset W_1 \subset \cdots \subset W_n = \mathbb{R}^n$$

where $\dim W_i < \dim W_{i+1}$ for all i. It is *complete* if $\dim W_i = i$ for all i.

We may represent each $W_\bullet = \{W_0 \subset W_1 \subset \cdots \subset W_n\}$ by an $n \times n$ matrix with row vectors $r_1, r_2, \dots, r_n$ where $W_i = \langle r_1, r_2, \dots, r_i \rangle$. Note that we need the span of the first i rows to have full rank.

So we may think about

$$F_n = \{n \times n \text{ matrices with the rank property}\} / \sim$$

where $M_1 \sim M_2$ if $\forall i$, the span of the first i rows is the same.

Allowable operations on matrices:

1. We may scale rows by a nonzero constant.
2. We may add r_i to r_j if $i < j$.

Note that we MAY NOT permute rows.

Now, what is the analog of our "reduced" row echelon form before?

$$\begin{pmatrix} 0 & 0 & 1 & * & * \\ 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 1 & * & * \\ 1 & *0 & * & * & \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & * & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Now the 1's from a permutation matrix. So now we have a $(3,1,5,2,4)$ as a permutation. Also, the demand that the first i rows have dimension i is equivalent to the $n \times n$ matrix has nonzero determinant.

So for $W \in S_n$, let Ω_w denote the set of all elements in the complete flag variety representable as above, where the positions of 1's from the permutation w. Here $*$ takes any in $\mathbb{R}$, so

$$\Omega_w \cong \mathbb{R}^{\# of *'s}.$$

As before, we may write

$$F_n = \coprod_{w \in S_n} \Omega_w.$$

Unfortunately, Ω_w is *also* referred to as a Schubert cell.

Exercise 9.11. What is the number of $*$'s in terms of w?

It is $\binom{n}{2} - \#\mathrm{inv}(w)$. Thus,

$$\Omega_w \cong \mathbb{R}^{\binom{n}{2} - \ell(w)}.$$

Proposition 9.12. *For all $w \in S_n$,*

1. $\Omega_w \cong \mathbb{R}^{\binom{n}{2} - \ell(w)}$
2. $X_\omega = \overline{\Omega_w} = \coprod_{v \ge w} \Omega_v$*, where $v \ge w$ is in the Bruhat order.*
3. $X_\omega \subset X_v$ *if and only if $w \ge v$ in the Bruhat order.*

So the face poset of the cell decomposition of F_n is opposite the poset of the Bruhat order.

Unfortunately, this CW complex is in fact not regular, so this does not answer Björner's question.

10 October 1, 2013

Today: matroids!

10.1 Matroids

There are several (cryptomorphic) ways to define matroids.

Definition 10.1. Let E be a finite set, and let $\mathcal{I}$ be a collection of subsets of E such that

(I1) $\varnothing \in \mathcal{I}$

(I2) $\mathcal{I}$ is downwards closed: if $I \in \mathcal{I}$ and $J \subset I$ then $J \in \mathcal{I}$.

(I3) If $X, Y \in \mathcal{I}$ and $|X| = |Y| + 1$, then $\exists x \in X - Y$ such that $Y \cup \{x\} \in \mathcal{I}$.

Then $(E, \mathcal{I})$ is a *matroid*. E is called the *ground set*, and $\mathcal{I}$ contains the *independent sets*.

Example 10.2. Let $E = \{1, 2\}$. The exhaustive list of possible $\mathcal{I}$'s is

- $\varnothing$
- $\varnothing$ and $\{1\}$.
- $\varnothing$ and $\{2\}$.
- $\varnothing$, $\{1\}$ and $\{2\}$.
- The power set of $\{1, 2\}$.

Definition 10.3. Two matroids are isomorphic if there is a bijection between their ground sets preserving independent sets.

10.2 Representable Matroids

Proposition 10.4. *Let A be a matrix over a field $\mathbb{F}$. Let E be the set of columns of A, and let $\mathcal{I}$ be the collection of subsets I of E such that the corresponding collection of column vectors are independent. Then $(E, \mathcal{I})$ is a matroid, denoted $M[A]$.*

Proof. The first two properties are obvious. For the second property, let $|X| = |Y| + 1$ for linearly independent subsets of columns X and$Y$, and let W be the subspace spanned by vectors in $X \cup Y$. Then $\dim W \geq |X|$. Now if $Y \cup \{x\}$ is linearly dependent for each $x \in X - Y$, then Y is a spanning set for W, contradiction. Hence $|X| \geq |Y|$. $\square$

Definition 10.5. A matroid $M[A]$ is called *representable* or *realizable*.

"Is the Missing Axiom of Matroid Theory Lost Forever?"

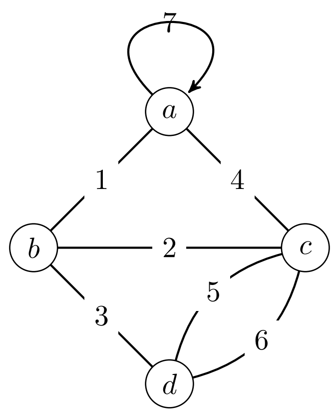

Figure 6: A finite graph G.

10.3 Graphical Matroids

Consider a graph G. Let E be the set of edges of G, and let $\mathcal{I}$ be the collection of all subsets I of edges such that I does not contain the edges of any cycle in G.

Example 10.6. In the figure, some of the cycles of G are 7, 124, 235, 56.

We show that this is a matroid, called the graphical matroid $M(G)$. One way to prove this is to verify all the axioms directly, this is a homework exercise.

Alternatively, we will show this is representable.

Definition 10.7. The incidence matrix of a garph is a $|V|\times|E|$ matrix $A = (A_{v,e})_{v\in V, e\in E}$ by

$$A_{v,e} = \begin{cases} 1 & \text{if } v \text{ and } e \text{ are incident, then } v \in e \text{ and } e \text{ is not a loop} \\ 0 & \text{otherwise} \end{cases}.$$

We like to think of this modulo two.

Example 10.8. For the graph in our notes:

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

Proposition 10.9. *If A is the incidence matrix of a finite graph G, then $M(G) \cong M[A]$ over $\mathbb{F}_2$.*

Proof. We wish to show that for any subset $I \subset E$, I contains cycles of G iff it corresponds to columns of A which are independent. It suffices to show that C is a minimal (by inclusion) cycle of G if and only if the corresponding columns are minimally dependent (again by inclusion).

If C is a loop, the corresponding column is the null vector. Otherwise, each vertex met by C is met by exactly two edges of C. It follows that the sum of the vectors is zero modulo two, hence the vectors are dependent. Conversely, let D be a minimally dependent collection of columns; if D is the zero column then the corresponding edge is a loop. Else the sum of the vectors in D is 0 (since we are working with $\mathbb{F}_2$) (since we are working with $\mathbb{F}_2$). Hence there exist two 1's in the ith position for each i; hence the corresponding edges form a cycle. $\square$

10.4 More Definitions

Definition 10.10. A subset of E which is not independent is called *dependent*.

Definition 10.11. A maximal (by inclusion) independent set is called a *basis*.

Definition 10.12. A minimal (by inclusion) dependent set is called a *circuit*.

Definition 10.13. Given a matroid on E, we define the *rank function* $r : 2^E \to \{0, 1, 2, \dots\}$ by setting $r(s)$ to the largest independent set contained in S.

Now we provide several other ways to define matroids.

Theorem 10.14. *Let $\mathcal{B}$ be a set of subsets of a finite set E. Then $\mathcal{B}$ is a collection of bases of a matroid on E if*

(B1) $\mathcal{B} \neq \varnothing$

(B2) $\forall B_1, B_2 \in \mathcal{B}$ *and* $x \in B_1 - B_2$ *then* $\exists y \in B_2 - B_1$ *such that* $(B_1 \cup \{y\}) - \{x\} \in \mathcal{B}$ *(basis exchange).*

Because B completely characterizes $\mathcal{I}$, then for any finite set E and $\mathcal{B}$ satisfying the conditions of the above theorem, we can construct a matroid with ground set E and $\mathcal{B}$ are the bases.

We can give a similar definition for circuits.

Definition 10.15. If E is a finite set and $\mathcal{C}$ is a collection of subsets of E called circuits. Then $(E, \mathcal{C})$ is a *matroid* if

(C1) $\varnothing \notin \mathcal{C}$,[17]

(C2) If $C \in \mathcal{C}$, and D is a strict subset of C, then $D \notin \mathcal{C}$, and

(C3) IF $C_1, C_2 \in \mathcal{C}$ are distinct, and $x \in C_1 \cap C_2$, then $(C_1 \cup C_2) - \{x\}$ contains a member of $\mathcal{C}$.

One last analogous definition using the rank function.

Definition 10.16. Let E be a finite set, and let $r : 2^E \to \{0, 1, 2, \dots\}$. Then (E, r) is called a matroid with *rank function* r if

(R1) $0 \leq r(x) \leq |X|$.

(R2) $X \subseteq Y \Rightarrow r(X) \leq r(Y)$.

(R3) $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$.

Remark 10.17. There is a notion of an orientable matroid, based on totally ordering E. There is a conjecture that "positively" orientable matroids are all representable. This was proved extremely recently (i.e. within the last week).

[17] I think I had a typo here earlier where it read $\varnothing \in \mathcal{C}$, but this is clearly absurd.

10.5 Proof of the Basis

Remark 10.18. Most of the proofs are elementary but tedious. There are about 100 different definitions.

Proposition 10.19. *All bases of a matroid have the same the same size.*

Proof. Suppose on the contrary that $B_1, B_2 \in \mathcal{B}$ with $|B_1| < |B_2|$. Then $I3$ implies that $\exists b \in B_2$ such that $B_1 \cup b$ is independent, contradiction. $\square$

Proof of the Basis Definition. First, we do the (easy) forwards direction. (B1) follows from (I1). For (B2), apply (I3) to $B_1 - x$ and B_2. Evidently $B_1 - x + \{y\}$ is independent. It must then be a basis, because all the bases have the same size.

For the other direction, suppose that $\mathcal{B}$ satisfies (B1) and (B2). First, we prove that all elements of B have the same size (this is not immediate since we don't have a matroid structure yet). Take $B \in \mathcal{B}$ with $|B|$ minimal. We claim that each other $A \in \mathcal{B}$ has $|A| = |B|$.

We proceed by induction on $|A - B|$. If $|A - B| = 0$, then $A \subseteq B$, but $|B|$ minimal forces $|A| = |B|$. Now suppose $|A - B| = k \geq 1$. Let $a \in A - B$. Axiom (B2) implies we can find $b \in B - A$ with $A - a \cup b \in \mathcal{B}$. Note that the size of $|(A - a \cup b) - B| = |A - B| - 1$; this is obvious. By the inductive hypothesis, we find $|A - a \cup b| = |B| \Rightarrow |A| = |B|$.

Now define

$$\mathcal{I} = \bigcup_{B \in \mathcal{B}} 2^B.$$

We claim that $(E, \mathcal{I})$ is a matroid now.

(I1) is obvious from (B1). (I2) is also straightforward by construction. For (I3), we wish to show that given $I, J \in \mathcal{I}$ with $|I| < |J|$, then $\exists j \in J - I$ with $I \cup j \in \mathcal{I}$. We proceed by induction on $r - |J|$, where r is the common size of the elements of $\mathcal{B}$.

Base case: If $r - |J| = 0$, then $J \in \mathcal{B}$. Choose $A \in \mathcal{B}$ containing I, and let $C = A \cap J - I$ and $B = A - (J \cup I)$. Because $|J| > |I|$ and $J \in \mathcal{B}$, it must be the case that $I \neq A$. Thus, either C or B is nonempty.

If $C \neq \varnothing$, let $a \in C$. Then $I \cup a \in \mathcal{I}$ and $a = J - I$. If $B \neq \varnothing$, then $\exists j \in J - A$ with $A - a \cup j \in \mathcal{B}$. Thus by basis exchange, $I \cup j \subset A - a \cup j \Rightarrow I \cup j \in \mathcal{I}$ and $j \in J - I$. This proves the inductive step.

Now assume $r - |J| = k - 1$; we wish to show the result is true for $r - |J| = k$. Consider $I, J \in \mathcal{I}$, where $|I| < |J|$, with $r - |J| = k$. We may select $A, B \in \mathcal{B}$ such that $I \subset A$ and $J \subset B$. By our prior base case, (I3) is true when $J = B$ is a basis, so we can find $b \in B - I$ such that $I \cup b \in \mathcal{I}$. We may assume Murphy's Law and thus $b \notin J$ as well, otherwise we area already done. Now $I \cup b \in\in \mathcal{I}$, and $b \notin J \Rightarrow J \cup b \subset B$. Therefore $I \cup b, J \cup b \in \mathcal{I}$, and $|I \cup b| < |J \cup b|$ and $r - |J \cup b| = k - 1$. But by induction, we can find $c \in (J \cup b) - (I \cup b) = J - I$ with $I \cup b \cup c \in \mathcal{I}$, then $c \in J - I$ and $I \cup c \in \mathcal{I}$ as desired.

$\square$

11 October 3, 2013

11.1 Polytopes

There are two ways to describe polytopes. First is using vertices.

Definition 11.1. A *polytope* is the convex hull of a finite set of points $V = \{v_1, V_2, \dots, v_n\}$ in $\mathbb{R}^d$. In other words,

$$P = \left\{ \sum_{i=1}^{n} a_i v_i \mid a_i \geq 0, \sum_{i=1}^{n} a_i = 1 \right\}.$$

Example 11.2. If $V = \{\pm e_1, \pm e_2, \pm e_3\}$ then P is an octahedron.

The other way is via inequalities.

Definition 11.3. A polytope as a bounded intersection of half-spaces defined by linear inequalities, as

$$P = \left\{ x \in \mathbb{R}^d \mid a_i \cdot x \leq z_i \forall i = 1, 2, \dots, m \right\}$$

where each $a_i \in \mathbb{R}^d$, $z \in \mathbb{R}$. Here, $a \cdot x \overset{\text{def}}{=} \sum a_i x_i$ is the usual dot product.

Example 11.4. The octahedron has eight facets $\pm x_1 \pm x_2 \pm x_3 \leq 1$.

Now we can define a ace of a polytope.

Definition 11.5. For $w \in \mathbb{R}^d$, we define the face

$$F_w = \{x \in P \mid w \cdot x \text{ is maximal}\}.$$

Note that this may consist of just a single point.

Example 11.6. The face with vertices $e_1, e_2, -e_3$ equals

$$F_{(1,1,-1)} = \{(x_1, x_2, x_3) \in P \mid (1, 1, -1) \cdot (x_1, x_2, x_3) \text{ is maximal}\}.$$

Naturally we may use $(a, a, -a)$ in place of $(1, 1, -1)$ for each $a > 0$.

Example 11.7. The face with vertices $\{e_1, e_2\}$ equals $F_{(1,1,0)}$, or in general, $F_{(u,u,v)}$ for any $u > v \geq 0$.

Example 11.8. The face which is $\{e_3\}$ is $F_{(0,0,-1)}$.

11.2 Matroids

Recall as last time that if $\mathcal{B}$ is a nonempty collection of subsets of a finite set E, then $\mathcal{B}$ is a collection of bases of a matroid if (and only if) it satisfies basis exchange.

Definition 11.9. The matroid polytope P_M of M, is defined as the convex hull in $\mathbb{R}^{|E|}$ of

$$\{e_{b_1} + \dots + e_{b_r} \mid \{b_1, \dots, b_r\} \in \mathcal{B}\}.$$

In the following examples, let $E = \{1, 2, 3\}$.

Example 11.10. If $B = \{\{1\}, \{2\}, \{3\}\}$, then we get a triangle with vertices e_1, e_2, e_3.

Example 11.11. Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ and consider $M(A)$. It has basis $\mathcal{B} = \{12, 23, 31\}$.[18] Then we get a triangle with vertices $e_1 + e_2$, $e_2 + e_3$, $e_3 + e_1$.

Example 11.12. If $\mathcal{B} = \{12, 13\}$ then we get a line segment joining $e_1 + e_3$ and $e_1 + e_2$. You can get this from a graphical matroid.

Remark 11.13. If a matroid M on E with $|E| = n$ has rank r, then $\dim P_M \le r - 1$. The reason is that P_M lies in the affine subspace

$$x_1 + x_2 + \cdots + x_n = r$$

because each of the vertices do.

Example 11.14. Let $M = (E, \mathcal{B})$, where $E = \{1, 2, 3, 4\}$ and $\mathcal{B} = \{12, 13, 14, 23, 24\}$. We get a "square pyramid" whose square base has vertices $13, 23, 14, 24$ and whose apex is 12.

Recall that given $M = (E, \mathcal{I})$, the rank function is $r : 2^E \to \{0, 1, \dots\}$ defined by setting $r(S)$ to be the largest independent set contained in S.

Theorem 11.15. *For any M, P_M is the set of all points $x \in \mathbb{R}^{|E|}$ such that*

(i) $x_i \ge 0$ *for each* $i \in E$

(ii) $\sum_{i \in S} x_i \le r(S)$ *for every subset* $S \subseteq E$.

(iii) Equality holds above when $S = E$.

Proof. We've checked the first and last already. It suffices to show the result for each vertex v_A of P_M, but

$$\sum_{i \in S} (v_A)_i = \#(S \cap A) \le \text{rank } S$$

where we use the fact that A is independent. □

11.3 Symmetric Basis Exchange

Proposition 11.16 (Symmetric Basis Exchange). *If $\mathcal{B}$ is a collection of bases of a matroid on E, then for any $B_1, B_2 \in \mathcal{B}$, if $b_1 \in B_1 - B_2$ then $\exists b_2 \in B_2 - B_1$ such that*

$$B_1 - b_1 \cup b_2, B_2 - b_2 \cup b_1 \in \mathcal{B}.$$

Theorem 11.17 (Gelford Goresky MacPherson Serganova). *Let $\mathcal{B}$ be a collection of k-subsets of $E = \{1, 2, \dots, n\}$. Let $P_{\mathcal{B}}$ be tho convex hull of v_B for each $B \in \mathcal{B}$. Then $(E, \mathcal{B})$ is a matroid if and only if every edge of $P_{\mathcal{B}}$ is parallel translates of $e_i e_j$ (here $i \ne j$). That is, if v_A and v_B are vertices of an edge, then $v_A - v_B = e_i - e_j$ for some $i \ne j$.*

Proof. First, if $w \in \mathbb{R}^n$ and $B \in \mathcal{B}$, we write $w(B)$ to denote $w \cdot v_B$.

Let v_A and v_B form an edge of $P_{\mathcal{B}}$. Then there exists a weight vector w such that A and B are the only w-maximal bases of $\mathcal{B}$. Let $a \in A - B$. Then symmetric exchange gives us a $b \in B - A$ such that $A - a \cup b, B - b \cup a \in \mathcal{B}$. Remark that

$$w(A - a \cup b) + w(B - b \cup a) = w(A) + w(B).$$

[18] A matroid where $B = \binom{E}{k}$ for some k is called a *uniform matroid.*

Because $w(A) = w(B)$ are w-maximal, this forces $A - a \cup b = B$ and $B - b \cup a = A$. This implies

$$v_A - v_B = e_a - e_b$$

as desired.

For the other direction, suppose v_A and v_B are vertices of $P_{\mathcal{B}}$ (not necessarily on an edge). We are given that all edges of $P_{\mathcal{B}}$ have the property that $v_C - v_D = e_i - e_j$ fore some $i \neq j$. Our goal is to show we can do a basis exchange with A and B.

First, we remark we may write $v_B - v_A$ as a positive linear combinations of the edges emanating from v_A (views as vectors). (This is obvious by convexity, because the span of the edges from v_A encompass the entire polytope). Let this span be $\sum \alpha_i E_i$, where $E_1, E_2, \dots$ are the edges emanating from v_A.

Suppose that some E_i appears in this span with $\alpha_i \neq 0$. Recall that $E_i = e_r - e_s$ for some $r \neq s$. **We claim that $s \in A - B$ and $r \in B - A$.** Let v_C be the endpoint of E_i other than v_A. Evidently

$$C = A \cup \{r\} - \{s\}$$

so that $s \in A$ and $r \notin A$. In particular, for all E_i, the negative coordinate of E_i is in A, while the negative coordinate of E_i is not. Now recall that $v_B - v_A$ is a vector in $\{0, \pm 1\}^n$. If $E_i = e_r - e_s$ occurs in the span, then $(v_B - v_A)_r > 0$ by the above statement. (After all, $r \notin A$, so all the contributions must be negative.) Similarly, $(v_B - v_A)_s < 0$. In other words,

$$(v_B - v_A)_r > 0 \Rightarrow r \in B$$
$$(v_B - v_A)_s < 0 \Rightarrow s \notin B$$

This completes the proof of the claim.

We've chosen $A, B \in \mathcal{B}$ and want to perform basis exchange. Choose $a \in A - B$. Then $(v_B - v_A)_a$ is negative. Since $v_B - v_A = \sum \alpha_i E_i$, then there exists E_i in the sum for which $(E_i)_a = -1$. Say

$$v_C - v_A = E_i = e_b - e_a.$$

Then $b \in B - A$ by our claim and $C = A \cup b - a$, so $A \cup b - a \in \mathcal{B}$. $\square$

11.4 Matroids of the Greedy Algorithm

Proposition 11.18. *Choose a matroid $M = (E, \mathcal{I})$ and a function $w : E \to \mathbb{R}$. Suppose we want to find a basis B of minimal weight, where*

$$w(B) \stackrel{def}{=} \sum_{b \in B} w(b).$$

Then the greedy algorithm works:

- *Start with $J \neq \varnothing$.*

- *Add to J a cheapest element $e \in E$ such that $J \cup e \in \mathcal{I}$.*

- *Repeat until we have a basis.*

Remark 11.19. This is related to finding a minimal-weight spanning tree of a connected graph G, where the edges are weighted. Then the bases of $M(G)$ are precisely the spanning trees of G. This implies that the greedy algorithm works.

Proof. Let $I = \{i_1, i_2, \dots, i_n\}$ is a basis with $w(i_1) \le w(i_2) \le \dots \le w(i_n)$; note that the algorithm picks the i's in that order. Consider a minimal-weight basis be $J = \{j_1, \dots, j_n\}$ with $w(j_1) \le w(j_2) \le \dots \le w(j_n)$.

Assume for contradiction that $w(I) > w(J)$, and let k be the smallest index for which $w(i_k) > w(j_k)$ (clearly this must exist). Then $\{i_1, \dots, i_{k-1}\}$ is independent, and $\{j_1, \dots, j_k\}$ is also independent. Then by the exchange axiom, there exists r such that $1 \le r \le k$ such that $i_{1,2}, \dots, i_{k-1}, j_r$ is independent. But

$$w(j_r) \le w(j_k) < w(i_k).$$

So the greedy algorithm should have chosen j_r instead of i_k, contradiction. $\square$

Remark 11.20. There is a very cool converse. If $\mathcal{I}$ is instead an arbitrary simplicial complex, and the greedy algorithm works for every function $w : E \to \mathbb{R}$, then $(E, \mathcal{I})$ is a matroid.

Proposition 11.21. *Pick a matroid $M = (E, \mathcal{B})$ and a function $w : E \to \mathbb{R}$, and let*

$$\mathcal{B}_w = \{B \in \mathcal{B}, w(B) \text{ is minimal}\}.$$

Then $(E, \mathcal{B}_w)$ is a matroid.

Proof. The set $F_w = \{x \in P_M \mid w \cdot x \text{ is minimal}\}$. (We can use minimal instead of maximal just fine). The vertices are precisely $\mathcal{B}_w$. The edges of P_M are translates of $e_i - e_j$, thus so are the edges of F_w. Hence by the converse of GGMS, F_w is a matroid polytope. $\square$

12 October 8, 2013

Today we will discuss operations on matroids.

12.1 A Lot of Definitions

Definition 12.1. Let $M = (E, \mathcal{B}_1)$ and $N = (F, \mathcal{B}_2)$. Then the *direct sum*, denoted $M \oplus N$, is the matroid with ground set $E \sqcup F$ and whose basis is the set

$$\{B_1 \sqcup B_2 \mid B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2\}.$$

Example 12.2. For graphs G_1 and G_2, the direct sum of their graphical matroids is just the graphical matroid on their disjoint union:

$$M(G_1) \oplus M(G_2) = M(G_1 \sqcup G_2).$$

Definition 12.3. Let $M = (E, \mathcal{B})$. The *orthogonal (dual)* matroid $M^* = (E, \mathcal{B}^*)$ is the matroid on E whose bases are the complements; i.e.

$$\mathcal{B}^* = \{E - B \mid B \in \mathcal{B}\}.$$

Example 12.4. For a graph G, $M(G)$ is planar if and only if G in planar; in that case, $M(G^*)$ is $M(G^*)$, where G^* is the dual graph.

For an example, let $G = K_3$ have edges 1, 2, 3. Then $\mathcal{B} = \{12, 23, 31\}$ and $B^* = \{1, 2, 3\}$.

Definition 12.5. Let $M(E, \mathcal{B})$ be a matroid and select $S \subset E$. Then the *restriction* of M to S, denoted $M|_S$, is the matroid on S whose independent sets are all independent sets of M contained in S; that is

$$\mathcal{B}(M|_S) = \{B \cap S \mid B \in \mathcal{B} \text{ and } |B \cap S| \text{ is maximal among all } B \in \mathcal{B}\}.$$

If $S \sqcup T = E$, we often write $M - T$ for $M|_S$ as well.

Example 12.6. For graphs G, if S is a subset of the edge set E, then

$$M(G)|_S = M(G')$$

where G' is formed by deleting edges in $E - S$ from of G.

Definition 12.7. Let $M = (E, \mathcal{B})$ be a matroid and choose $T \subset E$. The *contraction* of M by T, denoted M/T, is the matroid on $E - T$ whose independent sets are

$$\mathcal{I}(M/T) = \{S \subset E - T \mid S \cup J \in \mathcal{I}(E)\}$$

where J is an arbitrary maximal independent subset of T with respect to M. Equivalently,

$$\mathcal{B}(M/T) = \{B - T \mid B \in \mathcal{B} \text{ and } |B \cap T| \text{ is maximal among all } B \in \mathcal{B}\}.$$

Example 12.8. Let G again be a graph and $T = \{e\}$ be a singleton of any edge. Then $M(G)/T = M(G')$ where G' is formed by contracting the edge e in G.

Definition 12.9. If $M = (E, \mathcal{I})$ is a matroid, then

(i) a *loop* of M is a 1-element dependent set.

(ii) a *coloop* is an element of E which is in every basis.

Example 12.10. In a graphical matroid, loops are loops (edges joining a vertex to itself). Co-loops are bridges/cut-edges – those edges that, if deleted, increase the number of connected components by one.

12.2 Applications to Matroid Polytopes

Recall $P_{\mathcal{B}} = P_M$ is the convex hull of v_B as $B \in \mathcal{B}$, where $v_B = \sum_{b \in B} e_b$.

If M has ground set $\{1, 2, \dots, n\}$ and we have a weight vector $w = (w_1, w_2, \dots, w_n) \in \mathbb{R}^n$, then we can define the weight of a basis by $w(B) = \sum_{b \in B} w_b$. We showed that the greedy algorithm produces a maximal/minimal weight basis.

Proposition 12.11 (Ardila, Kl 2006). *Given M with ground set $\{1, 2, \dots, n\}$ and bases $\mathcal{B}$, any face of $\mathcal{P}_{\mathcal{B}}$ is a matroid a polytope. More specifically, let $w = (w_1, \dots, w_n) \in \mathbb{R}^n$ and consider the flag of sets*

$$\varnothing = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_k = [n]$$

such that $w_a = w_b$ for any a, b in $A_i - A_{i-1}$ and $w_a > w_b$ for $a \in A_i - A_{i-1}$, $b \in A_{i+1} - A_i$.

Then the face of $\mathcal{P}_{\mathcal{B}}$ whose vertices are the w-maximal bases of $\mathcal{B}$ is the matroid polytope of the matroid

$$\bigoplus_{i=1}^{k} (M\ |_{A_i})/A_{i-1}.$$

Proof. To choose a w-maximal basis, the greedy algorithm first picks $r(A_1)$ elements of A_1, then $r(A_2) - r(A_1)$ elements of $A_2 - A_1$ and so on. This algorithm, produces all possible w-maximal bases.

After $r(A_n)$ steps, the greedy algorithm has chosen a basis of A_{i-1}. In the next $r(A_i) - A_(i-1)$ steps, we need to choose elements which, when added to A_{i-1}, form a basis of A_i. The possible choices are precisely the bases of $(M|_{A_i})/A_{i-1}$. □

12.3 Connectedness

Definition 12.12. A matroid which cannot be written as a direct sum of two nonempty matroids is called *connected.*

Example 12.13. This corresponds to the connected notion for graphical matroids.

Fact 12.14. Any matroid can be written as a direct sum of connected matroids, called *connected components.*

Definition 12.15. Let $c(M)$ denote the number of connected components of a matroid.

Proposition 12.16. *For any matroid $M = (E, \mathcal{B})$, we have*

$$\dim \mathcal{P}_{\mathcal{B}} = |E| - c(M).$$

Proof for one direction. Set $k = c(M)$ and suppose $M = M_1 \oplus \cdots \oplus M_k$ where M_i has rank r_i on ground set E_i, where $E = E_1 \sqcup \cdots \sqcup E_k$.

Note that for each $i = 1, 2, \dots, k$, we have a constraint

$$\sum_{e \in E_i} x_e = r_i$$

in an analogous proposition which he had previously. There are k of these, so we deduce $\dim \mathcal{P}_{\mathcal{B}} \le |E| - k = |E| - c(M)$. □

Proposition 12.17. *Let M be a matroid on E. For two elements $a, b \in E$, set $a \sim b$ whenever there exist bases B_1, B_2 such that $B_2 = B_1 - a \cup b$. Alternatively, we can define $a \sim b$ if there exists a circuit containing a and b.*

Then $\sim$ is an equivalence relation, whose classes are the connected components of M.

12.4 The Tutte Polynomial

This was originally defined for graphs, but we can define it for matroids.

Definition 12.18. Let $M = (E, \mathcal{B})$. Let x, y be two independent variables. Define the function $T_M(x, y)$ as follows:

(i) If $E = \varnothing$ then $T_M(x, y) = 1$.

(ii) If $e \in E$ is not a loop or coloop, then

$$T_M(x, y) = T_{M-e}(x, y) + T_{M/e}(x, y).$$

(iii) If E consists only of i coloops and j loops, then

$$T_M(x, y) = x^i y^j.$$

The recurrence is known as the *deletion-contraction recurrence.*

Note that this is not obviously well-defined. However, the following theorem will demonstrate that it is.

Theorem 12.19. *If $M = (E, \mathcal{I})$ is a matroid,*

$$T_M(x, y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)}.$$

Remark 12.20. From the first formulation, it is obvious that $T_M(x, y)$ has positive coefficients, but this is not obvious from the theorem.

Proof. Write $R_M(x, y) = \sum_{A \subset E} x^{r(E)-r(A)} y^{|A|-r(A)}$. We wish to show that $R_M(x, y) = T_M(x+1, y+1)$, so it suffices to prove that R_M satisfies the deletion-contraction recurrence.

Let r' and r'' be the rank functions of $M - e$ and M/e respectively.

1. If e is not a coloop, then $r'(E-e) = r(E)$ because we can find a basis not including e. Thus, $r'(A) = r(A)$ for each A in the ground set $E - e$

2. If $e \in A$ is not a loop, then $r''(A - e) = r(A) - 1$; after all

$$\mathcal{B}(M/e) = \{B - e \mid B \in \mathcal{B} \text{ and } |B \cup e| \text{ is maximal}\}.$$

Consider

$$\begin{aligned} R_M(x, y) &= \sum_{A \subset E-e} x^{r(E)-r(A)} y^{|A|-r(A)} + \sum_{A : e \in A} x^{r(E)-r(A)} y^{|A|-r(A)} \\ &= \sum_{A \subset E-e} x^{r'(E-e)-r'(A)} y|A| - r'(A) + \sum_{\substack{A : e \in A \\ B = A-e}} x^{r''(E-e)+1-(r''(B)-1)} y^{|B|+1-(r''(B)+1)} \\ &= R_{M-e}(x, y) + R_{M/e}(x, y). \end{aligned}$$

Then do some other blah for the other parts of the recurrence. $\square$

How is the Tutte polynomial "universal"?

Definition 12.21. A function f from the set of all matroids to $\mathbb{C}$ is a *Tutte-Grothendieck*[19] *invariant* if the following conditions hold for all matroids M:

(i) If $E = \varnothing$, then $f(M) = 1$.

(ii) f assigns the same value A to each coloop, and the same value B to all loops. Furthermore, $f(M) = Af(M - e)$ for each coloop e, and $f(M) = Bf(M - e)$ for each loop e.

(iii) There exist nonzero constants α, β such that when e is neither a loop nor a coloop, then

$$f(M) = \alpha f(M - e) + \beta f(M/e).$$

Theorem 12.22 (Recipe Theorem)**.** *Let f be a Tutte-Grothendieck invariant. Then for all matroids M we have*

$$f(M) = \alpha^{|E|-r(E)}\beta^{r(E)}T_M\left(\frac{A}{\beta}, \frac{B}{\alpha}\right).$$

[19] Not sure why Grothendieck is in there...

13 October 10, 2013

13.1 Grassmonians Return

Recall that $\mathrm{Gr}_{dn}(\mathbb{R})$ represents d-planes in $\mathbb{R}^n$.

Definition 13.1. Given $I \in \binom{[n]}{d}$ and $A \in \mathrm{Gr}_{dn}(\mathbb{R})$ a full rank $d \times n$ matrix,

$$\delta_I(A) \stackrel{\text{def}}{=} \det \text{ of } d \times d \text{ submatrix of } A \text{ located in columns } I$$

Definition 13.2. Let $(\mathrm{Gr}_{dn})_{\geq 0}$ be the subset of $\mathrm{Gr}_{dn}(\mathbb{R})$ such that $\Delta_I(A) \geq 0$ for all I.

This is closely related to total positivity, cluster algebras, and scattering amplitudes.

Definition 13.3. For each $\mathcal{B} \subset \binom{[n]}{d}$, we define

$$S_{\mathcal{B}} = \{A \in \mathrm{Gr}_{dn}\mathbb{R} \mid \Delta_I(A) \neq 0 \Leftrightarrow I \in \mathcal{B}\}.$$

Very often this set is empty. In fact, $S_{\mathcal{B}} \neq \varnothing$ precisely when $\mathcal{B}$ is a collection of bases of matroids realizable over $\mathbb{R}$. This gives us a *matroid stratification*

$$\mathrm{Gr}_{dn}(\mathbb{R}) = \bigsqcup_{\mathcal{B} \subseteq \binom{[n]}{d}} S_{\mathcal{B}}.$$

These do not necessarily have a nice topology. In fact, we have

Theorem 13.4 (Mnev's Universality Theorem)**.** *The topology of $S_{\mathcal{B}}$ can be "as bad" as that of any algebraic variety.*

This led to a paper of Ravi Vakil entitled "Murphy's Law in Algebraic Geometry".

On the other hand, things become much nicer when we consider the nonnegative Grassmonian.

Definition 13.5. Let $S_{\mathcal{B}}^{\geq 0} = S_{\mathcal{B}} \cap (\mathrm{Gr}_{dn})_{\geq 0}$.

Theorem 13.6 (Postnikov)**.** *$S_{\mathcal{B}}^{\geq 0}$ is either empty or a topological cell.*

As a consequence, $(\mathrm{Gr}_{dn})_{\geq 0}$ is a disjoint union of cells.

13.2 Positroids

Definition 13.7. A matroid which has the form $M[A]$ for $A \in (Gr_{dn})_{\geq 0}$ is called a *positroid.*

Remark 13.8. By definition a positroid is realizable.

Example 13.9. Let

$$A = \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

One can check that $\Delta_{ij} \geq 0$ for all $i < j$, so this gives rise to a positroid.

Consider the illustration of the positroid in real space. It illustrates the following.

Proposition 13.10. *Let A be a matrix with column vectors $v_1, v_2, \dots, v_n$. Then $A \in (Gr_{dn})_{\geq 0}$ if and only if*

- *The vectors are cyclically ordered*

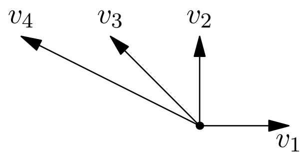

Figure 7: A positroid comes to life.

- *All vectors are on one side of the half-space with v_1 and $-v_1$.*

Consider the Plucker coordinates. For any $A \in GR_{dn}(\mathbb{R})$ and $1 \le i < j < k < \ell \le n$ and S a $(d-2)$ subset of $[n]$ disjoint from i, j, k, ℓ, we have

$$\Delta_{ikS}(A)\Delta_{j\ell S}(A) = \Delta_{ijS}(A)\Delta_{k\ell S}(A) + \Delta_{i\ell S}(A)\Delta_{jkS}(A).$$

In particular,

$$\Delta_{13}(A)\Delta_{24}(A) = \Delta_{12}(A)\Delta_{34}(A) + \Delta_{14}(A)\Delta_{23}(A).$$

Example 13.11. The matroid on $\{1, 2, 3, 4\}$ with bases $\{12, 13, 14, 23, 34\}$ is realizable over $\mathbb{R}$, but is not a positroid. Applying the Plucker relation would give 0 as the sum of two positive things.

13.3 Grassmann Necklace

Positroids are in bijection with the Grassmann necklace.

Definition 13.12. A *Grassmann necklace* of type (d, n) is a sequence

$$(I_1, \dots, I_n)$$

of d-subsets $I_k \in \binom{[n]}{d}$ such that for any $i \in [n]$,

- If $i \in I_i$ then $I_{i+1} = I_i - i \cup j$ for $j \in [n]$. (It is permissible that $i = j$.)
- If $i \notin I_i$ then $I_{i+1} = I_i$, where $I_{n+1} = I_1$.

Definition 13.13. The i-order $<_i$ is a total order

$$i <_i i+1 <_i \cdots <_i n <_i 1 <_i 2 <_i \cdots <_i i-1.$$

Definition 13.14. For any rank d matroid $M = ([n], \mathcal{B})$, let I_k be the lexicographically minimal basis of $\mathcal{M}$ with respect to $<_k$. Define

$$Neck(M) \stackrel{\text{def}}{=} (I_1, \dots, I_n).$$

Proposition 13.15 (Postnikov)**.** *The above $Neck(M)$ is indeed a Grassmann necklace (of type (d, n)) for any matroid M on $\{1, 2, \dots, n\}$ of rank d.*

Proposition 13.16. *When M is a positroid, we can recover M from its Grassmann necklace.*

Let $i \in [n]$. The *Gale order* of $\binom{[n]}{d}$ with respect to $<_i$ is the partial order $\le_i$ defined as follows: for any two d-subsets

$$S = \{s_1 <_i \cdots <_i s_d\} \quad \text{and} \quad T = \{t_1 <_i \cdots <_i t_d\}$$

we write $S \le_i T$ if and only if $s_j \le_i t_j$ for all j. (Here $S, T \subseteq [n]$.)

Theorem 13.17 (Postnikov, Oh)**.** *Let* $Neck = (I_1, \dots I_n)$ *be a Grassmann necklace of type* (d, n)*. Then the collection*

$$\mathcal{B}(Neck) \overset{def}{=} \left\{ B \in \binom{[n]}{d} \mid B \geq_j I_j \forall j \in [n] \right\}$$

is the collection of bases of rank d *positroids*

$$\mathcal{M}(Neck) \overset{def}{=} ([n], \mathcal{B}(Neck)).$$

Furthermore, for any positroid M*,* $\mathcal{M}(Neck(M)) = M$*.*

Summary: given any matroid, we look at the lexicographically minimal bases with respect to these shifted inequalities. Then this collection of bases is a Grassmann necklace. Many matroids give rise to the same necklace, but for positroids, we get a bijection to necklaces. The theorem of Postnikov and Oh gives the positroid from the necklace.

Corollary 13.18. *Let* M *be a matroid. Then each basis of* M *is also a basis of* $\mathcal{M}(Neck(M))$*.*

Proposition 13.19 (Ardila, Rinon, Williams)**.** *Let* $Neck = (I_1, \dots, I_n)$ *be a Grassmann necklace of type* (d, n) *and let* M *be the corresponding positroid. Then for any* $j \in \{1, 2, \dots, n\}$*, suppose the elements of* I_j *are*

$$a_1^j <_j a_2^j <_j \dots <_j a_d^j.$$

Then the matroid polytope $P_{\mathcal{M}}$ *can be described as follows:*

(i) $x_1 + x_2 + \dots + x_n = d$

(ii) $x_j \geq 0$ *for all* j

(iii) $x_{j+1} + x_{j+2} + \dots + x_{a_k^j - 1} \leq k - 1$ *for all* $j \in [n]$ *and* $k \in [d]$*.*

Here the third inequality is the interesting one. The general matroid polytope is described by the inequality $\sum_{i \in S} x_i \leq r(S)$ for all subsets $S \subseteq [n]$. In particular, we need 2^n inequalities to describe this in general. For this specific case we only need nd inequalities.

The converse also happens to be true: any matroid polytope constructed in the manner described in the proposition arises from a positroid.

Proof. Let P be the polytope described by the above inequalities. We wish to show $P = P_M$.

First we claim that the vertices of P are zero-one vectors. Rewrite the polytope P in terms of the "y-coordinates" defined by

$$y_i = x_1 + x_2 + \dots + x_i \quad i = 1, 2, \dots, n - 1.$$

The inequalities now have the form $y_i - y_j \leq a_{ij}$ for some integers a_{ij}. (We need the condition $x_1 + \dots + x_n = d$ here to convert something like $x_n + x_1$ into this form.) Now the matrix with row vectors $e_i - e_j$ is "totally unimodular" meaning that minors of submatrices are 0 or ± 1.

Then a classical result of Schrijver implies the vertices are zero-one vectors in y-coordinates, so the vertices have integer x-coordinates. It follows using (ii) and (iii) that the vertices have zero-one x-coordinates as well.

Since both P and P_M have zero-one vertices, it's enough to show they have the same set of zero-one points. For a zero-one vector e_B satisfying (i), the inequalities (3) are equivalent to the inequalities

$$B \geq_j I_j \quad \forall j$$

i.e. to the condition that $B \in \mathcal{B}(Neck)$. $\square$

13.4 A Refinement of the Proposition

Definition 13.20. Given $i, j \in [n]$, the *cyclic interval* $[i, j]$ is defined by

$$[i,j] = \begin{cases} \{i, i+1, \dots, j\} & \text{if } i \leq j \\ \{i, i+1, i+2, \dots, n, 1, 2, \dots, j-1, j\} & \text{otherwise} \end{cases}.$$

Theorem 13.21 (Ardila, Rincon, Williams)**.** *A matroid M of rank d on $[n]$ is a postroid if and only if its matroid polytope P_M is described by $x_1 + \cdots + x_n = d$ and*

$$\sum_{\ell \in [i,j]} x_\ell \leq a_{ij}$$

with $i, j \in [n]$ for some collection $\{a_{ij} \mid i, j \in [n]\} \subset \mathbb{R}$.

Proof. By the previous proposition, all positroid polytopes have the desired form. Now we want to prove the converse. Assume M is a rank d matroid on $[n]$ whose polytope P_M admits a description as in the theorem.

Let $r_{ij} = r_M([i,j])$; i.e. the rank of $[i,j]$ in M.

P_M satisfies the inequality $\sum_{\ell \in [i,j]} x_\ell \leq a_{ij}$, then we claim that $a_{ij} = r_{ij}$. Each vertex and hence each point in P_M satisfies $\sum_{\ell \in [i,j]} x_\ell \leq r_{ij}$ by rank. Equality is achievable; we can find a basis B such that $|B \cap [i,j]| = r_{ij}$. This establishes the claim.

Therefore P_M is described by $x_1 + \cdots + x_n = d$ and $x_i + x_{i+1} + \cdots + x_j \leq r_{ij}$.

Let $Neck = Neck(M) = (I_1, \dots, I_N)$ and let $M' = \mathcal{M}(Neck(M))$. We will prove that $M = M'$ by proving their basis are the same. One direction follows from a prior corollary: $\mathcal{B}(M) \subseteq \mathcal{B}(M')$. Consider $B \in \mathcal{B}(M')$. We'll show e_B satisfies the inequalities, which will complete the proof.

Consider $[i,j]$. Write $I_i = \{a_1 <_i a_2 <_i \cdots <_i a_d\}$ and let $k = |I_i \cap [i,j]|$. The definition of I_i implies that $k = r_{ij}$. Then $i \leq j \leq a_{k+1} - 1$ in cyclic order. The previous proposition implies that the vertex e_B of the positroid satisfies $x_i + x_{i+1} + \cdots + x_j \leq x_i + x_{i+1} + \cdots + x_{a_{k+1}-1} \leq k$. Since $k = r_{ij}$, e_B satisfies $x_i + \cdots + x_j \leq r_{ij} \forall [i,j]$ so e_B satisfies the inequalities of P_M which implies B is a basis of M. $\square$

Next time we will discuss oriented matroids.

14 October 15, 2013

Today: oriented matroids.

14.1 Directed Graphs

Let us first motivate matroids by discussing directed graphs.

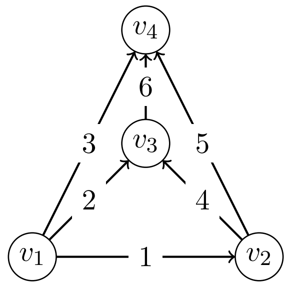

Figure 8: A directed graph.

Consider the set of *simple* cycles, that is, ignoring orientation. Each of these cycles is assigned either clockwise or counterclockwise.

Example 14.1. IN the clockwise cycle 1256, 2 and 6 are *positive* and 1 and 5 are *negative.*

We will write each signed circuit as $X = (X^+, X^-)$.

Remark 14.2. If (X^+, X^-) is a signed circuit, so is (X^-, X^+).

The signed circuits obtained from the digraph D in this way are

$$\mathcal{C} = \left\{ X = (X^+, X^-) : X \text{ a signed circuit of } D \right\}.$$

The pair $(E, \mathcal{C})$, or $\mathcal{M}_D$ or $\mathcal{M}(E)$, is called an *oriented matroid.*

Remark 14.3. If we drop the orientation of D and the signs on the circuits, we obtain the usual graphical matroid.

Definition 14.4. If X is a signed circuit, then $\underline{X}$ denotes the underlying circuit.

14.2 Oriented Matroid

Definition 14.5. $\mathcal{M} = (E, \mathcal{C})$ is an *oriented matroid* if the following axioms hold:

(C0) $\varnothing$ is not a signed circuit.

(C1) If X is a signed circuit, so is $-X$.

(C2) No proper subset of an underlying circuit is a circuit.

(C3) If X_0 and X_1 are signed circuits with $X_1 = -X_0$, and $e \in X_0^+ \cap X_1^-$, then $\exists X \in \mathcal{C}$ such that
$$X^+ \subseteq (X_0^+ \cup X_1^+) - e \quad \text{and} \quad X^- \subseteq (X_0^- \cup X_1^-) - e.$$

So $\mathcal{C}$ consists of objects of the form (X_i^+, X_i^-) where $X_i^+ \cap X_i^- = \varnothing$.
In the graph theoretical view, this is obvious. Just draw a picture.

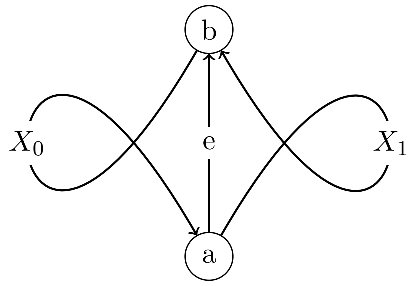

Figure 9: X_0 on left and X_1 on right. What's the third circuit?

14.3 Oriented matroids from vector configurations

Let $E = \{v_1, v-2, \dots, v_n\}$ be a set of vectors spanning some r-dimensional subspace with order, say $\mathbb{R}^r$. A linear dependence looks like

$$\sum_{i=1}^{n} \lambda_i v_i = 0 \quad \lambda_i \in \mathbb{R}.$$

The sets $\underline{X} = \{i \mid \lambda_i \neq 0\}$ corresponding to the minimal linear dependencies are circuits of an unoriented matroid. For the associated oriented matroid, we consider $X = (X^+, X^-)$ where

$$X^+ \overset{\text{def}}{=} \{i : \lambda_i > 0\}.$$

Proposition 14.6. *Let $\mathcal{C}$ be the collection of signed sets defined above. Then $\mathcal{M} = (E, \mathcal{C})$ is an oriented matroid.*

Let us recall the Plucker conditions before the next section.

Recall: Given a full rank $d \times n$ matrix A, we obtain a matroid $M[A]$. For $I \in \binom{[n]}{d}$, recall $\Delta_I(A)$ is the determinant of the $d \times d$ submatrix of A in columns I.

I is a basis of $M[A]$ if and only if $\Delta_I(A) \neq 0$.

The minors satisfy 3-term Plucker relationships: for $1 \le i < j < k < \ell \le n$ and $S \in \binom{[n]}{d-2}$ disjoint from $\{i, j, k, l\}$:

$$\Delta_{i\ell S}(A)\Delta_{j\ell S}(A) = \Delta_{ijS}(A)\Delta_{k\ell S}(A) + \Delta_{i\ell S}(A) + \Delta_{jkS}(A).$$

Compare this to Ptolemy's Theorem! Hyperbolic geometry...

14.4 Basis Axioms for Oriented Matroids

Let $v_1, v_2, \dots, v_n$ be columns of A, a $d \times n$ matrix.

Definition 14.7. We define the *basis orientation*or *chirotope* of A by

$$\chi(i_1, i_2, \dots, i_d) = \operatorname{sgn}\det(v_{i_1}, \dots, v_{i_d}) \in \{\pm 1, 0\}.$$

Example 14.8. Let $A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$.

Then $\chi(123) = 1$, $\chi(134) = -1$ and $\chi(124) = 0$.

Remark 14.9. χ is antisymmetric, e.g. $\chi(134) = -\chi(143)$.

Definition 14.10. Let $d \in \mathbb{Z}^+$ and let E be a finite set. A *chirotope* of rank d on E is a map $\chi : E^d \to \{-1, 0, 1\}$ such that

(B0) χ is not identically zero.

(B1) χ is alternating; that is

$$\chi(v_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(d)}) = \operatorname{sgn} \chi(v_1, \dots, v_d) \quad \forall v_i \in E, \sigma \in S_d.$$

(B2) If i, j, k, ℓ, if $y_3, \dots, y_d \in E$ if

$$\varepsilon \overset{\text{def}}{=} \chi(i, j, y_3, \dots, y_d)\chi(k, \ell, y_3, \dots, y_d) \in \{\pm 1\}$$

then either $\chi(j, k, y_3, \dots, y_d)\chi(i, \ell, y_3, \dots, y_d) = -\varepsilon$ or $\chi(j, \ell, y_3, \dots, y_d)\chi(i, k, y_3, \dots, y_d) = \varepsilon$.

This is unintuitive until we look at the Plucker relation. Now it rewrites as the following: if $\square = \varepsilon + \blacksquare$, either $\square = \varepsilon$ or $\blacksquare = -\varepsilon$. You can check this easily, because "inequalities" $\blacksquare \geq 0$ or $\square \leq 0$.

Well, of course,

Theorem 14.11. *Circuit axioms are equivalent to chirotope axioms.*

Proof. Actually hard. $\square$

Nonetheless, we still describe how to get the signed circuits from the chirotope. Given χ, we get the unsigned bases of $\overline{M}$ by looking at at subsets that χ assigns a nonzero value. From the unsigned bases we obtain the unsigned circuits of $\overline{M}$.

Each unsigned circuit $\underline{C}$ of $\underline{M}$ gives rise to a signed circuit C (up to sign) as follows: If $e, f \in \underline{C}$ are distinct, let $\sigma(e, f) \overset{\text{def}}{=} -\chi(e, S)\chi(f, S) \in \{\pm 1\}$ where (f, S) is any ordered basis of M containing $\underline{C} - e$.

Let $c \in \underline{C}$ and take C to be the signed circuit defined by

$$\begin{aligned} C^+ &= \{c\} \cup \{f \in \underline{C} - c \mid \sigma(c, f) = 1\} \\ C^- &= \{f \in \underline{C} - c \mid \sigma(c, f) = 1\} \end{aligned}$$

The claim that this is invariant under the choice of $c \in C$, up to sign. Again, the proof is not easy.

14.5 Matroid Grassmonian (MacPhersonian)

Names in this field...

Definition 14.12. Suppose $M = (E, \chi)$ and $M' = (E, \chi')$ are two rank d orientable matroids of E. Say M' is a *specialization* of M if $\chi(y_1, \dots, y_d) = \pm\chi'(y_1, \dots, y_d)$ whenever $\chi'(y_1, \dots, y_d) \neq 0$ for a fixed choice of $\pm$. (That is, $\pm$ is fixed for each $\langle y_i \rangle$. In still other words, we identify a chirotope with its negative.)

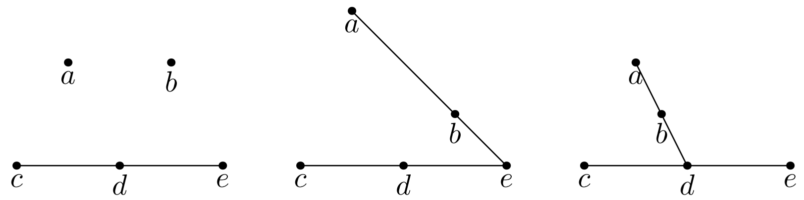

Figure 10: Some oriented matroids in $\mathbb{R}^2$.

We write $M \rightsquigarrow M'$ in this case. This is also called a *weak map* of matroids.

If M comes from a vector configuration this corresponds to putting vectors (or points) into a more special position while preserving orientations.

In the above figure $\mathcal{M}_1 \rightsquigarrow \mathcal{M}_2$, but $|mathcal M_1 \not\rightsquigarrow \mathcal{M}_3$ because $\chi(abe)$ changes.

Definition 14.13. Choose a set S of rank oriented matroids on $[n]$. Use the partial order of specialization, we obtain a topological space by taking the order complex.

Definition 14.14. Let S denote the set of all rank k oriented matroids on $[n]$. The poset is called the *matroid Grassmanian*, also called the MacPhersonian. It is denote $\mathrm{MacP}(k, n)$, and has order complex denoted $\|\mathrm{MacP}(k, n)\|$.

In 1993, MacPherson remarked that for $k \in \{1, 2, n-1, n-2\}$, this order complex is homeomorphic to $\mathrm{Gr}_{kn}(\mathbb{R})$, but otherwise the "topology is a mystery".

Then in 2003, Daniel Biss (MIT) announced that for all k and n, $\mathrm{MacP}(k, n)$ is homotopy equivalent to $\mathrm{Gr}_{kn}(\mathbb{R})$. This was quite exciting, published in the Annals, etc.

Then Mnev, in arXiv:0709.1291, pointed out a fatal flaw in the paper, which Biss had known about for a few years. Biss had used this to get his PhD, start a political campaign, etc.

Biss finally published a retraction in 2009. He is now a successful politician. I don't know what the moral of this story is.

15 October 17, 2013

Recall the definition of a *chirotope*. Recall that χ and $-\chi$ are considered equivalent.

Recall that given a $d \times n$ real matrix A (regarded as a sequence of columns $v_1, \dots, v_d$), we get a chirotope by

$$\chi(v_{i_1}, \dots, v_{i_d}) = \operatorname{sgn} \det \langle v_{i_1}, \dots, v_{i_d} \rangle \in \{0, \pm 1\}.$$

15.1 Positively Oriented Matroids

Definition 15.1. Let $M = ([n], \chi)$ be an oriented matroid of rank d. We say M is *positively oriented* with respect to the total order $1 < 2 < \cdots < n$ if $\chi(i, i_2, \dots, i_d) \ge 0$ whenever $i_1 < i_2 < \cdots < i_d$.

These were introduced and studied by Ilda de Silva in 1987. We abbreviate these as POM.

Now recall that the definition of a *positroid*: let A be a full rank $d \times n$ real matrix with nonnegative Plücker coordinates $\Delta_I(A) \ge 0$ for all $I \in \binom{[n]}{d}$. Equivalently A represents elements of $(\mathrm{Gr}_{dn})_{\ge 0}$. Then $M[A]$ is called a positroid, and the bases are

$$\mathcal{B}(M[A]) = \left\{ I \in \binom{[n]}{d} : \Delta_I(A) > 0 \right\}.$$

Evidently positroids are realizable by definition. On the other hand, POM's are not obviously realizable. Not too surprisingly,

Conjecture 15.2 (DeSilva 1987)**.** *POM's are realizable; i.e. they come from positroids.*

Theorem 15.3 (October 15, arXiv:1310.4159; Ardila-Rinon-Williams)**.** *The conjecture is true.*

15.2 Sketch of Proof

Recall the following statement, which is an exercise rediscovered by Ardila-Rinon. (See lecture 12.)

> Given M with ground set $\{1, 2, \dots, n\}$ and bases $\mathcal{B}$, any face of $\mathcal{P}_{\mathcal{B}}$ is a matroid a polytope. More specifically, let $w = (w_1, \dots, w_n) \in \mathbb{R}^n$ and consider the flag of sets
>
> $$\varnothing = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_k = [n]$$
>
> such that $w_a = w_b$ for any a, b in $A_i - A_{i-1}$ and $w_a > w_b$ for $a \in A_i - A_{i-1}$, $b \in A_{i+1} - A_i$.
>
> Then the face of $\mathcal{P}_{\mathcal{B}}$ whose vertices are the w-maximal bases of $\mathcal{B}$ is the matroid polytope of the matroid
>
> $$\bigoplus_{i=1}^{k} (M\mid_{A_i})/A_{i-1}.$$

Recall also

> A matroid M of rank d on $[n]$ is a postroid if and only if its matroid polytope P_M is described by $x_1 + \cdots + x_n = d$ and
>
> $$\sum_{\ell \in [i,j]} x_\ell \le a_{ij}$$
>
> with $i, j \in [n]$ for some collection $\{a_{ij} \mid i, j \in [n]\} \subset \mathbb{R}$.

Let us sketch the proof. We discuss only the case where $\mathcal{M}$ is connected, i.e. it cannot be written as a direct sum of two nontrivial oriented matroids. In other words, $\mathcal{M} \neq \mathcal{M}_1 \oplus \mathcal{M}_2$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are nonempty oriented matroids.

Lemma 15.4. *If $\mathcal{M}$ is connected, then the underlying unoriented matroid $M == \overline{\mathcal{M}}$ is connected as well.*

The converse is trivial.

Now recall that if N is a matroid on $[n]$, then $\dim P_N$ is n minus the number of connected components of N. In particular, if M is connected then $\dim P_M = n - 1$. Now assume for contradiction that M is not a positroid. Then M has a facet F whose equation is

$$\sum x_i = r_M(S)$$

where $S \subset [n]$ is not a cyclic interval.

F is a matroid polytope of a matroid with two connected components. By the proposition, F is defined by a vector $W = (w_1, \dots, w_n) \in \{0,1\}^n$. Then we find that F is a matroid polytope of the matroid with $M|_S \oplus M/S$. Because it has two connected components, $M|_S$ and M/S are both connected.

The vertices of F are bases which are w-maximal for $W = (w_1, \dots, w_n)$ with $w_i = 1$ if and only if $i \in S$. We want to produce a contradiction with the Plücker coordinates. If S is not a cyclic interval, then we can find $i < j < k, \ell$, in cyclic order, such that $i, k \in S$ an d$j, \ell \notin S$.

We know $M|_S$ is connected implies bases $A \cup i$ an d$A \cup k$ of $M|_S$ with basis exchange between i and k. Similarly, M/S connected implies bases $B \cup j$ and$B \cup \ell$ with a basis exchange between j and ℓ.

Then, in $M|_S \oplus M/S$, we have bases

$$A \cup B \cup T \text{ for } T \in \{\{i,j\}, \{i,\ell\}, \{j,k\}, \{k,\ell\}\}.$$

But now we claim $A \cup B \cup \{i,k\}$ is *not* a basis of $M|_S \oplus M/S$. For each of these T, the w dot product against these bases gets a contribution of $+1$ from each of T. On the other hand, $A \cup B \cup \{i,k\}$ is not a bases because its w-value is too large; the set $\{i,k\}$ gives a contribution of two.

Denote elements of $A \cup B$ by $y_3 < y_4 < \dots < y_d$. We claim that

$$\chi(i,j,y_3,\dots,y_d)\chi(k,\ell,y_3,\dots,y_d) = \chi(j,k,y_3,\dots,y_d)\chi(i,\ell,y_3,\dots,y_d).$$

This would be sufficient, because the Plücker relation would now read $0 = \bullet + \bullet$, where $\bullet \neq 0$ because $A \cup B \cup T$ are all bases.

Let $\chi(I)$ denote $\chi(i_1, i_2, \dots, i_d)$ where $i_1 < i_2 < \dots < i_d$ and $I = \{i_1, \dots, i_d\}$.

Since $\mathcal{M}$ is a POM, $\chi(I) \geq 0$ for all $I \in \binom{[n]}{d}$. Therefore,

$$\chi(a,b,y_3,\dots,y_d) = (-1)^r \chi(\{a\} \cup \{b\} \cup \{y_3,\dots,y_d\}) = (-1)^r$$

where r is the number of transpositions needed to put $a, b, y_3, \dots, y_d$ in increasing order.

WLOG $1 \leq i < j < k < \ell \leq n$ (otherwise just rebash, although there's also cyclic symmetric). Define

$$\begin{aligned} C_1 &= |(A \cup B) \cap [1, i-1]| \\ C_2 &= |(A \cup B) \cap [i+1 j-1]| \\ C_3 &= |(A \cup B) \cap [j+1, k-1]| \\ C_4 &= |(A \cup B) \cap [k+1, \ell-1]| \end{aligned}$$

Now recall the y_i are already in increasing order (and form $A \cup B$). We can easily compute

$$\chi(i, j, y_3, \dots, y_d) = (-1)^{2C_1 + C_2} = (-1)^{C_2}$$

because we need $C_1 + C_2$ moves to place j in the right position, and C_1 to move i to the correct position afterwards.

Similarly,

$$\begin{aligned}
\chi(k, \ell, y_3, \dots, y_d) &= (-1)^{C_1+C_2+C_3+C_4+C_1+C_2+C_3} = (-1)^{C_4} \\
\chi(j, k, y_3, \dots, y_d) &= (-1)^{C_1+C_2+C_3+C_1+C_2} = (-1)^{C_3} \\
\chi(i, \ell, y_3, \dots, y_d) &= (-1)^{C_1+C_2+C_3+C_4+C_1} = (-1)^{C_2+C_3+C_4}
\end{aligned}$$

15.3 MacPhersonian

Suppose $M = (E, \chi)$ and $M' = (E, \chi')$ are two rank k matroids on E. Recall that M' is a specialization of M if (after replacing χ with $-\chi$ if necessary) we have $\chi(y_1, \dots, y_k) = \chi'(y_1, \dots, y_k)$ whenever $\chi'(y_1, \dots, y_k) \neq 0$.

Then $M \rightsquigarrow M'$. Also called a *weak map*, but meh. A better term is *begets* or something.

Recall that the poset of all rank k OM's on $[n]$ is the MacPhersonian $\mathrm{MacP}(k, n)$ with order complex $\|\mathrm{MacP}(k, n)\|$, whose "topology is a mystery".

Definition 15.5. The positive MacPhersonian $\mathrm{MacP}^+(k, n)$ is a poset of all rank k POM's on $[n]$.

Theorem 15.6. *$\|\mathrm{MacP}^+(K, n)\|$ is homeomorphic to a closed ball.*

Proof. Recall that for each $\mathcal{B} \in \binom{[n]}{d}$,

$$S_{\mathcal{B}}^{>0} \overset{\text{def}}{=} \{A \in (\mathrm{Gr}_{dn}(\mathbb{R}))_{\geq 0} \mid \Delta_I(A) > 0 \Leftrightarrow I \in \mathcal{B}\}.$$

Each of these is nonempty if and only if $\mathcal{B}$ is a collection of bases of some positroid. These are *cells* of the positive Grassmonian.

We have a partial order $S_{\mathcal{B}}^{>0} \subset S_{\mathcal{B}'}^{>0}$ if and only if $B \subset B'$.

It turns out the poset is thin and EL-shellable if it is a face poset of regular CW complex homemorphic to a ball. (Williams 2007, Shelling Totally Nonnegative Flag Varieties.)

In particular, its order complex is homeomorphic to a ball. Now we have a bijection between positroids and POM's.

$\mathrm{MacP}^+(k, n)$ is a face poset of a regular CW complex homeomorphic to a ball, so $\|\mathrm{MacP}^+(k, n)\|$ is homeomorphic to a ball. $\square$

16 October 22, 2013

Polytopes!

16.1 Definitions

Definition 16.1. A *V-polytope* is the convex hull of a finite set of points.

Definition 16.2. An H-polyhedron is the intersection of finitely many closed half-spaces in some $\mathbb{R}^d$. It is called an H-polytope if it is bounded[20].

It turns out that these definitions are equivalent, so we were refer more generally to

Definition 16.3. A polytope is $P \subseteq \mathbb{R}^d$ which is either a V-polytope or (equivalent) an H-polytope.

Proof. Omitted. □

Definition 16.4. $P \subseteq \mathbb{R}^d$ and $Q \subseteq \mathbb{R}^e$ are affinely isomorphic, denoted $P \cong Q$, if there exists an affine map $f : \mathbb{R}^d \to \mathbb{R}^3$ a bijection.

An affine map is of the form $x \mapsto Ax + x_0$, where $x \in \mathbb{R}^d$, $x_0 \in \mathbb{R}^e$ and A is a $d \times e$ matrix.

Definition 16.5. A *face* of a polytope P is a intersection of P with a hyperplane H such that P contained in one of the two half-spaces on either side of H.

Note that $\varnothing$ counts as a face. Vertices are faces of dimension 0, edges are faces of dimension 1 and facets are faces of codimension 1.

16.2 Combinatorics of Polytopes

Now here is a different notion of isomorphism which we will investigate more.

Definition 16.6. P and Q are *combinatorially isomorphic* if there exists a bijection between their faces which preserves the inclusion relation.

What combinatorial information can be associated to polytopes?

- The face poset, i.e. the poset of faces ordered by inclusion.
- The f-vector, $(f_{-1}, f_0, f_1, \dots, f_d)$, where f_i is the number of i-dimensional faces, and $f_{-1} = 1$ (for the empty set).

So combinatorial isomorphism is precisely the isomorphism of face posets.

Proposition 16.7. *Face posets are lattices.*

Proof. The meet operation is the intersection. If F and G are faces of P, we just need to show $F \cap G$ is a face as well. Write

$$F = P \cap \{\underline{c} \cdot \underline{x} = c_0\}$$

and

$$G = P \cap \{\underline{b} \cdot \underline{x} = b_0\}$$

where $\underline{b}, \underline{c}$ are vectors and b_0, c_0 are scalars; further assume $c \cdot x \le c_0$ and $b \cdot x \le b_0$ are inequalities for P.

Then $(\underline{c} + \underline{b}) \cdot \underline{x} \le c_0 + b_0$ is an inequality for P. Then this gives $F \cap G$, because equality occurs here if and only if $\underline{x}$ has equality in both equations.

Hence the face poset of P is a meet-semilattice. Recall that any meet-semilattice with a $\hat{1}$ is a lattice, as desired. □

[20] There is no ray $\{x + ty \mid t \ge 0\}$ with $y \ne 0$

16.3 Examples of Polytopes

Example 16.8 (Simplex)**.** A d-simplex is a convex hull of any $(d+1)$ affinely independent points. The face lattice is the Boolean algebra.

Example 16.9 (Cube)**.** The d-dimensional hypercube is the set $C_d = \left\{x \in \mathbb{R}^d : -1 \le x_i \le 1\right\}$. It is the convex hull of $\{\pm 1\}^d$.

Definition 16.10. A polytope is *simplicial* if all proper faces are simplices (i.e. the facets have the minimal number of vertices given their dimension.)

Definition 16.11. A polytope is simple if each vertex is contained in the minimal number of facets (i.e. the dimension of P).

The above examples are prototypical of simplicial/simple polytopes. But now let's look at somewhat more interesting examples.

16.4 Cyclic Polytopes

Define the *moment curve* in $\mathbb{R}^d$ by $x \to \mathbb{R}^d$ with $t \mapsto x(t) \overset{\text{def}}{=} \left\langle t, t^2, \dots, t^d \right\rangle \in \mathbb{R}^d$.

Definition 16.12. The cyclic polytope $C_d(t_1, \dots, t_n)$ is the convex hull

$$\operatorname{conv}\left\{x(t_1), \dots, x(t_n)\right\}$$

of $n > d$ distinct points $x(t_i)$ if $t_1 < \dots < t_n$ on the moment curve.

Theorem 16.13 (Gale's evenness condition)**.** *Let $n > d \ge 2$. Choose $t_1 < \dots < t_n$. Then $C_d(t_1, \dots, t_n)$ is a simplicial d-polytope. Identify the point $x(t_i)$ with i; then a d-subset $S \subseteq [n]$ of the vertices forms a facet of $C_d(n)$ if and only if for each $i < j$ are not in S, the number of $k \in S$ between i and j is even.*

Proof. In what follows, let $\det 1 \left\langle x(t_0) < x_(t_1), \dots, x(t_d)\right\rangle$ for the Vandermonde determinant $\prod_{0 \le i < j \le d} (t_j - t_i) > 0$. (Aside: note that this matrix is totally positive.by the Vandermonde identity.)

In particular, the nonzero condition implies no $d = 1$ points on the moment curve are affinely dependent. As a result, the polytope is simplicial, and all facets contain exactly d points.

Now consider $S = \{i_1, \dots, i_d\} \subseteq [n]$. The hyperplanes H_s through the points $x(t_{i_s})$ is

$$\{H_s(z \in \mathbb{R}^d : F_s(z) = 0\}$$

where $F_s(z) = \det 1 \langle z, x(t_1), \dots, x(t_d)\rangle$. This is indeed a plane through the points because the determinant vanishes on all the $x(t_{i_s})$. Then H_s determines a face if the sign of $F_s(z)$ is the same for each z. It suffices to check the vertices.

Consider $z = x(t)$ on the moment curve. Now

$$F_s(z) = F_s(x(t))$$

is a polynomial in t of degree d. This polynomial vanishes for each $t = t_{i_s}$ so it has d zeros; that is

$$F_s(x(t)) = \text{constant} \prod_{i_s \in S} x_{i_s}$$

and sign analysis now trivializes the problem. ∎

As a result, the combinatorics of $C_d(t_1, \dots, t_n)$ does not depend on the actual values of $t_1, \dots, t_n$, only the order $t_1 < \dots < t_n$. Hence we refer to it as the *cyclic polytope* $C_d(n)$.

Why do we care?

Theorem 16.14 (Upper Bound Theorem, McMullen 1970)**.** *If P is a d-dimensional polytope with $n = f_0$ vertices, then for each k it has at most as many k-faces as the cyclic polytope; that is*

$$f_{k-1}(P) \le f_{k-1}(C_d(n)).$$

The proof uses shellability.

16.5 The Permutohedron

Definition 16.15. The permutohedron $\Pi_n \subset \mathbb{R}^n$ is the convex hull of all vectors $(\pi(1), \pi(2), \dots, \pi(n))$ where $\pi \in S_n$.

This is not full dimensional; it lies on the affine hyperplane $\sum x_i = \binom{n+1}{2}$. In general, it is $n-1$ dimensional.

The edges of Π_n are permutations which differ by a single transposition (multiplied on the left). Formally,

Definition 16.16. The (left) weak Bruhat order on S_n is the partial order with cover relations $u \lessdot v$ if there exists a simple reflection $s_i = (i\ i+1)$ such that $v = s_i u$ and $\ell(v) = \ell(u) + 1$.

Think of the permutations as functions, with u acting before s_i. This differs from the strong Bruhat order, where s does not have to be of the form $(i\,i+1)$, but can instead be any reflection.

The left Bruhat order is by value, not position. So $132 \lessdot 231$.

Then are earlier claims about edges is just

Proposition 16.17. *The edge graph of the permutohedron is the Hasse diagram of the weak Bruhat order (guess which one!)*

Theorem 16.18. *For any chain*

$$\varnothing \subsetneq A_1 \subsetneq A_2 \subsetneq \dots \subsetneq A_k \subset [n]$$

let $F(A_1, \dots, A_k)$ be the convex hull of vertices $\pi \in \Pi_n$, such that in $\pi = (\pi_1, \dots, \pi_n)$:

- *The numbers $\{1, 2, \dots, |A_1|\}$ are located in positions i where $i \in A_1$*
- *The numbers $\{|A_1| + 1, \dots, |A_2|\}$ are located in positions of $A_2 - A_1$*
- *The numbers $\{|A_2| + 1, \dots, |A_3\}|$ are located in positions of $A_3 - A_2$*

Then $F(A_1, \dots, A_k)$ is an $n-k-1$ dimensional face of the permutohedron, and every face has this from.

Example 16.19. $(3, 2, 5, 1, 6, 7, 4) \in F(\{2, 4\}, \{1, 2, 3, 4, 7\})$.

Proof. If P is any polytope in $\mathbb{R}^n$ and $w = (w_1, \dots, w_n) \in \mathbb{R}^n$, the subset

$$F_w \overset{\text{def}}{=} \{x \in P \mid w \cdot x \text{ is maximized }\}$$

is a face of P. Every face has this form.

Let $w = (w_1, \dots, w_n) \in \mathbb{R}^n$ Associate to w the chain $\varnothing \subsetneq A_1 \subsetneq \cdots \subsetneq A_k \subset [n]$ defined by $w_a = w_b$ for $a, b \in A_i - A_{i-1}$ and $w_a < w_b$ for $a \in A_i - A_{i-1}$ and $b \in A_{i+1} - A_i$. (Sound familiar?)

Then the permutations π which maximize the dot product $w \cdot \pi$ are those in this chain.

To finish, note that each of the A_i gives an "obvious" equation, showing the dimension is at most $n - k - 1$; then notice that removing a block is from an F chain leads to a proper containment, so we can go downwards as well. Yay! $\square$

17 October 24, 2013

Today we will be studying some results about f-vectors of simplicial polytopes.

17.1 Faces (hi Patrick Yang!)

Recall that a polytope is simplicial if each facet is a simplex. This implies the face lattice of the body of the polytope is a simplicial complex.

Recall the definition of a f-vector.

Definition 17.1. Define the f-polynomial by

$$f(X) = f_{d-1} + f_{d-2}x + \cdots + f_0 x^{d-1} + f_{-1}x^d.$$

Example 17.2. The f-polynomial of an octahedron is $8 + 12x + 6x^2 + x^3$.

Observe that $f(x-1) = x^3 + 3x^2 + 3x + 1$. There are some nice properties about this: as two examples, it is symmetric, and has nonnegative coefficients. These generalizes.

Theorem 17.3. *For any simplicial polytope, the f-polynomial is symmetric and has nonnegative coefficients.*

They're also unimodal, but we won't prove that.

17.2 Line Shellings

Recall that a (pure) simplicial complex is shellable if its facets can be arranged in a linear order $F_1, \dots, F_t$ such that the subcomplex $(\bigcup_{i=1}^{j-1} \overline{F_i}) \cap \overline{F_j}$ is pure of codimension 1 for all $j = 2, \dots, t$.

Equivalent, for each $i < j$, there exists some $\ell < j$ such that $F_i \cap F_j \subset F_\ell \cap F_j$ and such that $F_\ell \cap F_j$ is a facet of F_j.

Theorem 17.4. *Let $P \subseteq \mathbb{R}^d$ be a d-dimensional polytope and let $\underline{x} \in \mathbb{R}^d$ be a point outside of P. If $\underline{x}$ lie in general position*[21]*, then the body complex $\mathcal{C}(\partial P)$ has a shelling in which the facets that are visible from x come first.*

Here, a facet $F \subseteq P$ is visible from x if $\forall y \in F$, the closed line segment $[x, y]$ intersects P only at the point Y.

Proof. Given $\underline{x}$, choose a line ℓ through x such that

- ℓ hits the interior of the polytope
- whenever ℓ intersects a facet, it intersects its interior.
- ℓ is not parallel to a facet hyperplane.

Orient ℓ from P to x. Imagine P is a planet and we have a rocket which starts at the surface $\ell \cap F$. Ignite the rocket. Looking back down at the planet, we initially see only the facet we started from. As we move out into space, we see facets added to our vision by one by one. This gives us an ordering $F_1, F_2, F_3, \dots$ of facets. When we are far enough, "half" of the facets are visible. Then we wrap around infinity and all these facets disappear, and the remaining half becomes visible. As we slowly descend back to the planet, facets disappear one by one; add the vanishing facets to the sequence as they disappear.

You can check this is a shelling. □

[21] not in an affine hull of the facet

Definition 17.5. Such shellings are called line shellings.

Proposition 17.6. *If $F_1, \dots, F_t$ is a line shelling, then so is $F_t, \dots, F_1$.*

17.3 Proof of Result

Let us prove our theorem from earlier.

Recall that $\mathcal{C} = \mathcal{C}(\partial P)$ is the abstract simplicial complex on the vertices of P that we get by looking at proper faces of P.

Let $d = \dim(P)$, so facets $\mathcal{F} \subseteq \binom{V}{d}$. Choose a shelling order $F_1, F_2, \dots$ on the facets and define a restriction R_j of the face F_j

$$R_j \overset{\text{def}}{=} \{v \in F_j \mid F_j - v \subseteq F_i \text{ for some } 1 \le i < j\}.$$

Remark 17.7. The proof that follows works for any polytope, not just a simplicial complex.

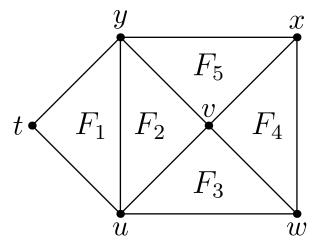

Figure 11: A shelling of a simplicial complex.

Example 17.8. In the figure, $R_1 = \varnothing$, $R_2 = \{v\}$, $R_3 = \{w\}$, $R_4 = \{x\}$, and $R_5 = \{x, y\}$.

At each step, consider the faces that we are adding at each step.

1. We add y, u, z, yz, uy, uz, uyz.
2. We add v, vy, vu, vyu.
3. We add w, wv, wu, wuv.
4. We add x, xv, xw, xvw.
5. We add xy, xvy.

Claim 17.9. When we build up $\mathcal{C}$ according to the shelling, the new faces we add at the jth step are precisely the vertex sets G with $R_j \subseteq G \subseteq F_j$.

Proof. Certainly a new face must be a subset of F_j. If it misses a vertex $v \in R_j$, then by definition of R_j it was already contained in a previous facet. Together these imply $R_j \subseteq G \subseteq F_j$.

Conversely, suppose G satisfies $R_j \subseteq G \subseteq F_j$ and assume for contradiction that it's not new; $G \subseteq F_i$ for some $i < j$. This is where the shelling will come in: by the definition of shellability there exists $\ell < j$ such that $F_i \cap F_j \subset F_\ell \cap F_j = F_j - \{w\}$.

Since $F_j - w < F_\ell$, $w \in R_j$. Since $R_j \subset G \subset F_i \cap F_j \subseteq F_j - w \Rightarrow w \notin R_j$, contradiction. $\square$

Our claim now implies that the shelling gives a partition

$$I_1 \sqcup \cdots \sqcup I_S$$

of faces of $\mathcal{C}$ into intervals of the form $I_j \stackrel{\text{def}}{=} \{G : R_j \subseteq G \subseteq F_j\}$.

From this we can read off the f-vector. If $|R_j| = i$ then there are $\binom{d-i}{k-i}$ faces of dimension $k-1$ contained in I_j. Therefore,

$$f_{k-1} = \sum_{j=1}^{s} \binom{d-|R_j|}{k-|R_j|}.$$

Definition 17.10. Define $h_i = h_i(\mathcal{C}) = \#\{1 \le j \le s : |R_j| = i\}$. Then we can define the h-vector of a simplicial polytope is

$$\overline{h} = (h_0, h_1, \dots, h_d).$$

The h-polynomial is then defined by $h_d + h_{d-1}x + \cdots + h_1 x^{d-1} + h_0 x^d$.

Note that this is independent of the choice of shelling. Clearly, $h_i \ge 0$. Furthermore,

$$f_{k-1} = \sum_{i=1}^{k} h_i \binom{d-i}{k-i} \Leftrightarrow f(x) = \sum_{i=0}^{d} h_i (x+1)^{d-1} = h(x+1).$$

Thus, $f(x-1)$ indeed has nonnegative integer coefficients.

This argument works for any shellable simplicial complex; any such complex has an h-vector. However, we will require the condition for $h(x)$ to be symmetric.

As an interlude, one can "nicely" compute the h-vector from the f-vector using what is called *Stanley's trick*. Note that because $f(x) = h(x+1)$, this is just high school algebra. See `http://gilkalai.wordpress.com/2009/04/02/eran-nevo-the-g-conjecture-i/`.

17.4 Proof of the second part

Let us now prove the second part, which we name here.

Theorem 17.11 (Dehn-Sommervill equations)**.** *The h-vector of the boundary of a simplicial polytope satisfies $h_k = h_{d-k}$ for $k = 0, \dots, d$.*

Remark 17.12. We only need half of the h-vector to recover the rest, by the above theorem. Hence, some people define the g-vector by $g_i = h_{i+1} - h_i$, where we only use half the components.

There exists a complete characterization of g-vectors that can arise.

Remark 17.13. We can write any symmetric polynomial about x^i as a linear combination of $x^i(x+1)^j$ as j ranges. Writing the h-vector in this from gives what is called the *gamma vector*.

Proof. Again choose a line shelling $F_1, F_2, \dots, F_s$ for a polytope. Note that $F_s, F_{s-1}, \dots, F_1$ is also a shelling. Furthermore, if F_i comes earlier than F_j in the first shelling then it comes later in the second shelling.

Recall that

$$\operatorname{Res}(F_j) = R_j \stackrel{\text{def}}{=} \{v \in F_j \mid F_j - v \subseteq F_i \text{ for some } 1 \le i < j\}.$$

We claim that $\operatorname{Res}(F_j)$ in the second shelling is $F_j - R_j$ in the first shelling. This follows from the fact that the set $F_j - v$ (a ridge) lies in precisely two facets of the polytope.

From here it is not hard to see that by reversing the shelling, we have reversed the h-vector – which is independent of the shelling! $\square$

Corollary 17.14 (Euler-Poincare formula). *If P is a convex polytope, then*

$$-f_1 + f_0 - f_1 + f_2 \cdots + (-1)^{d-1} f_{d-1} + (-1)^d f_d = 0.$$

This becomes $V - E + F = 2$ when $d = 3$.

We'll prove this for the simplicial polytopes using the Dehn-Sommervill relations. (The formula is true much more generally.) We can derive that

$$h_k = \sum_{i=0}^{k} (-1)^{k-1} \binom{d-i}{d-k} f_{i-1}.$$

In particular, when $k = 0$, we get

$$h_0 = \sum_{i=0}^{0} (-1)^i \binom{d-i}{d-k} f_{i-1} = f_{-1}.$$

Also,

$$\begin{aligned} h_d &= \sum_{i=0}^{d} (-1)^{d-i} \binom{d-i}{d-d} f_{i-1} \\ &= \sum_{i=0}^{d} (-1)^{d-1} f_{i-1} \end{aligned}$$

Equating $h_0 = h_d$ gives the above formula.

18 October 29, 2013

No class.

19 October 31, 2013

Today we will talk about simple polytopes, and how to tell a simple polytope from its graph.

Definition 19.1. Recall that a d-dimensional polytope is *simple* if each vertex is incident to exactly d edges (or equivalently, d facets)

In a simple polytope, for each vertex v and each i-subset of edges incident to it, there is an i-dimensional face using precisely these edges among edges incident to v. An example is the cube.

19.1 Graphs of Polytopes

Definition 19.2. Given a polytope P, let $G(P)$ be the *graph* of P; i.e. the graph whose vertices are $V(P)$ and whose edges are precisely the endpoints of edges in P.

This is also called the *1-skeleton* of the polytope. It's what left if you ignore all faces of dimension two or higher; in particular, it's the bottom two layers of the face lattice of P.

In 1970, Perles made the following conjecture. It was proved in 1987 by Blind and Mini. Kalai later gave a proof that we will observe.

Proposition 19.3 (Blind and Mini, 1987). *Given that P is simple, one can reconstruct P from $G(P)$.*

To do this, we first develop the tool of acyclic orientations.

19.2 Acyclic Orientations

Definition 19.4. Given a graph G, an *acyclic orientation* of G is an orientation with no directed cycle.

Definition 19.5. A *sink* in an oriented graph is a vertex with no edges directed away from it.

Remark 19.6. Given any acyclic orientation $\mathcal{O}$ on a graph, we get a partial order on $V(P)$ by $x \le_{\mathcal{O}} y$ if there exists an $\mathcal{O}$-oriented path from x to y. It's trivial to check this is indeed partial order.

Remark 19.7. If $\mathcal{O}$ is an acyclic orientation of a finite graph $G = (V, E)$, then for any $A \subseteq V$, the restriction of $G(P)|_A$ has a sink with respect to $\mathcal{O}$.

Proof. Follow some directed path in A. Because the orientation is acyclic, we must arrive at a vertex we cannot leave; i.e. a sink. $\square$

Definition 19.8. An acyclic orientation $\mathcal{O}$ of $G(P)$ (the graph of a polytope) is *good* if for each nonempty face F of P, $G(F)$ has exactly one sink. Otherwise, we say $\mathcal{O}$ is *bad*.

Proposition 19.9. *$G(P)$ has a good acyclic orientation.*

Proof. Choose a linear functional[22] $\ell : \mathbb{R}^d \to \mathbb{R}$ that is injective on the vertices and orient an edge $u \to v$ if and only of $\ell(u) < \ell(v)$. Then every face has precisely one sink, namely the "highest" point. $\square$

[22]For example, take P and put it in space such that each vertex is at a different height.

First, we characterize good versus bad orientations of $G(P)$. Let $\mathcal{O}$ be an acyclic orientation of $G(P)$.

Remark 19.10. If x is a vertex of $G(P)$ of indegree k then x is a sink in 2^k faces of P, one for each subset of the edges incident to x.

Let $h_k^{\mathcal{O}}$ denote the number of vertices of $G(P)$ with indegree k (in $\mathcal{O}$), and define

$$f^{\mathcal{O}} = \sum_{i=0}^{d} 2^i h_i^{\mathcal{O}} = \sum_{\text{vertex} x} \# \text{ faces of } P \text{ with } x \text{ a sink.}$$

Every face will be counted at least one for an arbitrary acyclic orientation, because every restriction has a sink.

Let f be the total number of nonempty faces of P; i.e. the "true" number. Since each face has at least one sink, we deduce that

$$f^{\mathcal{O}} \geq f$$

and with equality if and only if $\mathcal{O}$ is good. To distinguish between good and bad acyclic orientations, we could therefore list all of them; the good ones will be the minimal ones since good acyclic orientations exist.

Definition 19.11. Given $A \subseteq V(P)$ and $\mathcal{O}$ an acyclic orientation of $G(P)$, we say A is *initial* with respect to $\mathcal{O}$ if any edge $u \to v$ of $G(P)$ with one vertex in A and one vertex not in A points out of A.

These initial subsets are precisely the order ideals in our poset from earlier.

19.3 The Algorithm

Theorem 19.12 (Kalai)**.** *An induced connected k-regular subgraph H of $G = G(P)$ is a k-face of P if and only if its vertices are initial with respect to some good acyclic orientation $\mathcal{O}$ of G.*

We now have a very slow algorithm.

1. Consider all acyclic orientations $\mathcal{O}$ and compute $f^{\mathcal{O}}$.
2. The good orientations are those for which $f^{\mathcal{O}}$ is minimized (equal to f).
3. Take the order ideals in each good acyclic orientations.
4. Look for the connected regular subgraphs.

"If your goal is to reconstruct a 100-dimensional polytope from its graph, I think your first step should be to give up" – Hood.

Proof. First, consider a k-face F. Put the polytope into space such that the k-face is at the bottom. Then the linear functional gives us what we want.

For the other direction, consider such a subgraph H. Let H be a connected k-regular subgraph and let $\mathcal{O}$ be a good acyclic orientation such that $V(H)$ is initial with respect to $\mathcal{O}$. Let x be the sink of H with respect to $\mathcal{O}$. By definition of good, x exists and is unique.

So there are k edges in H pointing to x. Since $G(P)$ is a graph of a simple polytope there is a k-face of P containing those k edges. Since $G(P)$ is a graph of a simple polytope there is a k-face of P containing those k edges. Hence all vertices of F are $\leq x$ with respect to $\mathcal{O}$. But $V(H)$ is initial with respect to $\mathcal{O}$ so $V(H)$ contains all vertices $\leq x$ as well. Hence $V(F) \subseteq V(H)$. But both H and $G(F)$ are k-regular and connected; this forces $V(F) = V(H)$ as desired. $\square$

19.4 Examples of Simple Polytopes

We saw the permutohedron last time. Let's discuss the associahedron.

19.4.1 Associahedron

The associahedron has vertices corresponding to the $\frac{1}{n}\binom{2n-2}{n-1}$ of balanced strings of parentheses (think Catalan). The edges correspond to one application of the associative law (hence the name). The case $n = 4$ is illustrated. Another way to view this is in terms

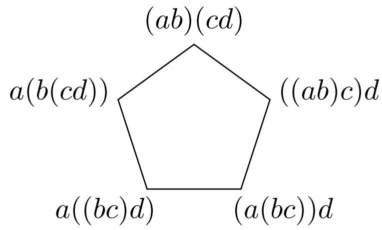

Figure 12: The associahedron when $n = 4$.

of triangulations. This corresponds to flipping diagonals.

One can show that this is in fact the graph of a polytope.

19.4.2 Graph Associahedron

We now present a vast generalization. For each graph G, we can build the graph associahedron P_G.

- If G is the chain on n vertices, one obtains an associahedron.
- If G is the complete graph, one obtains the permutohedron.
- If G is a cycle on n vertices, one obtains the cyclohedron.
- If G is a star, one obtains the stellohedron.

Here is the construction. Suppose G has n vertices; we will construct $P_G \subset \mathbb{R}^{n-1}$. Draw a simplex with n vertices. Select a bijection τ between the facets of a simplex and with vertices of G.

Now for each *connected subset* S of size $n-1$ in G, F_S is a vertex of a simplex. Shave the simplex at that vertex (i.e. cut off a corner). Then do the same for connected subsets of size $n-2$ (now shaving edges). Continue; we do this for subsets of size $n-1, n-2, \dots, 1$.

Example 19.13. Let G be the graph on $[4]$ with $E(G) = \{12, 13, 14\}$. Consider a tetrahedron with faces labelled 1, 2, 3, 4.

We want to show we can get the face lattice of P_G from G.

Definition 19.14. A *tube* is a connected subset of G.

Definition 19.15. A *tubing* is a collection of tubes $\{t_1, t_2, \dots, t_k\}$ of tubes of G such that

- No two tubes are adjacent; i.e. their union is also a tube.

- No two tubes intersect unless one is a strict subset of the other.

Theorem 19.16 (Carr, Devadoss)**.** *The face lattice of P_G is the poset of tubings ordered by reverse containment.*

We also want to show we can get the edges. If v_1 is incident to the facets $f_1, \dots, f_d$ then v_2 must be incident to all the same facets except one.

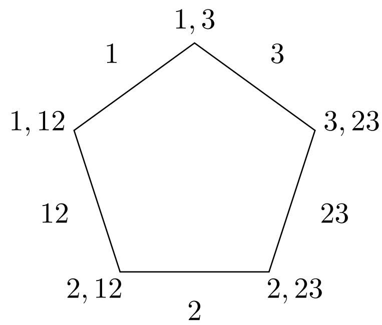

Figure 13: The associahedron when $n = 4$ realized by a chain.

20 November 5, 2013

Today we will do some final things about face vectors and talk about the cd-index.

20.1 Review

Question. Can one characterize f-vectors of simplicial d-polytopes?

Recall that the f-vector of a d-polytope is $(f_{-1}, f_{)}, f_1, \dots, f_{d-1}$ where f_i is the number of i-dimensional faces.

We can encode the f-vector by an h-vector $(h_0, \dots, h_d)$ obeying $h_i \ge 0$ and $h_k = h_{d-k}$. We found that

$$f_{k-1} = \sum_{i=0}^{k} h_i \binom{d-i}{k-i} \Leftrightarrow h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_k.$$

Remark 20.1. Additionally, this lets us define h vectors for any polytope, not necessarily simplicial. (Note that our original definition required shelling but the one above does not.)

20.2 The g-Vector

Definition 20.2. The g-vector of a simplicial d-polytope is defined by

$$g(P) = \left(g_0, g_1, \dots, g_{\left\lfloor \frac{1}{2}d \right\rfloor}\right)$$

where $g_0 = h_0 = 1$ and $g_k = h_k - h_{k=1}$ for $1 \le k \le \left\lfloor \frac{1}{2}d \right\rfloor$.

Note "simplicial" polytope.

By the Dehn-Sommervill equations the g-vector uniquely determines the h vector. Hence classifying f-vectors is equivalent to classifying h-vectors

For any fixed $n, k \ge 0$ there exists a unique binomial expansion of n in the form

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \dots + \binom{a_2}{2} + \binom{a_1}{1}$$

where $a_k > a_{k-1} > \dots > a_2 > a_1 \ge 0$. (To do so, choose a_k as large as possible, and proceed greedily.)

Definition 20.3. Let $\partial_k(n+1) = \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \dots + \binom{a_2}{1} + \binom{a_1}{0}$ where the a_i are as above.

Definition 20.4. An *M-sequence* is a sequence of nonnegative integers h_i such that $\partial_k(h_k) \le h_{k-1}$ for all k.

These are ...strange definitions. "It would probably take another hour-long lecture to give some intuition for where these numbers come from".

20.3 The g-Theorem

Theorem 20.5 (g-Theorem, Billera & Lee, Stanley '79)**.** *A sequence $g = (g_0, \dots, g_{\left\lfloor \frac{1}{2}d \right\rfloor})$ of nonnegative integers is the g-vector of a simplicial d-polytope if and only if it is an M-sequence.*

Billera & Lee gave a construction for any M-sequence, while Stanley at around the same time showed the converse with the hard Lefschetz theorem theorem. More about this in Ziegler.

20.4 The β-index

Question. What if we want to understand not only the number of faces of each dimension of a polytope, but also the number of chains of faces of fixed dimension?

Definition 20.6. Let P be an n-dimensional polytope and let $S \subseteq \{0, 1, \dots, n-1\}$. Write $S = \{s_1 < s_2 < \dots < s_k\}$. The *flag f-vector* of P is the vector with components

$$f_S = \#\{F_1 \subset F_2 \subset \dots \subset F_k\}$$

where F_i is a face of dimension s_i.

By convention, $f_\varnothing = 1$.

Definition 20.7. The *flag β-vector* (or just β-vector) of P is the vector with components

$$\beta_s = \sum_{T \subseteq S} (-1)^{|S-T|} f_T.$$

Equivalently, $f_S = \sum_{T \subseteq S} \beta_T$.

This is sometimes called the *generalized h-vector*, although it isn't a full generalization.

Example 20.8. Let P be a hexagonal prism.

We have $f_{01} = 36$ because we pick an edge and a pair $(12 \cdot 3 = 36.)$ Similarly, $f_{02} = 12 \cdot 3 = 36$ and $f_{12} = 18 \cdot 2 = 36$. Finally, $f_{012} = 12 \cdot 3 \cdot 2 = 72$.

S	f_S	β_S
$\varnothing$	1	1
0	12	11
1	18	17
2	8	7
01	36	7
02	36	17
12	36	11
012	72	1

. . . well. Positive and symmetric. Simple polytopes are too special. Let's try a non-simple polytope.

Example 20.9. Let P be a square pyramid. Then the table is as follows.

S	f_S	β_S
$\varnothing$	1	1
0	5	4
1	8	7
2	5	4
01	16	4
02	16	7
12	16	4
012	32	1

Well then.

Theorem 20.10 (Stanley)**.** *When P is a polytope, the β-vector is positive and symmetric.*

The proof is very similar to that for the h-vector. One constructs a reversible shelling as an interpretation for the β-vector, from which this follows.

In fact, this even holds if we replace "polytope" with "S-shellable regular CW decomposition of a sphere". Here S-shellable is some modified shellability (since there are a bunch of versions of shellability).

20.5 The cd-index

Let us encode the β-vector with a polynomial.

Let a, b be noncommutative variables. For $S \subseteq \{0, 1, \dots, n-1\}$ write

$$U_S = u_0 u_1 \dots u_{n-1}$$

where $u_i = a$ if $i \notin S$ and $u_i = b$ otherwise.

Example 20.11. If $n = 3$ and $S = \{0, 2\}$, we have $u_S = aba$.

Definition 20.12. The $ab-$index of P is defined[23] by

$$\Psi(P) = \sum_S \beta_S u_s.$$

Example 20.13. For the hexagonal pyramid,

$$\begin{aligned}
\Psi(P) &= bbb + 11abb + 17bab + 7bba + 7aab + 17aba + 11baa + aaa \\
&= (a+b)^3 + 10abb + 16bab + 6bba + 6aab + 16aba + 10baa \\
&= (a+b)^3 + 10(abb + bab + aba + baa) + 6(bab + bba + aab + aba) \\
&= (a+b)^3 + 10(ab+ba)(a+b) + 6(a+b)(ab+ba)
\end{aligned}$$

It turns out that the ab-index can be written entirely in terms of $c = a + b$ and $d = ab + ba$. That is,

Theorem 20.14 (Bayer-Klappen, Stanley)**.** *For P be a polytope, then $\Psi(P) \in \mathbb{Z}\langle c, d\rangle$.*

Here the angle brackets indicate noncommutative variables. These coefficients are positive for polytopes. We can even define cd-indices for Eulerian posets, but in that case the coefficients need not be positive.

Example 20.15.

$$\Psi(\text{hexagonal prism}) = c^3 + 10dc + 6cd.$$

In case it's not already obvious,

Definition 20.16. This is called the cd-index.

This is much more compact than β_S.

20.6 Generalizations to Posets

Earlier we saw that we can replace P with the regular CW decompositions of a sphere. So we may still define a β-vector, and therefore a cd-index. More generally, we may try to replace P with a nice poset (e.g. an Eulerian one).

Theorem 20.17 (Bayer-Klappen)**.** *Let P be a graded poset. Then P has a cd-index with integer coefficients if and only if the β-vector of P satisfies the generalized Dehn-Sommerville relations.*

Definition 20.18. A poset P is *Eulerian* if it is graded, has a minimum $\hat{0}$ and maximum $\hat{1}$, and every interval has the same number of elements of even rank or odd rank.

Equivalently, $\forall u \le v$ we have $\mu(u, v) = (-1)^{\ell(u,v)}$.

[23] What is that symbol? It's certainly not Ψ.

Example 20.19. Face posets of regular CW-decompositions of spheres are Eulerian.

Theorem 20.20 (Stanley)**.** *If P is an Eulerian poset, then $\Psi(P) \in \mathbb{Z}\langle c, d\rangle$; i.e. the cd-index makes sense.*

Here are some results on the non-negativity of the cd-index.

1. (Stanley) It is true for S-shellable posets (including polytopes).

2. (Karu) It is true for Gornestein* posets, including regular CW-spheres.

3. It is not true for all Eulerian posets.

It is conjectured that it is nonnegative for all regular CW complexes (though it is known that the cd-index does in fact exists).

And of course: we would like a combinatorial interpretation of the cd-index.

Recall that the graph-associahedra P_G are polytopes associated to graphs G, and the face lattice of P_G has an explicit description in terms of tubes and tubings.

21 November 7, 2013

Discrete Morse Theory was developed by Robin Forman in 1998. It is a technique for understanding the topology for CW complexes, but it is easiest to use for simplicial and regular CW complexes.

Recall that regular CW complexes are nice because the order complex of their face poset is homeomorphic to the complex itself, implying they can be viewed combinatorially.

Morse Theory: see Millner. Forman developed an analogue of Morse theory for combinatorial settings. Shellability generally fails because posets can have order complexes which are not wedges of spheres (surprise!) but discrete Morse theory can be applied more generally.

21.1 Discrete Morse function

Note: we do this for regular CW complexes, but we can generalize.

Let K be a simplicial complex or regular CW complex, and (in abuse of notation) let K also denote the set of cells of the CW complex. If α is a cell of dimenison p, we will write $\alpha^{(p)}$. Finally, we can view K as a poset, where cells α, β have $\alpha < \beta \Leftrightarrow \alpha \in \overline{\beta}$.

Roughly, a discrete Morse function is a function taking the cells of K to real numbers which assigns higher numbers to higher dimensional cells with at most one exception. More precisely,

Definition 21.1. A function $f : K \to \mathbb{R}$ is called a *discrete Morse function* if for every $\alpha^{(p)} \in K$ the following properties hold:

(i) $\#\left\{\beta^{(p+1)} > \alpha \mid f(\beta) \le f(\alpha)\right\} \le 1$.

(ii) $\#\left\{\gamma^{(p-1)} > \alpha \mid f(\gamma) \ge f(\alpha)\right\} \le 1$.

Remember that this is for *regular* CW complexes! The definitions become more contrived for non-regular CW complexes.

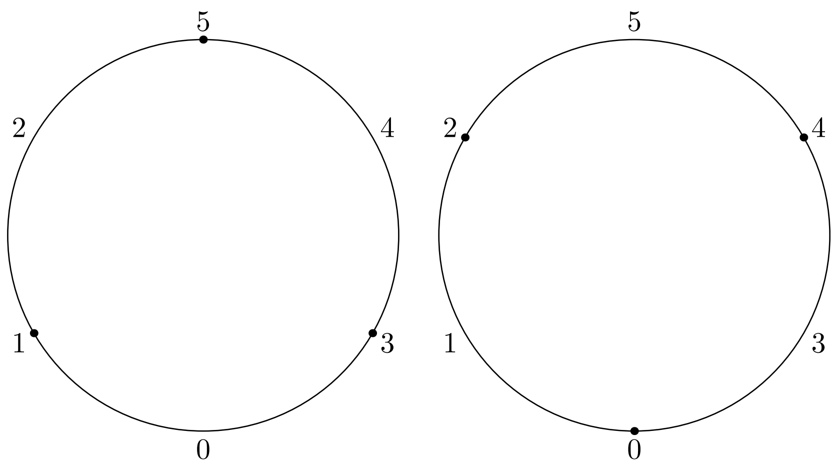

Figure 14: Decompositions of the 1-sphere. The left is a non-example, the right is an example of a discrete Morse function.

Definition 21.2. Given a discrete Morse function, cell $\alpha^{(p)}$ is *critical* (with respect to that Morse function) if the two sets in the previous definition are in fact both empty.

Example 21.3. The critical cells in the figure are the ones with values 0 and 5.

Here is the main theorem of discrete Morse theory.

Theorem 21.4 (Forman)**.** *Suppose that K is a regular CW complex (e.g. a simplicial complex) with a discrete Morse function. Then K is homotopy equivalent to a CW complex with exactly one cell of dimension p for each critical cell of dimension p.*

So our goal is to try and find a discrete Morse function with the minimal number of cells. This will give the most information.

Example 21.5. In our example, there is one critical cell of dimension 0 and one of dimension 1, so the theorem implies that our S^1 is homotopy equivalent to a CW complex with one 0-cell and one 1-cell. (In this case, it's obvious.)

Note that regularity is not preserved.

Corollary 21.6. *If K has a discrete Morse function with exactly one critical cell of dimension 0, then K is contractable.*

21.2 Lemmata and Proofs

Lemma 21.7. *In a discrete Morse function, the cardinalities of $\left\{\beta^{(p+1)} > \alpha \mid f(\beta) \le f(\alpha)\right\}$ and $\left\{\gamma^{(p-1)} < \alpha \mid f(\gamma) \ge f(\alpha) \le 1\right\}$ cannot both be 1.*

We will refer to this as $(\star)$.

Proof. Assume for the sake of contradiction that $\gamma^{(p-1)} < \alpha^{(p)} < \beta^{(p+1)}$, but $f(\gamma) > f(\alpha) > f(\beta)$. Since K is a regular CW complex, there exists another cell α' such that $\gamma < \alpha' < \beta$, since the boundary of β is a sphere and we're in a regular CW complex. (Think of a diamond in a poset).

Now we obtain $f(\beta) > f(\alpha')$ because β already has a lesser face with higher value, namely α. Hence $f(\alpha') < f(\beta)$. Similarly, $f(\gamma) < f(\alpha')$. So we obtain a chain $f(\gamma) < f(\alpha') < f(\beta)$, yet $f(\gamma) > f(\alpha) > f(\beta)$, contradiction. $\square$

Idea of the proof of Forman's Theorem: the discrete Morse function gives a way to build CW complexes by attaching the cells in the order prescribed by the function; i.e. first adding the cells which are assigned the smallest value.

Given K with a discrete Morse function f, and any $c \in \mathbb{R}$, define the level subcomplex by

$$K(c) \overset{\text{def}}{=} \bigcup_{\alpha \mid f(\alpha) \le c} \bigcup_{\beta \le \alpha} \beta.$$

The second $\bigcup$ basically means we include the closure $\overline{\alpha}$ rather than just α.

Then the theorem follows from two lemmas.

Lemma 21.8. *If there are no critical cells α with $f(\alpha) \in (a, b]$ then $K(b)$ is homotopy equivalent to $K(a)$. (Actually $K(b)$ collapses to $K(a)$.)*

Lemma 21.9. *If there is a single critical cell α with $f(\alpha) \in (a, b]$ then there is map $f : S^{d-1} \to K(a)$, where d is the dimension of α, where $K(b)$ is homotopy equivalent to $K(a) \cup_F B^d$; i.e. we have glued on another d-cell using the attaching map f.*

Note: in the example that follows, we abuse notation by referring to cells by their f-values.

Example 21.10. In our circle example as before, $K(0)$ is a single point. $K(2) = K(1)$ is the arc from 0 to 2 inclusive, while $K(4) = K(3)$ is the major arc from 4 to 2 inclusive. $K(5) = S^1$ though.

This happens to give a shelling, but shh!

Some intuition: to construct $K(1)$ from $K(0)$ we first add the edge 1. It is not critical because it has a codimension 1 face which has a higher value, namely 2. In order for $K(1)$ to be a subcomplex, we must add that face. So the edge 1 in $K(1)$ has a free face; i.e. a face which is not the face of anything else already existing cells.

If the other endpoint of 1 was already in $K(0)$, we couldn't have retracted that edge. In general, given a regular CW complex with a discrete Morse function, when we go from one level subcomplex to another, the noncritical cells are added in pairs, each containing a cell plus a free face. This follows from $(\star)$.

General idea for lemma 1: Suppose $K_2 \subset K_1$ and K_1 has two cells α and β not in K_2, where β is a free face of α. Then K_2 can be retracted to K_1. So $K_1 \sim K_2$.

General idea for lemma 2: what happens when we add a critical cell? (We went from $K(4)$ to $K(5)$ in our example.) Because the cell is critical, its entire boundary alreday lies in a previous subcomplex. So this is the map $F : S^{(d-1)} \to K(4)$.

21.3 How to Find Discrete Morse Functions

Recall that the discrete Morse function is a certain kind of function from the cells in K to the reals. Chari gave an equivalent description in terms of matchings of the Hasse diagram.

Definition 21.11. A *matching* of a graph $G = (V, E)$ is a subset $M \subseteq E$ such that each vertex appears in at most one of edge of M. It is called *perfect* if each vertex appears exactly once.

The matchings need not be perfect.

Definition 21.12. Let K be a regular CW complex with a discrete Morse function f. Let $F(K)$ be the Hasse diagram of the face poset of K. Define a matching $M(f)$ on the face poset whose edges correspond to the pairs of cells α, β such that $\alpha \lessdot \beta$ yet $f(\beta) \leq f(\alpha)$.

This is a matching by $(*)$.

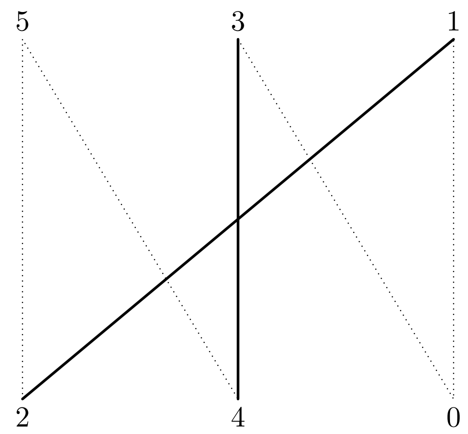

Figure 15: An example of an acyclic matching from our recurrent example.

Definition 21.13. A matching of the Hasse diagram is called *acyclic* if after we orient all matched edges up and all other edges down, the resulting directed graph is acyclic.

Proposition 21.14 (Chari)**.** *A subset C of cells of a regular CW complex is the set of critical cells of some discrete Morse function f if and only if there is a matching M on the Hasse diagram such that M is acyclic and C is the set of unmatched vertices.*

Definition 21.15. An acyclic matching of a Hasse diagram is a *Morse matching.*

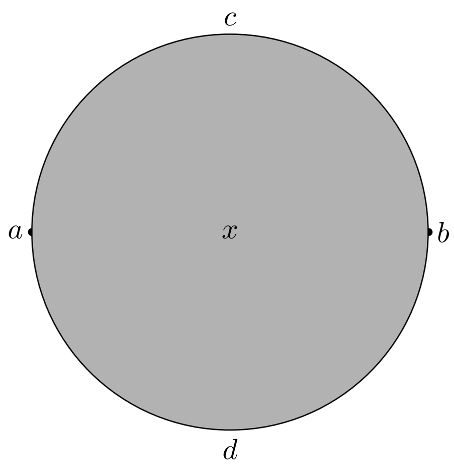

Figure 16: The two-ball as a CW-complex.

Example 21.16. Consider B^2 constructed as shown, with two 0-cells a, b, two 1-cells c, d and the two-cell x. The matching $\{xc, db\}$ is acyclic, and leaves only a unmatched. Hence B^2 is contractible.

22 November 12, 2013

Recall that given a rank r matroid $\mathcal{M} = ([n], \mathcal{B})$, the matroid (basis) polytope P_M is a convex hull of the indicator vectors of its basis (namely, $e_B = \sum_{b \in B} e_b$ for each $B \in \mathcal{B}$.)

Recall the GGMS theorem: every edge of P_M is parallel to $e_i - e_j$ for some $i \neq j$.

22.1 Coxeter Reflections

Place a mirror perpendicular to each edge (bisecting it). Note that the distance from the origin O to each basis is $\sqrt{r \cdot 1^2 + (n-r) \cdot 0^2} = \sqrt{r}$. Hence O lies on each of these mirrors.

We want to look at reflections across the mirrors.

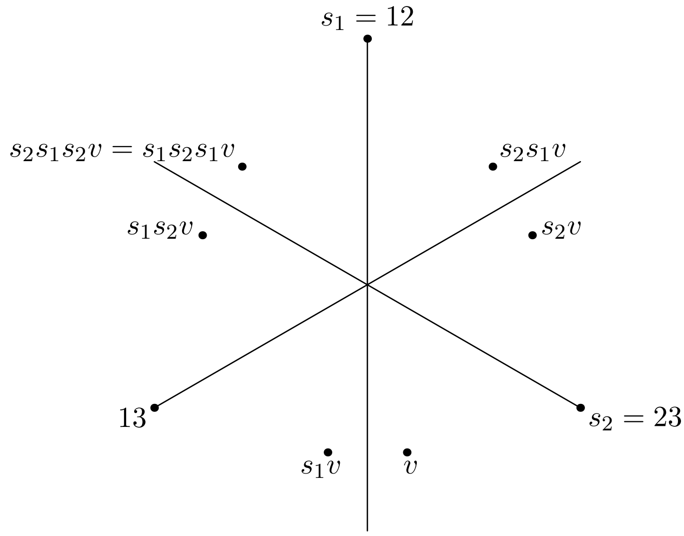

Figure 17: Reflections

Example 22.1. Let $\mathcal{M} = ([3], \mathcal{B} = \{12, 13, 23\})$. Then the matroid polytope is a equilateral triangle and the mirrors are each of the perpendicular bisectors.

Let s_1 be the reflection across e_{12} and s_2 be the reflection across e_{23}. We find that the group generated by the s_i's is finite: it has presentation

$$\left\langle s_1, s_2 \mid s_1^2 = s_2^2 = 1, (s_1 s_2)^3 = 1 \right\rangle.$$

More generally, how does reflection act on a vector? Consider reflections s_α in the mirror perpendicular to α. Let v be an arbitrary vector and write

$$v = x + \lambda \alpha$$

where x lies in the mirror and $\lambda \in \mathbb{R}$. Then the reflection of v is

$$s_\alpha v = x - \lambda \alpha = v - 2\lambda\alpha.$$

Now let us consider the inner product $(\alpha, v) = (\alpha, x + \lambda\alpha) = \lambda(\alpha\alpha)$. We get the familiar relation

$$s_\alpha v = v - 2\frac{(\alpha, v)}{(\alpha, \alpha)}\alpha.$$

22.2 Specializing to the Matroid Polytope

Now suppose $\alpha = e_i - e_j$, and let $v = (v_1, \dots, v_n)$. Then

$$s_\alpha v = v - 2\frac{(e_i - e_j, v)}{(e_i - e_j, e_i - e_j)}(e_i - e_j) = v - \frac{2(v_i - v_j)}{2}(e_i - e_j)$$

which is equal to

$$v - (v_i - v_j)(e_i - e_j) = (v_1, \dots, v_j, \dots, v_i, \dots, v_n)$$

i.e. the v_i and v_j get swapped. Hence reflection across the mirror perpendicular to $e_i - e_j$ swaps the i and j coordinates.

If a matroid $\mathcal{M}$ is connected, then every $e_i - e_j$ appears among edges of P_M.

Recall that for any two elements $a, b \in [n]$ we put $a \sim b$ whenever there are bases B, B' of M such that $B' = (B - \{a\}) \cup \{b\}$. This is an equivalence relation, and equivalence classes are the connected components of $\mathcal{M}$.

This implies that when $\mathcal{M}$ is connected, then every a, b is an edge $e_B - e_{B'}$ as an edge, so all transpositions are in the reflection group. Therefore we get all transpositions and hence obtain

Proposition 22.2. *If $\mathcal{M}$ is connected, the reflection group generated by the reflections across the mirrors is the symmetric group S_n.*

In the general case, the reflection group arising from an arbitrary matroid is the product of smaller sub-symmetric groups.

22.3 The Symmetric Group

Recall that the symmetric group has a presentation with generators $S_1, S_2, \dots, S_{n-1}$ (where $S_i = (i; i+1)$) and relations

$$\begin{aligned} s_i^2 &= 1 \quad \forall i \\ (s_i s_{i+1})^3 &= 1 \quad \forall i \\ (s_i s_j)^2 &= 1 \quad \forall \left|j - i\right| \ge 2 \end{aligned}$$

The Coxeter-Dynkin diagram of this is the chain on n vertices, with all edges weights equal to three.

Definition 22.3. A *Coxeter group* is a group with the presentation $\langle r_1, \dots, r_n \mid (r_i r_j)^{m_{ij}} = 1 \rangle$, where $m_{ii} = 1$ and $m_{ij} \ge 2$ (or $m_{ij} = \infty$) for $i \ne j$.

Question 22.4. When is a Coxeter group finite?

The answer is kinda complicated. See Humphreys or Wikipedia. Here you go: `http://en.wikipedia.org/wiki/Coxeter_group#Classification`.

These happen to be related to Platonic solids; i.e. the polytopes whose faces are congruent convex regular polyhedron. The symmetry groups of these polytopes are the following Coxeter groups.

- Simplex: Type A_n
- Hypercube/cross-polytope: BC_n
- Dodecahedron/icosahedron: H_3
- 24-cell: F_4
- 120-cell and 600-cell: H_4.

22.4 Coxeter Matroid Polytopes

Definition 22.5. A *Coxeter matroid polytope* is a polytope whose edge-mirror reflections generate a finite group.

These were introduced by Gelfand and Serganova in 1987. See "Coxeter Matroids" by Borovik-Gelfand-White.

Remark 22.6. Vertices of Coxeter matroid polytopes play the role of bases.

Before giving the definition of a Coxeter matroid, we provide yet another definition of a matroid.

Definition 22.7. Define a partial order on $\binom{[n]}{k}$ as follows. For any $A, B \in \binom{[n]}{k}$ with $A = \{i_1 < \cdots < i_k\}$ and $B = \{j_1 < \cdots < j_k\}$, we say $A \le B$ if and only if $i_k \le j_k$ for each k.

Remark 22.8. $\binom{[n]}{k}$ is in bijection with Young diagrams contained in a $k \times (n-k)$ rectangle. Label the south-east border from $1, 2, \dots, n$ and consider the vertical steps. This order then corresponds to containment.

Definition 22.9. Let $\omega \in S_n$. Define another partial order on $\binom{[n]}{k}$ by $A \overset{\omega}{\le} B$ if and only if $\omega^{-1}(A) \le \omega^{-1}(B)$. This is the *Gale order induced by w.*

Theorem 22.10 (Gale)**.** *Let $\mathcal{B} \subseteq \binom{[n]}{k}$. Then $\mathcal{B}$ is the set of bases of a matroid if and only if $\mathcal{B}$ satisfies the following maximality property: For every $\omega \in S_n$, $\mathcal{B}$ contains a unique member $A \in \mathcal{B}$ which is maximal in $\mathcal{B}$ with respect to $\overset{\omega}{\le}$.*

22.5 Flag Matroids

These are actually special cases of the Coxeter matroids.

Recall that every point in the Grassmanian $\mathrm{Gr}(k, n)$ gives rise to a rank k matroid on the ground set $1, 2, .., n$. In fact, this is a special case of a flag variety.

Definition 22.11. Fix n and some positive integers $1 \le k_1 < k_2 < \cdots < k_m \le n$. Let $\mathbb{F}$ be a field. Then the *partial flag variety* $\mathrm{Fl}_n^{k_1,\dots,k_m}$ is the set of all partial flags

$$\{V_1 \subset V_2 \subset \cdots \subset V_m \subset \mathbb{F}^n \mid \dim V_i = k_i\}.$$

Example 22.12. When $m = 1$, $\mathrm{Fl}_n^{k_1} = \mathrm{Gr}(k_1, n)$.

Definition 22.13. When $m = n$, we have $\mathrm{Fl}_n^{1,2,\dots,n}$. This is called the *complete flag variety.*

Now let's define the combinatorial analogue.

Definition 22.14. A *combinatorial flag* is a strictly increasing sequence $F^1 \subset F^2 \subset \cdots \subset F^m$ of finite subsets of $[n]$. Let $k_i = \#F^i$. Then $(k_1, \dots, k_n)$ is called the *rank* of the flag.

Definition 22.15. The collection of all combinatorial flags of $(k_1, \dots, k_n)$ in $[n]$ is denoted $F_n^{k_1,\dots,k_n}$.

Now let's define an analogue of the Gale ordering. For every $\omega \in S^n$, we define the Gale ordering $\le^\omega$ on $F_n^{k_1,\dots,k_m}$ as follows: if $F = (F^1, \dots, F^m)$ and $G = (G^1, \dots, G^m)$ are two flags, we say $F \le^\omega G$ if and only if $F^i \overset{\omega}{\le} G^i$ for all i.

Example 22.16. Suppose $m = n$, and we're considering the complete combinatorial flags $(F^1, \dots, F^n)$ and $(G^1, \dots, G^n)$. By definition, $\#F^i = \#G^i = i$. So we can encode both F and G as permutations $(f_1, \dots, f_n)$ and $(g_1, \dots, g_n)$ where $F^i = \{f_1, f_2, \dots, f_i\}$ and similarly for G.

Then $F \overset{\omega}{\leq} G$ for $w = e$ corresponds precisely to the Bruhat order.

Definition 22.17. A collection $\mathcal{F}$ of combinatorial flags of rank $(k_1, \dots, k_m)$ is called a *flag matroid* if and only if $\mathcal{F}$ satisfies the maximality property: for every $\omega \in S_n$, the collection $\mathcal{F}$ contains a unique element which is maximal in $\mathcal{F}$ with respect to the ordering $\overset{\omega}{\leq}$.

Remark 22.18. One gets a (realizable) flag matroid from any point in a (partial) flag variety. Let $U_1 \subset U_2 \subset \cdots \subset U_m \in \mathbb{F}^n$ be a flag of subspaces of $\mathbb{F}^n$ of dimensions $k_1, \dots, k_m$.

Then each subspace U_1 represents a matroid of rank k_i.

Theorem 22.19. *The combinatorial flags*

$$\{B_1 \subset B_2 \subset \cdots \subset B_m \mid B_i \text{ a basis of } M_i\}$$

form a flag matroid.

22.6 Closing Stuffs

Flag matroids are a strict subset of Coxeter matroids. Here is the definition of a Coxeter matroid, even though it might not make sense.

Definition 22.20. Let P be a standard parabolic subgroup in a finite reflection group W and let $\mathcal{M} \subset W^p$, the set of minimal length coset representatives. We say that $\mathcal{M}$ is a *Coxeter matroid* if for any $w \in W$ there is a unique $A \in \mathcal{M}$ such that for all $B \in \mathcal{M}$, $B \overset{\omega}{\leq} A$.

22.7 Tableau Criterion for the Bruhat Order

Aside.

We have $(f_1, \dots, f_n) \leq (g_1, \dots, g_n)$ in the Bruhat order if and only if each $1 \leq i \leq n$the increasing rearrangement of $\{f_1, \dots, f_i\}$ is componentwise at most the increasing rearrangement of $\{g_1, g_2, \dots, g_n\}$.

Example 22.21. Suppose we wish to check whether $513624 \geq 312456$. We write the following for the permutations:

$$\begin{aligned} 5 &\geq 3 \\ 15 &\geq 13 \\ 135 &\geq 123 \\ 1356 &\geq 1234 \\ 12356 &\geq 12345 \\ 123456 &\geq 123456. \end{aligned}$$

Then the inequality is true if and only each of the $\binom{6+1}{2} = 21$ components are componentswise greater.

23 November 14, 2013

23.1 Hood Chatham: Toric Varieties

23.1.1 Toric Varieties

Toric varieties are nice geometric objects.

Definition 23.1. A *lattice* is a finitely generated free abelian group.

Example 23.2. $\mathbb{Z}^n$ is a lattice.

Definition 23.3. A *torus* is $N \otimes_{\mathbb{Z}} \mathbb{C}^* \approx (\mathbb{C}^*)^n$. A *toric variety* is a complex variety $X \supseteq (\mathbb{C}^\times)^r$ as a dense subset.

Example 23.4. Toric varieties include $(\mathbb{C}^*)^n$, $\mathbb{C}^n$, and $\mathbb{P}^n$.

Here $\mathbb{C}^* = \mathbb{C} - \{0\}$.

I no longer have any idea what's happening. Let me copy down some stuff from the board. (What's a co-character?)

$$\mathrm{Hom}(\mathbb{C}^*, (\mathbb{C}^*)^2) \cong \mathbb{Z}^2$$

with

$$s \mapsto (s^{a_1}, s^{a_2}).$$

We want to extend co-characters. Somehow.

Want to add in coordinate axes to complete $(\mathbb{C}^*)^2$.

23.1.2 Cone

$N \subseteq N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition 23.5. A rational polyhedral cone of N is the $\mathbb{R}_{\ge 0}$ span of a finite subset $S \subseteq N$.

These act a lot like polytopes.

Given a cone $\sigma \subseteq N_{\mathbb{R}}$, let $M = \mathrm{Hom}(N, \mathbb{Z})$ be the dual lattice of $N_{\mathbb{R}}$. Then the dual cone $\sigma^v \subseteq M_{\mathbb{R}}$ is defined by

$$\sigma^v = \{f \in M_{\mathbb{R}} \mid v \in \sigma, f(v) \ge 0\}.$$

The f are functionals.

darn I wish I knew more higher math

23.1.3 Fans

Definition 23.6. A fan of N is a collection of Σ of cones of $N_{\mathbb{R}}$ each such that

(i) all cones have 0 as a face

(ii) for each cone $\sigma \in \Sigma$, if τ is a face of σ then $\tau \in \Sigma$

(iii) if $\sigma, \sigma' \in \Sigma$ then $\sigma \cap \sigma'$ is a face of each.

Theorem 23.7. *Every normal toric variety is associated to a fan… in some way.*

Example 23.8. Consider projective two-space

$$\mathbb{P}^2 = (\mathbb{R}^3 - \{(0,0,0)\})/\sim$$

where $(x, y, z) \sim (\lambda x, \lambda y, \lambda z)$ for $\lambda \in \mathbb{C}^*$.

We have a map $(t_1, t_2) \mapsto (1, t_1, t_2)$ taking $(\mathbb{C}^*)^2 \to \mathbb{P}^2$. Consider a map $s \mapsto (1, s^a, s^b)$ and consider the limiting behavior as $s \to 0$.

- If $a, b > 0$, the limit is $(1, 0, 0)$.
- If $a < 0$ and $a < b$, we have $(1, s^a, s^b) = (s^{-a}, 1, s^{b-a})$ which approaches $(0, 1, 0)$.
- If $b < 0$ and $b < a$, we similarly get $(0, 0, 1)$.

The facets of the cones are the positive a-axis, positive b-axis, and $y = x$ (where $x \le 0$).

23.2 Emmanuel Tsukerman: Parametrizing Totally Nonnegative Flag Varieties

Main result follows.

Theorem 23.9 (ET)**.** *TNN flag varieties are cool.*

We don't have enough time to give a proof, so we provide intuition instead.

The theory of TNN flag varieties

- is an extension of total positivity
- comes up in high energy physics. See, e.g. "scattering amplitudes and the positive Grassmanian".
- appears in integrable systems

Recall the real flag variety consists of

$$\{V_1 \subset V_2 \subset \cdots \subset V_n \dim V_i = i\}.$$

Define $G = \mathrm{SL}_n(\mathbb{R})$ and let B^+ denote the upper triangular matrices in G.

Exercise 23.10. Show that G/B^+ is the set of flag varieties.

For each $1 \le i \le n-1$, we define a homomorphism $\varphi_i : \mathrm{SL}_2 \to \mathrm{SL}_n$ by taking the identity $n \times n$ matrix and inserting the argument into it, with the upper-left corner in the (i,i) position. Then define

$$x_i(m) = \varphi_i \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \qquad y_i(m) = \varphi_i \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix} \qquad s_i = \varphi_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Define U^- to be the set of matrices with 1's on the main diagonal and 0's below it.

Let $U^-_{\ge 0}$ denote the semigroup generated by the $y_i(p)$, where $p \in [0, \infty)$.

Definition 23.11 (TNN Flag Variety)**.** $(G/B^+)_{\ge 0} = \overline{\left\{ uB^+ \mid u \in U^-_{\ge 0} \right\}}$.

Let W denote the Weyl group of G, and let $R_{v,w}$ is the "Richardson variety" for some $v \le w$ both in S_n. Define

$$R^{>0}_{v,w} \overset{\text{def}}{=} R_{v,w} \cap \left(G/B^+\right)_{\ge 0}.$$

Definition 23.12. Let $w \in W$ with $w = s_{i_1} \dots s_{i_m}$ be a reduced expression. A subexpression v consists of replacing some of the s_i's with 1.

Example 23.13. $v = 1s_2 1 1 s_2 s_3$ is a subexpression of $w = s_3 s_2 s_1 s_3 s_2 s_3$.

Let $v_{(k)}$ denote the product of the leftmost k letters of v.

Definition 23.14. Letting $<$ be the Bruhat order,

$$\begin{aligned} J_v^0 &= \left\{ k \in [m] \mid v_{(k-1)} < v_{(k)} \right\} \\ J_v^+ &= \left\{ k \in [m] \mid v_{(k-1)} = v_{(k)} \right\} \\ J_v^\bullet &= \left\{ k \in [m] \mid v_{(k-1)} > v_{(k)} \right\}. \end{aligned}$$

Example 23.15. If $v = 1s_2 1 1 s_2 s_3$ then $J_v^0 = \{2, 6\}$, $J_v^+ = \{1, 3, 4\}$, and $J_v^\bullet = \{5\}$.

Definition 23.16. A subexpression is *non-decreasing* if $J_v^\bullet = \varnothing$.

Definition 23.17. A subexpression is *distinguished* if $v_{(j)} \le v_{(j-1)} s_{i_j}$ for all $j \in [m]$. Here the i_j come from w.

Definition 23.18. A subexpression is *positive distinguished* if it is distinguished and non-decreasing.

Definition 23.19. Let $v = 1s_2 1 1 s_2 s_3$ and $w = s_3 s_2 s_1 s_3 s_2 s_3$ as before. Check that v is distinguishable.

Lemma 23.20. *Given $v \le w$, there exists a unique positive distinguished subexpression v_+ for $v \in w$.*

Okay here is the main result.

Theorem 23.21. *Suppose $w = s_{i_1} \dots s_{i_m}$, $v \le w$, and v_+ is a positive distinguished subexpression. Then*

blah

23.3 Steven Karp: Shelling TNN Flag Varieties

We will be talking about reflection orders, introduced by Dyer.

Let $z = \{(i\ j) \mid 1 \le i < j \le n\}$ denote the set of transpositions in S_n.

Definition 23.22. A reflection order is a total order on $(z, \preceq)$ such that $(i\ k)$ is between $(i\ j)$ and $(j\ k)$ for $i < j < k$.

Example 23.23. When $n = 3$, the two reflection orders are $12 \preceq 13 \preceq 23$ and $23 \preceq 13 \preceq 12$.

As an example, a lexicographic order is a reflection order. Of course, we want more than that.

Definition 23.24. For $I \subset [n-1]$, then $\langle I \rangle = \langle (i\ i+1) \mid i \in I \rangle \subseteq S_n$.

Example 23.25. If $n = 4$, $I = \{1, 3\}$ then $\langle I \rangle = \langle (1, 2), (3, 4) \rangle$ which is a subgroup of order 4.

Here is the key lemma.

Lemma 23.26 (Dyer)**.** *For disjoint $I, J \subseteq [n-1]$ there exists a reflection order such that the transpositions in $\langle I \rangle$ come first and those in $\langle J \rangle$ comes last.*

w	$\#v : v \le w$
12 34	1
13 24	2
14 23	4
23 14	4
24 13	8
34 12	14

Table 1: The sum is 33.

Example 23.27. If $I = \{1, 3\}$ and $J = \{2\}$, then one reflection order which works is

$$(3\ 4) \preceq (1\ 2) \preceq (1;4) \preceq (1\ 3) \preceq (2\ 4) = preceq(2\ 3).$$

Applications:

- One can find an EL-labeling of S_n. Label each edge of the Hasse diagram as follows: if $x \lessdot y$, and $y = tx$, then we label the edge with $t \in z$. (We did this in class). This turns out to be an EL-labeling with respect to *any* reflection order.

  One can even generalize this to EL-label any Coxeter group.

- We can EL-label the face poset of $\mathrm{Gr}_{k,n}^{\ge 0}$ (Williams). The parametrizations are due to Retsch. Its faces are enumerated by postroids, Grassmann necklaces. To tie these into reflection order, we use a parametrization of partial flag varieties.

Definition 23.28. An $(n-k, n)$-Grassmannian permutation in S_n is $w \in S_n$ such that has at most one descent; furthermore, this descente must be at the $n-k$th position. In other words,

$$w(1) < \dots < w(n-k) \text{ and } w(n-k+1) < \dots < w(n).$$

Equivalently, w is a minimal length coset representative of the symmetric group modulo the subgroup generated by all adjacent transpositions other than the kth one.

Example 23.29. $w = 23568147$ is a $(5, 8)$-Grassmannian.

So our next application is

Theorem 23.30 (Retsch)**.** *The cells of $Gr_{k,n}^{\ge 0}$ are indexed by pairs (v, w) such that $v \le w$ and W is $(n-k, n)$ Grassmanian.*

Example 23.31. Take $k = 2, n = 4$. We obtain the table below.

So how do we relate positroids to transpositions? The answer is decorated permutations.

Definition 23.32. For $A \in \mathrm{Gr}_{k,n}$ define the decorated permutation $\pi^{:}_A$ by

$$\pi^{:}_A(i) = j \pmod n$$

where $j \le i$ is "maximal" such that $A^{(i)}$ is contained in the span of $A^{(i-1)}A^{(i-2)}, \dots, A^{(j)}$ where all indices are taken modulo n.

Each fixed point is labeled with a dot or a cycle, according to whether we need to go all the way back or not.

Example 23.33. Let $A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$. We get $\pi^{:}_A = 3421$. Pick each column and cycle backwards until the current column is in a span of these columns.

It's not obvious this is a permutation, but it is.

Example 23.34. Let $B = \begin{bmatrix} 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. We obtain $\pi^{:}_B = \dot{1}4\hat{3}2$.

This explains how to label the fixed points. Label everything to the left of a divider with a dot...

24 November 19, 2013

qq flu.

25 November 21, 2013

25.1 Kim Dae Young - Discrete Morse Theory

Recall some stuff about discrete Morse theory.

We will talk about gradient vector fields and the Morse complex.

Question 25.1. Given a simplicial complex K, how can we find a discrete Morse function?

$f(\alpha) = \dim \alpha$ works, but is not interesting.

We will pair up noncritical cells. If $\alpha^{(p)} < \beta^{(p+1)}$ and $f(\alpha) \geq f(\beta)$, then we draw an arrow from α to β. We thus observe that each simplex α of K is either the head/tail of an arrow if it is noncritical (and neither iff it is critical).

Definition 25.2. A *discrete vector field* V on K is a collection $\left\{\alpha^{(p)} < \beta^{(p+1)}\right\}$ of simplices of K such that every simplex is contained in at most one pair of V.

Definition 25.3. For a given discrete vector field V on K, a *V-path* (or *gradient path*) is a sequence of simplices

$$\alpha_0^{(p)}, \beta_0^{(p+1)}, \alpha_1^{(p)}, \beta_1^{(p+1)}, \dots, \beta_r^{(p+1)}, \alpha_{r+1}^{(p)}$$

such that for each $n = 0, 1, \dots, r$, $\{\alpha_n < \beta_n\}$ in V and $\beta_n > \alpha_{n+1} \neq \alpha_n$.

Let K be a simplicial complex with a discrete Morse function f. Let $C_p(K, \mathbb{Z})$ denote the space of p-simplicial chains and $\mu_p \subseteq C_p(K, \mathbb{Z})$ bet he span of critical p-simplices.

Theorem 25.4. *There are boundaries $\tilde{\partial}_d : \mu_d \to \mu_{d-1}$ such that $|tilde\partial_{d-1} \circ \tilde{\partial}_d = 0$ for all d and such that the resulting differential complex*

$$0 \to \mu_n \xrightarrow{\tilde{\partial}_n} \mu_{n-1} \to \cdots \to \mu_1 \xrightarrow{\partial_1}\to 0$$

can compute the homology of K.

Theorem 25.5. *Choose any orientation of each simplex of K. For any critical $(p+1)$-simplex β,*

$$\tilde{\partial}\beta = \sum_{\text{critical } p\text{-simplex } \alpha} c_{\alpha,\beta}\alpha$$

where

$$c_{\alpha,\beta} = \sum_{\gamma \in \Gamma(b,a)} m(\gamma)$$

where $\Gamma(b, a)$ is the set of V-paths from a maximal face of β to α and $m(\gamma) = \pm 1$ depending on whether given γ the orientation of β induces our chosen orientation on α.

25.2 Joe Kileel - The Tropical Semiring

The word "tropical" is a reference to Brazil and nothing else.

Definition 25.6. The tropical semiring is $(\mathbb{R} \cup \infty, \oplus, \otimes)$ where $a \oplus b = \min\{a, b\}$ and $a \otimes b = a + b$.

Example 25.7. $1 \otimes (4 \oplus 5) = 1 \otimes 4 = 5$.

Remark 25.8. The identity of $\oplus$ is ∞; the identity of $\otimes$ is 0.

This satisfies all the axioms of a ring other than additive inverses.

Proposition 25.9 (Freshman's Dream)**.** *$(a \oplus b)^n = a^n \oplus b^n$ for all a and b.*

Here is a motivation: if you want to assign n workers to n jobs, and c_{ij} is the cost of assigning worker i to job j, the minimum cost is given by the tropical permanent

$$\bigoplus_{\pi \in S_n} \otimes_{i=1}^n C_{i\pi(i)}$$

Now we take tropical polynomials, say

$$p(x) = x^2 \oplus 2x \oplus 5.$$

This can be written as $\min\{2x, 2+x, 5\}$; the "roots" are the points with kinks.

25.3 Josh Wen - Moment maps and matroids

Set $I_n = \{1, 2, \dots, n\}$ and let $B_k(I_n) = \binom{I_n}{k}$.

Let $X, Y \in \mathrm{Gr}_k(\mathbb{C}^n)$, $J \in B_k(I_N)$, and $\mathbb{C}^J = \bigoplus_{i \in J} e_i$. We will write $X \sim Y$ if $\dim(X \cap \mathbb{C}^J) = \dim(Y \cap \mathbb{C}^J)$.

Oh no I heard Schubert cells I give up.

26 November 26, 2013

26.1 Evan Chen: Acyclic Orientations modulo Click Sequences

Did not go overtime!

26.2 Qingyun Wu: Two Poset Polytopes

Let $P = \{x_1, \dots, x_n\}$.be a poset.

Definition 26.1. $\mathbb{R}^P$ denotes the set of all functions $f : P \to \mathbb{R}$.

Definition 26.2. The *order polytope* corresponds is the subset of $\mathbb{R}^P$ given by

$$\{f : 0 \le f(x) \le 1 \text{ and } f(x) \le f(y) \quad \forall x \le_P y\}.$$

Definition 26.3. The *chain polytope* is given by

$$\left\{ g \in \mathbb{R}^p : \min_{x \in P} g(x) \ge 0 \text{ and } g(y_1) + \dots + g(y_k) \le 1 \text{ for any chain } y_1, \dots, y_k \right\}.$$

http://link.springer.com/content/pdf/10.1007/BF02187680.pdf

26.3 Henry Maltby: The Greedy Algorithm on Matroids

Example 26.4. Prim and Kruschev find minimal spanning trees.

Okay, to matroids. We have a matroid

$$M = (E, \mathbb{I})$$

and a weight function $w : E \to \mathbb{R}^+$.

1. Start with $J = \varnothing$.
2. Add to J the cheapest element $e \in E$ such that $J \cup \{e\}$ is still independent.
3. Repeat until we have a basis.

The gives us a w-minimal bases, where the weight of a basis B is given by

$$w(B) = \sum_{b \in B} w(b).$$

This corresponds exactly to spanning trees (forests) if M is a graphical matroid.

Recall that we had the nice result that if $\mathcal{I}$ is a simplicial complex and the greedy algorithm works for all weight functions $w : E \to \mathbb{R}^+$ then $(E, \mathcal{I})$ is a matroid.

We wish to generalize this. So we build poset matroids.

Let P be a poset and $\mathcal{I}$ be a set of filters of P. We wish to have the following properties.

1. If $Y \in \mathcal{I}$ and $X \subseteq Y$ then $X \ni \mathcal{I}$.
2. If $X, Y \in \mathcal{I}$ with $|X| < |Y|$ there exists y maximal in $Y - X$ such that $x \cup \{y\} \in \mathcal{I}$.

Observe that if is a poset with no relations, then this is precisely our original definition of matroid.

We again impose a weight function $w : P \to \mathbb{R}^+$ preserving the order of w. Then we can do a greedy algorithm again to find a minimal basis.

1. Start with $J = \varnothing$.
2. Add to J the cheapest element x such that $J \cup \{e\} \in \mathcal{I}$.
3. Continue to a basis.

Spanning acyclic subcomplexes! Greedoids.

27 December 3, 2013

All three speakers discuss "positroids and non-crossing partitions".

27.1 Anastasia Chavez

Goals for this talk: cover

- Positroid M
- Matroid polytope Γ_M
- The poset NC_n^d
- State main theorem

Definition 27.1. Suppose A is a real $d \times n$ rank d matrix such that all maximal minors are nonnegative. Call A *totally nonnegative*, and the representable matroid $M(A)$ as *positroid.*

We will be thinking about matroids via the basis definition.

Example 27.2. Let M with ground set $\{1, 2, 3\}$ and bases $\{12, 13, 23\}$. A working A is

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

This is good because all determinants are one. Note this is not true if we swap the 1 and -1 in the third column.

Lemma 27.3. *Let M be a positroid on $E = \{1 < 2 < \cdots < n\}$. Then for any $1 \le a \le n$, M is also a positroid on the ordered ground set $E = \{a < a+1 < \cdots < n < 1 < \cdots < a-1\}$.*

Proposition 27.4. *Suppose we have a decomposition of $[n]$ into two cyclic interval $A = [\ell+1, m]$ and $A' = [m+1, \ell]$. Consider the positroid M over A, M' over A'. Then $M \oplus M'$ is a positroid over the ordered set $\{1 < \cdots < n\}$.*

Recall the definition of a matroid polytope. Now let's define a non-crossing partition.

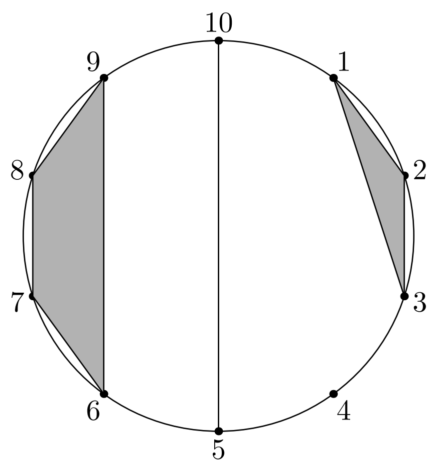

Figure 18: A non-crossing partition $\{1, 2, 3\} \sqcup \{4\} \sqcup \{5, 10\} \sqcup \{6, 7, 8, 9\}$.

Definition 27.5. A *non-crossing partition* is defined as follows.

Theorem 27.6. *Let M be a positroid over $[n]$ and let $S_1, \dots, S_t$ be the ground set of the connected components of M. Then $\Pi_M = \{S_1, \dots, S_t\}$ form a non-crossing partition of $[n]$, called the non-crossing partition of M.*

Definition 27.7. We can weight each of our partitions by assigning a nonnegative integer $0 \le w_i \le |S_i|$ to each S_i in the partition. We write this as S^w.

Definition 27.8. NC_n^d is the poset of non-crossing partitions of $[n]$ of weight d (meaning S^w has the sum of w_i equal to d). The cover relation $S^w \lessdot T^v$ occurs when S is a refinement of T and merging blocks of S to form a block of T corresponds corresponds to adding weights.

Now we have the main theorem.

Theorem 27.9 (Williams et al)**.** *If M is a rank d positroid on $[n]$ then the face poset of Γ_M is an induced subposet of $NC_n^d \cup \{\hat{0}\}$.*

27.2 Jun Hong

Bijecting positroids to Grassmann necklaces and decorated permutations.

Definition 27.10. A *Grassmann necklace* of type (d, n) is a sequence

$$(I_1, \dots, I_n)$$

of subsets of $[n]$ each with cardinality d, such that

- $i \in I_i \Rightarrow I_{i+1} = I_i - i + j$ for some $j \in [n]$
- $i \notin I_i \Rightarrow I_{i+1} = I_i$.

We can go from positroids to Grassmann necklaces by taking the lexicographic minimum $I_k = \min_{<_k} \mathcal{B}$. Here $<_k$ is the cyclic order $k <_k k+1 <_k \cdots <_k n <_k 1 <_k \cdots <_K k-1$. The other direction goes by

$$\mathcal{B} = \left\{ B \in \binom{[n]}{d} \mid B \ge_k I_j \forall j \right\}.$$

Theorem 27.11. *Given a Grassmann necklace $(I_1, \dots, I_n)$ we get a positroid $M = ([n], \mathcal{B})$ where $\mathcal{B}$ is defined above.*

...

27.3 Benson Au

Positroids and free probability.

What is free probability? It can be described as "noncommutative probability plus free independence."

Classical probability studies $(\Omega, \mathcal{F}, P) \Leftrightarrow (L^\infty(\Omega, P), E)$ which confuses me because I happen to not know what L^∞ is. There's an operation $*$ which is an involution conjugate linear anti-isomorphism. We have

$$E[f^* f] \ge 0 \text{ with equality iff } f \equiv 0.$$

Definition 27.12. A $*$-prob space is a pair $(\mathcal{A}, e)$ where $\mathcal{A}$ is a unital $*$-algebra over $\mathbb{C}$ and $\phi \in A_{+,1}$ in the sense that $\phi(a^*a) \ge 0$ and $\phi(1) = 1$. ϕ is called the *expectation operator*.

Example 27.13. $(L^\infty(\Omega, P), E)$ is one. For a second example, take

$$(M_n(\mathbb{C}), \tfrac{1}{n}\mathrm{tr})$$

where $M_n(\mathbb{C})$ is the set of $n \times n$ deterministic matrices.

Let's copy down more stuff.

Definition 27.14. Assume $*$-probability space (A, ϕ) with a family $(A_i)_{i\in}I$ of unital $*$-subalgebras. This family is *free* or $*$-*free* if $\phi(a_1, a_2, \dots, a_n) = 0$ whenever $\phi(a_j) = 0$ for all j, and if $a_j \in A_{k(j)}$ then $k(j) \ne k(j+1)$.

Okay I quit I have no idea what's happening. It will probably make sense in four years. Or if I know what classical probability theory, lol.

28 December 5, 2013

28.1 Moor Xu: Eulerian Posets

Definition 28.1. A graded poset P with rank function ρ is *Eulerian* if

$$\mu(s,t) = (-1)^{\rho(s,t)}$$

for any $s \le t$ in P. Here $\rho(s,t) = \rho(t) - \rho(s)$.

Recall that $\mu = \zeta^{-1}$ from very early lectures, so this is equivalent to $\sum_{u \in (s,t)} (-1)^{\rho(s,u)} = 0$; that is,

> P is Eulerian if any interval we have the same number of odd and ever rank elements.

Example 28.2. Chains are non-examples because any interval of length 3 has two elements of some parity and one of the other. On the other hand, the standard Bruhat order on S_3 is Eulerian.

Example 28.3. The face poset of any convex polytope is Eulerian. In particular, the Boolean lattice B_n is Eulerian. More generally, any CW-decomposition of a sphere is Eulerian.

We try to generalize our polytope results to Eulerian posets.

28.1.1 h-polynomial

First, we try to generalize the notion of an h-polynomial. For P an Eulerian poset, we can write down a polynomial $h(P,x)$, constructed inductively. Sketch of construction below.

First, if $P = \{1\}$ we define

$$h(1,x) = g(1,x) = 1.$$

If P is a poset of rank $n+1 > 0$, then we set

$$h(P,x) = h_0 + h_1 x + \cdots + h_n x^n$$

and

$$g(P,x) = h_0 + (h_1 - h_0)\,x + \cdots + (h_n - h_{n-1}) x^{\lfloor \frac{1}{2} n \rfloor}.$$

Then define $h(P,x) = \sum_Q g(Q,x)(x-1)^{n-\rho(Q)}$.

This is called the toric h-polynomial. This coincides with the h-vector for a polytope.

The Dehn-Sommerville equations generalize to $h_i = h_{n-i}$ in this case, and this coincides with the h-vector for a polytope.

28.1.2 cd-index

Definition 28.4. If P is a graded poset, let f_i be the number of elements of rank i, and let the f-vector be $f(P) = (f_1, f_2, \dots, f_n)$.

Definition 28.5. Let $S \subseteq [n]$ with $S = \{a_1 < a_2 < \cdots < a_k\}$. Then the *rank-selected subposet* P_S of P is the set of

$$P_S = \{t \in P : \rho(t) \in S\} \cup \left\{\hat{0}, \hat{1}\right\}.$$

Then set $\alpha_P(S)$ be the number of maximal chains in P_S. This is the flag f-vector of P.

Definition 28.6. The flag h-vector is

$$\beta_p(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_P(T).$$

Define $u_s = u_1 u_2 \dots u_n$ for each $S \subseteq [n]$ by

$$u = \begin{cases} a & i \notin S \\ b & i \in S. \end{cases}$$

Let $\Psi_P(a,b) = \sum_{S \subseteq [n]} \beta_p(s) u_s$. The variables do not commute.

Theorem 28.7. *If P is an Eulerian poset, $\Psi_P(a,b)$ can be written as a polynomia in variables $c = a+b$ and $d = ab+ba$. This is the* cd-index.

It is not always true that the coefficients are nonnegative, but this is often the case. Stanley showed that the *cd*-index is positive for S-shellable posets, and conjectured that they were positive for larger classes. Karua (2006) showed this true for Govenstein$*$ posets.

28.2 Ryan Thorngren: The Hard Lefschetz Theorem and Polytopes

Given a polytope P of dimension d, we can form the face vector $\vec{f}$ where the jth component is the number of faces of dimension j, where $0 \le j \le d-1$.

Question 28.8. What characterizes this f-vector?

We have that

$$\sum (-1)^j f_j = 1 - (-1)^d.$$

The following theorem is true.

Theorem 28.9. *This is the only linear relation satisfied for all $\vec{f}$.*

Proof. The cases $d = 1, 2$ are clear, so we induct. Suppose we have some other relation

$$\sum_j \alpha_j f_j = \beta.$$

Construct a pyramid p^* and double-pyramid p^{**} (whatever that is). Allegedly we can compute

$$\vec{f}(P^*) = (1 + f_0, f_0 + f_1, \dots, f_{d-1} + 1)$$

and

$$\vec{f}(P^{**}) = (2 + f_0, 2f_0 + f_1, \dots, 2f_{d-1}).$$

Subtracting these two from our hypothetical relation, we obtain that

$$\sum_{0 \le j \le d} \alpha_{j+1} f_j(P) = \alpha_d - \alpha_0.$$

Now apply the inductive hypothesis. □

Question 28.10. What about simplicial P?

Remark 28.11. In this case, the incidence algebra is controlled: every k-face is adjacent to exactly $\binom{d-j}{d-k}$ j-faces where $j \le k$. Every k-face is contractible, so

$$\chi = 1 = \sum_{i \le j \le k} (-1)^j \#\ j\text{-faces adjacent to some } k\text{-face}.$$

From this, we can derive the Dehn-Sommerville relations, that is

$$f_k = \#k\text{-faces} = \sum_{0 \le j \le k} (-1)^j \binom{d-j}{d-k} f_j.$$

Theorem 28.12. *These are the only linear relations.*

These are still not sufficient: even if an abstract f-vector satisfies these conditions, it is not necessarily the face vector of a simplicial P.

28.3 Justin Chen: How to Shell a Monoid

Definition 28.13. $\Lambda \subseteq \mathbb{N}^d$ will be a submonoid, finitely generated by generators $\{\alpha_1, \dots, \alpha_n\}$. Define $k[\Lambda] = k[z^\lambda \mid \lambda \in \Lambda] \subseteq k[z_1, \dots, z_d]$.

Define a homomorphism $k[x_1, \dots, x_n \to k[\Lambda]$ by $x_i \mapsto z^{\alpha_i}$. Let $I(\Lambda)$ be the kernel, so that

$$k[x_1, \dots, x_n]/I(\Lambda) \cong k[\Lambda].$$

Definition 28.14. Define the numbers

$$\beta_i^\lambda(k) = \dim_k \operatorname{Tor}^{k[\Lambda]}(k,k)_\lambda.$$

Proposition 28.15.

$$I(\Lambda) = \langle x_{i_1} x_{i_2} \dots x_{i_r} - x_{j_1} \dots x_{j_s} \mid \alpha_{i_1} + \dots + \alpha_{i_r} = \alpha_{j_1} + \dots + \alpha_{j_s} \rangle.$$

Theorem 28.16 (L-S)**.** $\beta_i^\lambda(k) = \dim_k \tilde{H}_{i-2}(\Delta(\lambda), k)$ *where* $\Delta(\lambda)$ *is the order complex of* $(0, \lambda)$.

Define a partial order $\le_\Lambda$ on Λ by $\alpha \le_\Lambda \beta \Leftrightarrow \beta - \alpha \in \Lambda$.

Definition 28.17. $k[\Lambda]$ is Koszul if the minimal free resolution of k (over $k[\Lambda]$) is linear.

Theorem 28.18. $k[\Lambda]$ *is Koszul if and only if* $\Delta(\lambda)$ *are Cohen-Macauley, that is for all* $\lambda \in \Lambda$, $\tilde{H}_i((\mu_1, \mu_2), k) = 0$ *except in the top degree.*

Theorem 28.19 (Thouh, Vu 2013)**.** $v(d,n) - \{a\}$ *is Koszul except if* $a = (0, \dots, 2, d-2)$ *or a permutation.*