Math 179: Graph Theory

Evan Chen

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Notes for the course M179: Introduction to Graph Theory, instructed by Wasin So.

1 January 23, 2013

1.1 Logistics

Website: `www.math.sjsu.edu/~so` contains most of the information. Three handouts: syllabus, calendar, homework

Textbook: *A First Course in Graph Theory*. "The reason I choose this book is because it's cheap. All the graph theory books are isomorphic." We will cover ten chapters.

The grade will consist of:

Homework (20%) 10 assignments. Each chapter will have its own homework; 5 problems for each chapter. Solutions will be posted afterwards. Two assignments will be dropped.

Project (10%) Paired.

Test (30%) Two tests, 15% each. Already on calendar.

Final (40%) Final exam. Wednesday, May 15, 2013, 2:45 PM.

The usual scale applies. However $A \geq 90\%$, $A+ \geq 94\%$ and $A- \geq 87\%$.

1.2 Introductory Question

Problem. Which letters can be drawn without lifting the pencil or double tracing?

`BCDLMNOPQRSUVWZ`, not `AEFGHIJKLTXY`. Yeah this is just Eulerian tours, okay.

Problem. Four color theorem on CA map.

Three colors is not sufficient if there is an odd cycle whose vertices are all connected to some other vertex.

Problem. Instant insanity.

Solution. What matters is the opposite color.

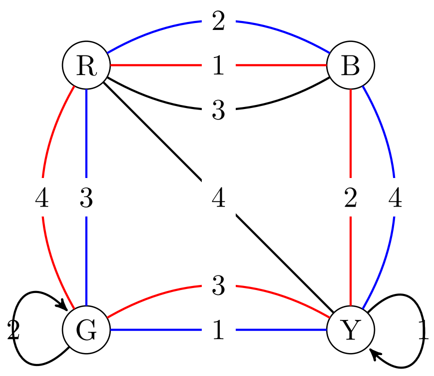


Figure 1: Instant Insanity

We need only find four cycles whose edges each have 1,2,3,4, which are disjoint; they are colored in the above diagram. This corresponds to a solution. □

1.3 Definitions

Definition. A graph G is a set of vertices V along with a set of edges E.

All three problems can be abstracted into graphs as described.

2 January 28, 2012

2.1 Today's Topics

Chapter 1:

- graph
- subgraph
- Walk, trail, path
- Closed walk, circuit, cycle
- Connected graph
- Disconnected graph

2.2 Preliminary Definitions

Definition. A (simple) graph G is an ordered pair (V, E), where V is a nonempty set, and E is a collection of 2-subsets of V. V is sometimes call deth vertex set of G , and E is called the edge set of G.

Example. Let $V = \{1, 2, 3\}$ and $E = \{\{1, 2\}, \{1, 3\}\}$. This gives a graph G_1.

Example. Let $V = \{1, 2, 3\}$ and $E = \{\{1, 2\}, \{2, 3\}\}$. This gives a graph G_2.

Remark. Such G is called a labelled graph.

2.3 Isomorphism

For human beings, this is not very nice. So, we visualize a graph as a picture. These

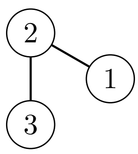


Figure 2: G_1

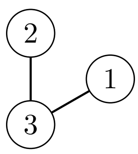


Figure 3: G_2

two graphs are different, since their edges are different. But if we eliminate the labelling (i.e. we take the unlabelled graph) then these graphs are not the same. In other words, these graphs are *isomorphic*.

We want to study graphs, structurally, without looking at the labelling. That is, the more interesting properties of a graph do not rely on the labelling. But labels are useful for implementation, or if we want to introduce some sort of asymmetry.

Sometimes it is hard to tell that graphs are isomorphic. Really. It's NP-hard.

Definition. The *order* of a graph G is $|V|$. The *size* of G is $|E|$.

Definition. A *trivial graph* is a graph with order 1. An *empty graph* is a graph of size 0.

Note that a graph must have at least one vertex by definition. But a graph can certainly have no edges!

For now we are not permitting loops, so trivial graphs are necessarily empty.

2.4 Subgraph

Definition. Let G be a graph. H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Example. Let G_3 be the graph with $V = \{1, 2, 3\}$ and $E = \{\{1, 2\}, \{2, 3\}, \{3, 1\}\}$ Then G_1, G_2 are subgraphs of G_3 but G_1 is not a subgraph of G_2. The subgraph is *proper* if either of these inclusions is strict.

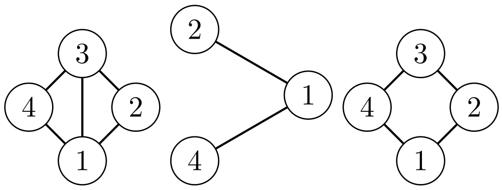
This is a fairly strong condition, but here are some other conditions.

Definition. H is a *spanning* subgraph if $V(H) = V(G)$.

Example. G_1 is a spanning subgraph of G_3.

Most important definition:

Definition. H is an *induced subgraph* if $uv \in E(H) \iff uv \in E(G) \forall u, v \in V(H)$.



In the above figures, the second but not the third graph is an induced subgraph of the first.

2.5 Walks

Definition. A $u-v$ walk in a graph G is a finite sequence of vertices $(u = v_0, v_1, \cdots, v_k = v)$ such that v_i and v_{i+1} are adjacent for each $0 \le i \le k - 1$.

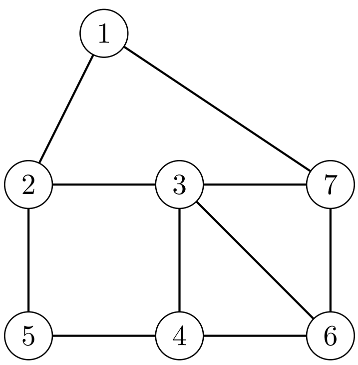


Figure 4: Yet another graph

Example. The following are all $1 - 7$ walks:

- $(1, 2, 3, 4, 5, 2, 3, 7)$
- $(1, 7)$
- $(1, 2, 3, 7)$
- $(1, 2, 3, 6, 4, 3, 7)$
- $(1, 7, 1, 7)$

Note that $(1, 7, 7)$ is not a walk since $\{7, 7\}$ is not an edge. Also, $(1, 7, 1)$ is a $(1, 1)$ walk.

Definition. A walk is closed if and only if $u = v$. Otherwise, it is open.

Definition. A *trail* is an open walk without repeating vertices.

Definition. A *path* is an open walk without repeating vertices.

Note that paths are trails, but not vice-versa.

Definition. A circuit is a closed walk without repeating edges.

Definition. A cycle is a closed walk without repeating vertices, other than the initial and terminal vertices.

Theorem 2.1 (Chartrand and Zhang, 1.6)**.** *If there is a $u - v$ walk between two vertices, then there is also a $u - v$ path.*

Proof. Take the minimal walk. If it's not a path trim it, contradiction. □

2.6 Connectedness

Definition. A graph G is *connected* if $\forall u, v \in V(G)$ there exists a $u - v$ path.

Definition. Let G be a connected graph. Then $d(u, v)$ is the smallest length of any $u - v$ path if $u \neq v$, or 0 if $u = v$.

Example. In figure 4,

$$\begin{aligned}
d(1, 2) &= 1 \\
d(1, 3) &= 2 \\
d(1, 4) &= 3 \\
d(1, 5) &= 2 \\
d(1, 6) &= 2 \\
d(1, 7) &= 1
\end{aligned}$$

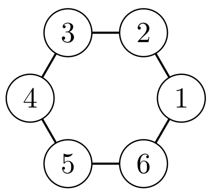


Figure 5: A cycle graph

In the cycle graph C_6 (figure 5), we have $d(1, 3) = 2$ and $d(1, 4) = 3$.

Definition. The diameter of a connected graph G is defined as

$$G = \max_{u \neq v} d(u, v).$$

Example. The diameter of 4 is 3.

Remark. By construction, $d(u, v) \leq \operatorname{diam}(G) \forall u, v \in V(G)$.

Remark. The *six degrees of separation* suggests that $\operatorname{diam}(G) \approx 6$, where V is the set of people, and E represents acquaintance.

How do we check if a graph is connected?

Theorem 2.2 (Chartrand and Zhang, 1.10)**.** *If* $\operatorname{ord} G \geq 3$ *then* G *is connected if and only if there exist vertices* $u \neq v$ *in* G *such that* $G - u$ *and* $G - v$ *are both connected.*

Proof. First we prove the condition is sufficient. We wish to show x and y are connected for any $x, y \in V(G)$. If $\{x, y\} \neq \{u, v\}$ it's trivial; otherwise, take some vertex w not equal to either $\{u, v\}$ and walk from u to w then v. This is possible since $\operatorname{ord} G \geq 3$.

For the converse, consider a connected graph G. Pick a diameter $u - v$ of G. We claim this works. Assume on the contrary that $G - u$ is not connected. Then $\exists x, y \in V(G)$ such that all $x - y$ paths contain u. Now consider v. If x and v are not connected in $G - u$ then $x - u - v$ is longer than $u - v$. So x and v are connected. Similarly y and v are connected. So x and y are connected. □

To wrap up:

Definition. A graph is disconnected if it is not connected.

So we can measure the disconnected-ness by looking at *connected components.*

Definition. Given a graph G, then we define $\sim$ by $u \sim v$ when there is a $u - v$ path. Then $\sim$ is an equivalence relationship, and its classes are the connected components.

3 January 30, 2013

Reminder: homework due next Monday.

3.1 Today's Topics

- Graph classes
- Graph operators

Last time we learned about

- connected graph (diameters)
- disconnected graphs (number of components)

3.2 Classes of graphs

Here are some "prototypcial" graphs.

Empty graph We let E_n denote the empty graph with order n and size 0. This graph is disconnected if and only if $n \geq 2$.

Path graph We let P_n be the graph of order n and size $n-1$. You can guess what this is: It is connected with diameter $n-1$.

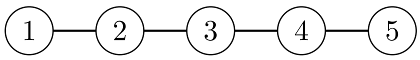


Figure 6: The graph P_5

Cycle graph We let C_n denote the graph of order n and size n which consists of a single cycle. Note that $n \geq 3$. It is connected with diameter $\left\lfloor \frac{1}{2} n \right\rfloor$.

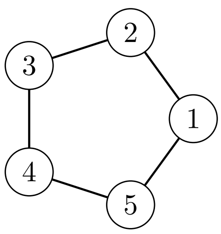


Figure 7: The graph C_5

Complete Graph We let K_n denote the complete graph, with order r and size $\binom{n}{2}$. It is extremely connected with diameter 1 (for $n \geq 2$). Note that this is the only class of connected graphs with diameter 1.

Note that $E_1 = K_1 = P_1$, $K_2 = P_2$ and $K_3 = C_3$.

3.3 Bipartite Graphs

Some more general graphs:

Bipartite graph A graph G is bipartite if $V(G) = U \sqcup V$ for some nonempty U, V such that if $u_1u_2, v_1v_2 \notin V(G)$ for $u_{,}u_2 \in U$, $v_1, v_2 \in G$. In other words, the induced subgraphs $G[U]$ and $G[W]$ are completely disconnected. In still other words, G has chromatic number at most 2 and $|V(G)| \geq 2$.

For example, E_n is trivially bipartite for $n \geq 2$. Similarly, P_n is bipartite if $n \geq 2$. (In fact, all trees are bipartite.) More interestingly, C_n is bipartite if and only if n is even. And of course, K_n isn't bipartite unless $n = 2$.

Complete bipartite graph For integers s, t the graph $K_{s,t}$ is a bipartite graph with s vertices of one color and t vertices of the other color, and any two vertices of the opposite colors are joined. The order is $s + t$ and size is st. The diameter is at most 2.

Theorem 3.1 (Chartrand and Zhang, 1.12)**.** *A graph G is bipartite if and only if G has no odd cycle.*

Proof. If there is an odd cycle we clearly lose (just take a cycle $C = (c_0, c_1, \cdots, c_{2k+1})$ with $c_{2k+1} = c_0$). Color chase.

To show the condition is sufficient, split into connected components. Fix a vertex v_0. Then color a vertex v red iff $d(v_0, v)$ is even and blue otherwise. In particular, v_0 is red. This works as follows: consider the paths from v_0 to u and v, where u, v are the same color. Take a vertex k on these paths (possibly equal to v_0) for which the geodesics from k to u and k to v are disjoint. This is clearly possible. ☐

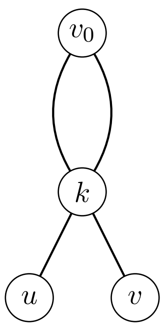


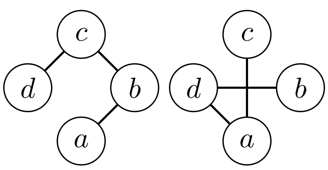
Figure 8: Long paths

In general it is pretty hard to show that there are no odd cycles.

3.4 Graph Operations

Graph complement Given $G = (V, E)$, the complement $\bar{G}$ is defined by $V(\bar{G}) = V(G)$ and $E(\bar{G}) = \{uv : uv \notin E(G)\}$.

For example, the following two graphs are complements:



Note that if $\bar{E}_n = K_n$. It is not necessarily easy to tell if the complement of a connected graph is connected, but we have a homework problem

Theorem 3.2 (Chartrand and Zhang, 1.11)**.** *If G is disconnected, then $\bar{G}$ is connected and* $\operatorname{diam}(G) \le 2$.

Finally, note, of course that $\overline{\overline{G}} = G$.

Graph union Given G and H with disjoint vertex sets, the graph union of G and H, denoted $G \cup H$, is defined by $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. Note that $k(G \cup H) = k(G) + k(H)$. (Here $k(G)$ is the number of components.)

Graph joint Given G and H with disjoint vertex sets, then the graph joint, denoted $G + H$, is defined by

$$\begin{aligned} V(G+H) &= V(G) \cup V(H) \\ E(G+H) &= E(G) \cup E(H) \cup \{uv : u \in V(G), v \in v(H)\} \end{aligned}$$

For example, $K_{s,t} = E_s + E_h$ and $\bar{K}_{s,t} = K_s \cup K_t$.

In general, $\overline{G+H} = \bar{G} \cup \bar{H}$, or $G + H = \overline{\bar{G} \cup \bar{H}}$. So a joint is always disconnected (by, say, Chartrand and Zhang, 1.11).

Graph product Given G and H with disjoint vertex sets, the graph product $G \times H$ is defined by:

$$\begin{aligned} V(G \times H) &= V(G) \times V(H) \\ E(G \times H) &= \{(u,x)(v,x) \mid uv \in V(G)\} \cup \{(x,u)(x,v) \mid uv \in V(H)\} \end{aligned}$$

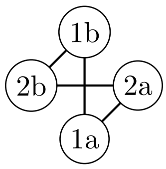


Figure 9: $K_2 \times K_2$, with vertex sets $\{1,2\}$ and $\{a,b\}$

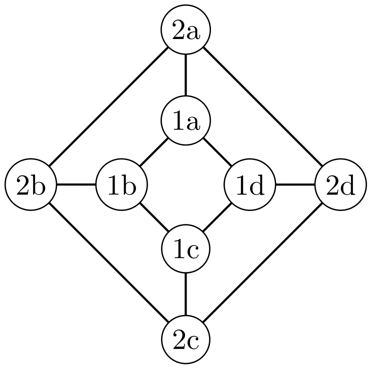


Figure 10: $K_2 \times C_4$

Note that $(G_1 \cup G_2) \times H = (G_1 \times H) \cup G - 2 \times H)$.

The hypercube can be realized as $Q_n = \underbrace{K_2 \times K_2 \times \cdots \times K_2}_{n \text{ times}}$.

4 February 4, 2013

4.1 Today's Topics

Chapter 2.

- degree
- First Theorem of Graph Theory
- high degree $\implies$ connectedness
- regular graph

4.2 The Degree of a Vertex

This is a *local parameter* because it is defined with respect to a single vertex.

Definition. Given a graph G and $v \in V(G)$, u is said to be a *neighbor* of v if $uv \in E(G)$. The set of all neighbors is called the *neighborhood* of v, denoted $N(v)$.

Definition. The *degree* of a vertex v, denoted $\deg v$ or $d(v)$, is the cardinality of the neighborhood of v.

Some special names.

Definition. We say v is an *isolated vertex* if $\deg v = 0$. We say it is an *end vertex* (or leaf) if $\deg v = 1$.

Definition. Let $\delta(G) = \min_{v \in V(G)} \deg v$ denote the minimum degree.
Let $\Delta(G) = \max_{v \in V(G)} \deg v$ denote the maximum degree.

Trivially, $0 \le \delta(G) \le \deg v \le \delta(G) \le n - 1$.

4.3 Sum of degrees

Theorem 4.1 (First Theorem in Graph Theory, Chartrand and Zhang, 2.1)**.** *For any graph G,*

$$\sum_{v \in V(G)} \deg v = 2\left|E(G)\right|.$$

Proof. Clear. □

Corollary 4.2. *The number of vertices of odd degree is even.*

Example. Let G be a graph with 14 vertices and 27 edges. Moreover, the number of vertices of degree 3 is x and the number of vertices of degree 4 is 6, and the number of vertices of degree 5 is z. Find x and z.

Solution. We have that $x + 6 + z = 14$ and $3x + 4 \cdot 6 + 5z = 2 \cdot 27$. Solving, $x = 5$ and $z = 3$. □

Now, can we construct a graph with said degrees? On Wednesday we will find conditions for which there exists a graph with a certain multiset of degrees.

Example. In any graph, there exist two vertices have the same degree.

Solution. Classic. □

Occasionally, if $\deg v = |V(G)| - 1$, then sometimes we call v a *universal vertex*.
How many edges do we need to guarantee a connected graph on n vertices?

Answer. $\binom{n-1}{2} + 1$. Consider K_{n-1}.

This is weak, though. So here is a better criterion.

Theorem 4.3 (Chartrand and Zhang, , 2.4)**.** *If* $\deg u + \deg v \geq n - 1$ *for all* $u, v \in V(G)$, *then* G *is connected. Moreover,* $\operatorname{diam} G \leq 2$.

Proof. If u and v are disconnected, then look at the neighborhoods of u and v. They must overlap, since $G - \{u, v\}$ has just $n - 2$ vertices but $|N(u)| + |N(v)| = n - 1$. This forms a path of length 2. □

Random thought: in terms of n, find the best k such that if $\deg u + \deg v + \deg w \geq k \forall u, v, w \in V(G)$, then G is connected.

4.4 Regular Graph

Definition. A graph is *k-regular* if $\deg v = k$ for all $v \in V(G)$.

For example, E_n is 0-regular, and K_n is $(n-1)$-regular. Furthermore, C_k is 2-regular. A special 3-regular graph is the *Peterson graph.*

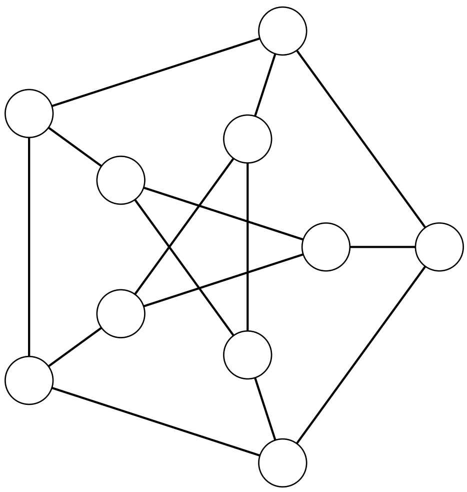


Figure 11: The troll face.

Definition. Given n and $0 \leq r \leq n - 1$ where rn is even, define $H_{r,n}$ the *Harary graph* to be an r-regular graph on n vertices, as follows: label the vertices $0, 1, \cdots, n-1$ modulo n, and

- If r is even, then join each vertex k to $k - \frac{1}{2}r$ to $k + \frac{1}{2}r$ (excluding self).
- If r is odd, then join each vertex k to $k - \frac{1}{2}(r-1)$ to $k + \frac{1}{2}(r-1)$, then to $k + \frac{1}{2}n$.

Exercise. Is the Petersen graph a Harary graph?

Theorem 4.4 (Existence Theorem for Regular Graph, Chartrand and Zhang, 2.6)**.** *Given* $0 \leq r \leq n - 1$, *there exists an* r*-regular graph on* n *vertices if and only if* rn *is even.*

Proof. To construct an example, use the Harary graph. If rn is odd then apply the First Theorem directly to yield a contradiction. □

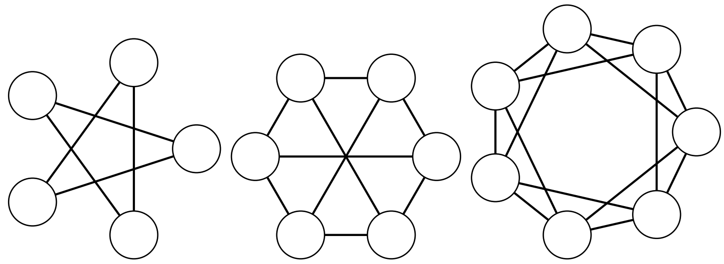


Figure 12: $H_{2,5}$, $H_{3,6}$, and $H_{4,7}$

4.5 Subgraphs of Regular Graphs

Clearly, every graph of order n is a *subgraph* of K_n. The question "is every graph of order n an induced subgraph of K_N?" is also no. But now, consider the question:

Is every graph of order n an induced subgraph of some regular graph?

Answer: yes.

Example. P_n is an induced subgraph of C_{n+1}.

Theorem 4.5 (Chartrand and Zhang, 2.7)**.** *G is always an induced subgraph of a $\Delta(G)$-regular graph H.*

Proof. We perform induction on $n = \Delta(G) - \delta(G)$. The base case $n = 0$ is already clear.

Make a copy of G, call it G'. Then join all the vertices of minimal degree to each other from G to G'. This decreases n by one, so we are done by induction. $\square$

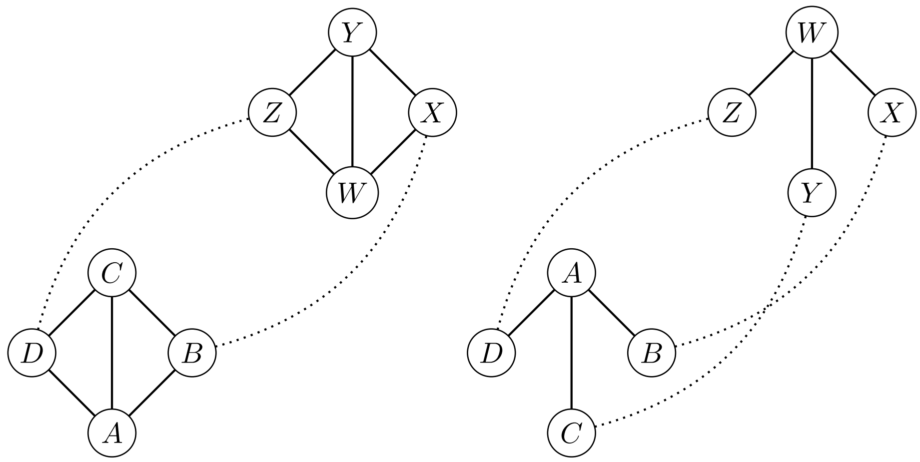


Figure 13: Duplicating graphs G. This drops $n = 1$ to $n = 0$ and $n = 2$ to $n = 1$, respectively.

5 February 6, 2013

5.1 Homework Review

OOPS darn I keep forgetting graphs are loopless. (Also I should read the question.)

For 1.17 one can use the following lemma:

Lemma. *If nonempty* $U, W \subseteq V(G)$ *have an empty intersection, then there exists* $p_0 \in U$, $p_k \in W$, *and a path* $p_0 - p_1 - \cdots - p_{k-1} - p_k$ *for which* $p_i \notin U \cup V$ *for each* $1 \le i \le k-1$.

Proof. Pick the shortest such path. □

For 1.18, the graph with all twelve vertices is actually pretty nice:

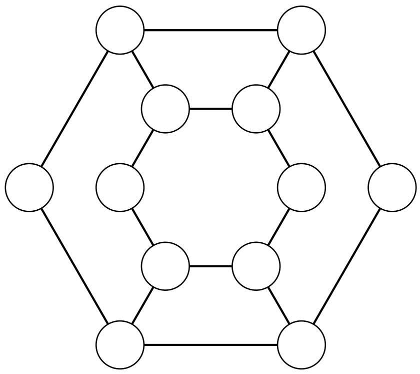


Figure 14: All of 1.18

A few remarks about 1.22: the graph $G \times K_2$ is called a G*-prism*.

5.2 Today's Topics

Chapter 2:

- Degree sequence
- Graphical integer sequence
- Hakimi-Harvel Theorem

5.3 Degree Sequences

Definition. The degree sequence of a graph G is the sequence of degrees of G, arranged in nonincreasing order.

Example. The degree sequence of P_4 is $(2, 2, 1, 1)$.

The degree sequence of C_6 is $(2, 2, 2, 2, 2, 2)$. So is $C_3 \cup C_3$.

The degree sequence of K_4 is $(3, 3, 3, 3)$.

Definition. Given a nonincreasing sequence s of nonnegative integers, we call it *graphical* if there is a graph whose degree sequence is s.

What are some necessary conditions for a graphical sequence of length n? If the sequence is $d_1 \ge d_2 \ge \cdots \ge d_n$, then

(i) $d_1 + d_2 + \cdots + d_n$ is even (and at most $n(n-1)$, from the second condition.)

(ii) $0 \le d_i \le n-1$ for each i.

(iii) $d_2, d_3, \cdots, d_{d_1+1}$ are all nonzero.

(iv) If $d_i = n-1$ then $d_n \ge i$.

(v) If $d_j = 0$ then $d_1 \le j-2$.

Example. $(3,3,3,1)$ is not graphical. Note that $d_3 = n-1$ (here $n=4$) which would force $d_4 \ge 3$, which is false. For similar reasons, $(7,7,4,3,3,3,2,1)$.

OK, we can make more necessary conditions. What are the sufficient conditions?

(i) If $d_i = r$ for all $i = 1, 2, \cdots, n$ such that rn is even, then such a graph exists.

(ii) If $d_1 \ge d_2 \ge \cdots \ge d_n$ is graphical, then $n-1-d_n \ge n-1-d_1$ is graphical. (Complement everything).

5.4 Hakami-Havel

Theorem 5.1 (Hakimi-Havel Theorem)**.** *A sequence $d_1 \ge d_2 \ge \cdots \ge d_n$ is graphical if and only if*

$$d_2 - 1, d_3 - 1, \cdots, d_{d_1+1} - 1, d_{d_1+2}, \cdots, d_n$$

is graphical.

Example. Is $3,3,2,2,1,1$ graphical?

Solution. Descending, we get $2,1,1,1,1$, then $1,1,0,0$ which is graphical, so the answer is yes. We can easily build upwards, to. □

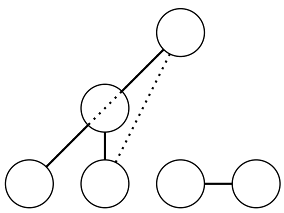


Figure 15: 3,3,2,2,1,1

Let us wrap up our example from last time as well.

Example. Is the sequence $(d_i)_{i=1}^{14} = (5,5,5,4,4,4,4,4,4,3,3,3,3,3)$ graphical?

Solution. Use the Hakimi-Havel Theorem twice to get $(d_i'')_{i=1}^{12} = (3,3,\cdots,3)$ which is clearly doable. □

Proof of Hakimi-Havel. The sufficient part is easy; just construct it by adding an nth vertex and connecting it to the right vertices.

For the necessary part, we claim that there exists a graph G with degree $d_1 \ge \cdots \ge d_n$ such that the neighbors of v_1 are $v_2, v_3, \cdots, v_{d_1+1}$, where $\deg v_i = d_i$ for each i. Assume the contrary. Pick G with the property that

$$\sum_{v \in N(v_1)} \deg v$$

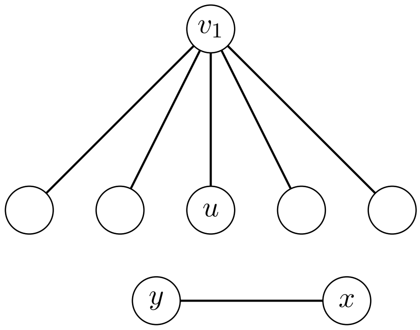


Figure 16: Graph switching

is maximal.

Evidently there exist vertices $u \in N(v_1)$ and $y \notin N(v_1)$ for which $\deg y > \deg u$. Hence, due to degrees, there is some vertex $x \in N(y)$ such that $x \notin N(u)$. Then delete the edges uv, xy and add the edges ux and vy. This maintains the degree sequence but increases the sum, contradiction. □

6 February 11, 2013

6.1 Today's Topics

Chapter 3:

- Isomorphic graphs.

6.2 Isomorphism

Definition. Two graphs G and H are *isomorphic* if and only if there exists a biejction $f : V(G) \to V(H)$ such that $uv \in E(G) \iff f(u)f(v) \in E(H)$. Such an F is called an *isomorphism*.

As an example, the following two graphs are isomorphic:

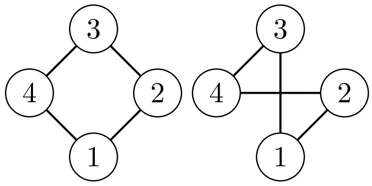


Figure 17: Two isomorphic graphs. A bijection is $f = (4\,3) \in S_4$.

Musing: how many such f exist?

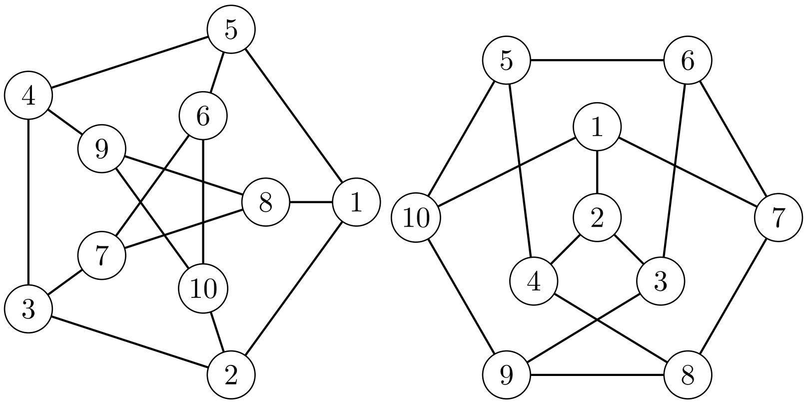


Figure 18: Petersen again! This thing

6.3 Disproving isomorphism

Graph isomorphism is NP. In other words, it's pretty hard.

Of course, some trivial optimizations; isomorphism preserves:

- the multiset of degrees (in particular, the order and sizes)
- number of connected components (in particular, connectedness)
- diameters
- ...just about everything

Example. Are $K_3 \cup K_3$ and C_6 isomorphic?

$n = 1$	E_1
$n = 2$	E_2, P_2
$n = 3$	$E_3, P_2 \cup E_1, P_3, K_3$
$n = 4$	$E_4, K_2 \cup E_2, K_2 \cup K_2, K_{1,2} \cup E_1, K_3 \cup E_1, K_{1,3}, P_4, C_4, \overline{K_{1,2} \cup E_1}, K_4 - uv, K_4$
	$\vdots$
$n = 9$	$274,668 graphs$

Table 1: All graphs of order n

Of course not. But the multiset of degrees is the same.

The textbook provides an essentially useless "necessary and sufficient" condition for isomorphism.

Theorem 6.1 (Chartrand and Zhang, 3.1)**.** *G is isomorphic H if and only if $\overline{G}$ is isomorphic to $\overline{H}$.*

Proof. Clear. □

Example. Is $K_{3,3}$ isomorphic to $K_3 \times K_2$?

The complements are $K_3 \cup K_3$ and C_6, so the answer is negative.

6.4 General definition of isomorphism

Definition. Two graphs are *isomorphic* if there exist a labellings on them such that they are isomorphic.

This is just a technicality.

Now, we can define an equivalence relationship on the sot of all graphs $\mathcal{G}$ by $G \simeq H$ iff G is isomorphic to H.

Therefore we get equivalence classes. So we can drop the label (because of isomorphism classes) and represent graphs by an unlabeled representative. So yay.

Oh yeah and now the number of labellings is finite, so we can talk about "how many graphs of order n". Here we are counting equivalence classes.

6.5 Classes of graphs with small orders

If we restrict our attention to connected graphs, $n = 9$ has only 11,117 classes.

6.6 Self-complementary Graphs

Definition. G is *self-complementary* if G is isomorphic to $\overline{G}$.

The only such graph for $n = 1$ is E_1. No examples exist for $n = 2, 3$ or in general for $n \in (4\mathbb{Z} + 2) \cup (4\mathbb{Z} + 3)$ (since we need $\binom{n}{2}$ to be even.)

Exauhsting the possibities for $n = 4$, we see that only P_4 works. (Hint: only consider those with 3 edges.)

For $n = 5$, C_5 works.

For $n = 8$, consider P_4. Draw edges between all the A's and B's etc. to get supergraphs. Now join the B and C vertices.

In fact any working $n \equiv 0, 1 \pmod 4$ can be constructed using this trick.

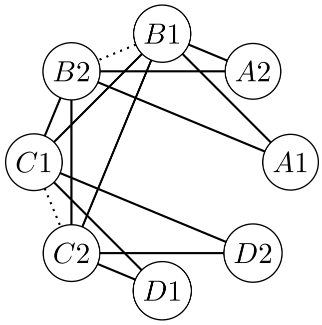


Figure 19: A construction for $n = 8$

Example. Is there a nontrivial disconnected self-complementary graph?

Of course not.

7 February 13, 2013

7.1 Homework Review

gj you didn't fail.

7.2 Today's topics

Chapter 3:
Reconstruction Conjecture

7.3 Card Game

"I am thinking of a graph."

Given the graph G, we are shown the induced subgraphs $G - \{v\}$ for each $v \in G$. For our example:

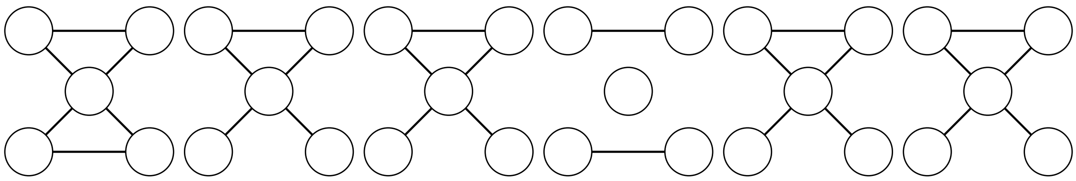


Figure 20: G has order 6, and dropping each of the vertices yields the following figures.

For this particular graph, it is not hard to use the fourth card to reconstruct the graph:

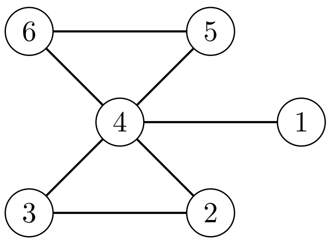


Figure 21: The solution to the problem

Definition. For any vertex $v \in V(G)$, the graph $G - v$ is called a *card* of G. The *deck* of G refers to the collection of all cards.

7.4 Reconstruction Conjecture

Is it possible to reconstruct the graph G from its deck?

In fact, there is a counter-example already at $n = 2$; note that K_2 and E_2 have the same deck! So we modify this to give:

Problem (Reconstruction Conjecture)**.** If G has at least three vertices, then G can be reconstructed from its deck.

This problem is still open.

7.5 Recognizable Properties

Definition. A property/parameter is called *recognizable* if it can be determined from the deck of a graph.

What properties are recognizable? Let there be n cards (i.e. G has order n) and let the cards be $C_1, C_2, \cdots, C_n$. Let m_i be the size of C_i, and let m be the size of G.

Fact 7.1. The order is recognizable.

Proof. This is trivial. □

Fact 7.2. The size is recognizable, and therefore the degree sequence. In particular, regularity is recognizable.

Proof. Observe that $m_i = m - \deg v_i$. Summing up, we find that

$$\sum m_i = \sum m - \sum \deg v_i = nm - 2m \implies m = \frac{1}{n-2} \sum m_i.$$

Therefore we obtain m, and we can find the degree sequence by looking at each m_i individually. □

Fact 7.3. Connectedness is recognizable.

Proof. Apply Theorem 2.2 directly. □

Fact 7.4. Being bipartite is recognizable; G is bipartite if and only if

(i) All cards are bipartite, and

(ii) G is not an odd cycle.

Proof. If G is a cycle, then it's trivial. If not, odd cycles are detected when any vertex not in the cycle is deleted. □

Exercise. Show explicitly that the number of connected components of G is recognizable by giving an algorithm.

7.6 Another Deck

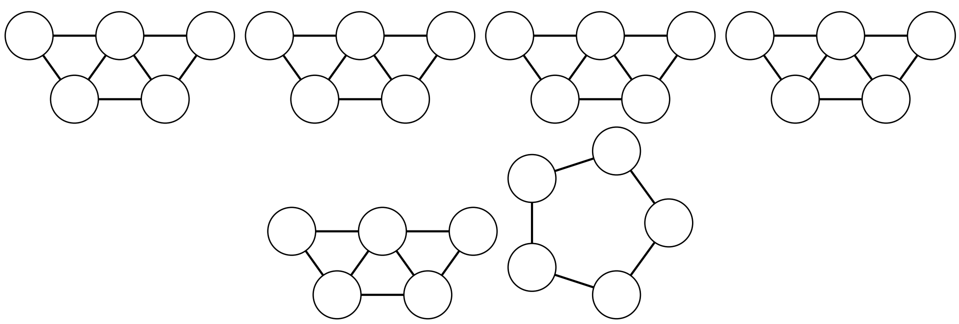


Figure 22: Another deck.

The solution is pretty clearly $C_5 + K_1$.

7.7 Determining number of components in a deck

Let's take an example $G = K_4 \cup C_3$. We get

- Four cards $C_3 \cup C_3$.
- Three cards $K_4 \cup C_2$.

For a disconnected graph G, look at the largest connected component that appears among all the cards. Then we see that this must be a component of the original graph, and now we can go downwards!

8 February 18, 2013

8.1 Today's Topics

Chapter 4

- Tress

Definition. $n(G)$ will denote the order of G, and $m(G)$ will denote the size. Also, $k(G)$ will denote the number of components.

8.2 Bridges

Definition. An edge e is a *bridge* if the $k(G - e) > k(G)$.

Example. In figure 23, the dotted edge is a bridge.

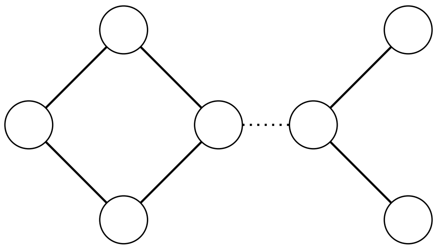


Figure 23: An example of a bridge.

In P_n, every edge is a bridge. In C_n, none of them are.
In this sense, none of the bridges in the Bay Area is actually a bridge.

Fact (Chartrand and Zhang, 4.1)**.** Given a connected graph G, an edge $e \in E(G)$ is a bridge if and only if $e \notin C$ for any cycle $C \subseteq G$.

Proof. Clear. □

8.3 Trees

Definition. A *tree* is an acyclic graph (i.e. without cycles).

Fact (Chartrand and Zhang, 4.2)**.** A graph has a unique path between any two vertices if and only if it is a tree.

Proof. Clear. □

Fact (Chartrand and Zhang, 4.3)**.** T has at least 2 vertices of degree 1.

Proof. Consider a path of length $\text{diam}(T)$. The endpoints must have degree 1. □

8.4 The Size of a Tree and Consequences

Fact. In any tree, $m(T) = n(T) - 1$.

Proof. Induction on $n \geq 2$. The base case $n = 2$ is clear. The inductive step $n \geq 3$ is obvious. □

Definition. A *forest* is an acyclic graph. Note that it need not be connected; hence its components are trees.

Fact (Chartrand and Zhang, 4.6)**.** For any forest $\mathcal{F}$, $m(\mathcal{F}) = n(\mathcal{F}) - k(\mathcal{F})$.

Proof. If $\mathcal{F} = T_1 \cup T_2 \cup \cdots T_k$ for trees T_i then

$$\sum_{1 \le i \le k} m(T_i) = \sum_{1 \le i \le k} (n(T_i) - 1) .$$ □

Fact. If G is a connected graph, then $m(G) \ge n(G) - 1$. Equality occurs if and only if G is a tree.

Proof. Cut open all the cycles in G to obtain a tree G', where $m(G') \le m(G)$. But then $m(G') = n(G) - 1$. Equality occurs only if no cycles were cut. □

Combining all of these results:

Theorem 8.1 (Characterization of Trees)**.** *Let G be a graph of order n and size m. Then any two of the following conditions imply the third:*

(i) G is connected

(ii) G is acyclic

(iii) $m = n - 1$

Furthermore, these conditions together imply G is a tree.

8.5 Spanning Tree

Definition. A *spanning tree* of a connected graph is a subgraph $T \subseteq G$ for which $V(T) = V(G)$ and T is a tree.

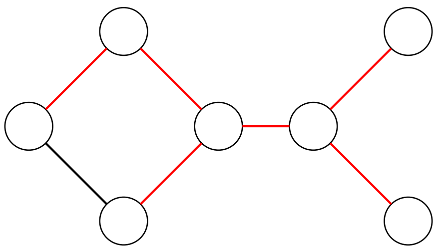


Figure 24: A spanning tree, highlighted in red

Now, K_n has *many* different spanning trees.

Fact 8.2. The number of labeled spanning trees of K_n is n^{n-2}.

We can define the following matrices: The adjacency matrix is given by

$$A(G) : a_{ij} = \begin{cases} 1 & \text{if } i, j \text{ adjacent} \\ 0 & \text{otherwise} \end{cases} .$$

$$D(G) : d_{ij} = \begin{cases} \deg i & \text{if i=j} \\ 0 & \text{otherwise} \end{cases} .$$

Then define the Laplacian matrix $L(G) = D(G) - A(G)$.

Theorem 8.3. *Delete any row and column of $L(G)$ to get a matrix $L'(G)$. The number of spanning trees of G is* $\det L'(G)$*, regardless of which row and column is selected.*

9 February 20, 2013

AMC 12B too hard...

9.1 Today's Topics

Chapter 5

- Spanning trees
- Minimal-weight spanning trees
  - Kruskel's Algorithm
  - Prim's Algorithm

9.2 Test 1 Synopsis

Test 1 on Monday. Six questions, 90 minutes.

1. Basic
2. Degree sequences (Hakimi-Havel)
3. Isomorphic Graph
4. Reconstruction Conjecture
5. Minimal-weight spanning trees
6. Proof: reproduce a proof either presented in class or on the homework

9.3 Spanning Trees

We have already seen that a graph contains "many" spanning trees.

Theorem 9.1 (Chartrand and Zhang, 4.10)**.** *Every connected graph has a spanning tree.*

Proof 1. Cut open any cycles. The result is a spanning tree. □

Proof 2. Start with a trivial graph v. Pick edges incident with v to form a connected subgraph H. Continue to pick edges joining $V(H)$ and $V(G - H)$. Eventually we get a connected graph with $n - 1$ edges, so this is a spanning tree. □

Proof 3. Start with an edge in $E(G)$. Beginning with the empty graph on $V(G)$, keep adding edges in such a way as to avoid cycles, until $n - 1$ edges have been added. □

The second and third proofs are the basic idea for Prim's Algorithm and Kruskel's Algorithm, respectively.

9.4 Weighted Graphs

First, definitions.

Definition. A *weighted graph* is a graph with a weight function $w : E(G) \to \mathbb{R}$.

Often the range is actually $\mathbb{Z}$.

Definition. Given a subgraph H of G, the *weight of H* is defined as $\sum_{e \in E(H)} w(e)$.

9.5 Minimum Weight Spanning Tree Problem

Problem. Given a weighted connected graph G with weight function w, find the minimum possible weight of a spanning tree.

Note that this is well-defined, since the set of spanning trees is finite and nonempty. The two algorithms presented here are greedy.

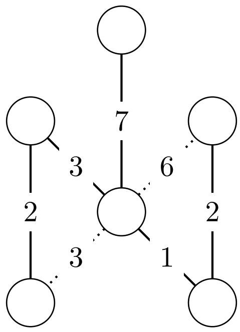


Figure 25: An example of a graph. Spanning tree is the solid line.

As another example, consider the following graph.

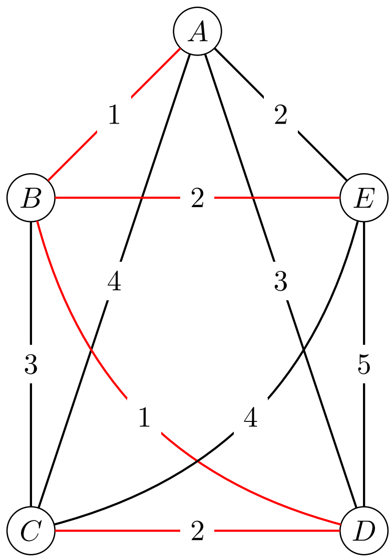


Figure 26: Another graph; spanning tree is colored red.

9.6 Kruskal's Algorithm

Algorithm 9.2 (Kruskal)**.** Given a weighted connected graph,

- Start with an edge with minimum weight.
- Add edge with minimum weight from the remaining edge while avoiding cycles.
- Repeat $n-2$ times.

For the graph given in figure 26, the following sequence produces a spanning tree. A possible example follows:

1. Pick edge AB.
2. Pick edge BD.
3. Pick edge BE.
4. Pick edge DC.

Before giving the proof, let us make some observations. Let $\{e_1, e_2, \cdots, e_{n-1}\}$ be the edges, in order.

We observe that $G(\{e_1, e_2, \cdots, e_i\})$ is a forest for all i.

Also, we need the following lemma.

Lemma 9.3. *Let T be a spanning tree of a connected graph G. IF $e \notin E(T)$, then $\exists f \in E(T)$ such that*

$$T' = T - f + e$$

is a spanning tree.

We actually need the idea in the proof, not the lemma itself.

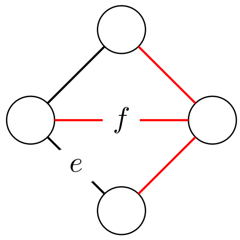


Figure 27: An example of the lemma. Spanning tree in red.

Proof of 9.2. Suppose T_k is generated by Kruskal's, with edges $E_k = (e_1, e_2, \cdots, e_{n-1})$ in that order, but is not a minimum weight spanning tree.

Then $\exists T$ with $w(T) < w(T_k)$ such that $E(T)$ contains $e_1, \cdots, e_i$ but not e_{i+1}; pick the T where this i is maximal. We will now construct a spanning tree T' with $w(T') < w(T_K)$, and moreover, $E(T')$ contains $e_1, e_2, \cdots, e_i, e + i + 1$; this will give a contradiction.

We "invoke" the lemma. Since $e_{i+1} \notin T$, we can find an $f \in T$ such that

$$T' = T - f + e_{i+1}$$

is a spanning tree, and $f \notin \{e_1, \cdots, e_i\}$. This is possible because in the lemma we pick f from a cycle, and there is no cycle containing all edges in E_k.

WE claim that $w(T') < w(T_k)$. It suffices to prove that $w(f) \geq w(e_{i+1})$. If this is not the case, we have $w(f) < w(e_{i+1})$; but this is impossible because we picked the edges greedily! (There is no cycle since $e_1, \cdots, e_i, f \in T$.) □

9.7 Prim's Algorithm

Algorithm 9.4 (Prim's Algorithm)**.** Given a connected graph G,

- Start with a trivial graph H on an arbitrary vertex.
- Among all edges between $V(H)$ and $V(G - H)$, pick one with the minimal weight and add it to form a new H.

- Repeat until $V(H) = V(G)$.

The result is a minimal weight spanning tree.

Proof. In book. □

10 Test 1 Solutions

Score: 100/100.

Recorded here are the distinct solutions.

3. Considering diameters also works. Planarity works for 3b. Bipartite also works.

Theorem 10.1. *If G is a connected graph with all edges having all distinct weight then the minimal spanning tree is unique.*

Proof. Use the proof of Kruskal and use strict inequalities everywhere. □

11 March 4, 2013

(finishes that fairly long homework exercise)

11.1 Today's Topics

Chapter 5:

- Cut vertex
- Nonseparable graph
- blocks

11.2 Cut vertices

Definition. v is a cut vertex of G if $k(G-v) > k(G)$.

Example. C_n has no cut vertices ($n \geq 3$).

Example. $K_{1,n}$ has exactly one cut vertex ($n \geq 2$).

Example. P_n has $n-2$ cut vertices (where $n \geq 3$).

Question. Is there a graph with all cut vertices?

11.3 Consequences

Fact 11.1 (Chartrand and Zhang, 5.1)**.** If uv is a bridge of a connected G, then v is a cut vertex if and only if $\deg v \geq 2$.

Proof. Clear. □

Easy consequences:

- Non end-vertices of a tree are cut vertices.
- If G has order at least 3, and G has a bridge, then G has cut vertex.

Fact 11.2 (Chartrand and Zhang, 5.3)**.** Let v be a cut-vertex of a connected graph G. If u, w lie in different components of $G - v$ then v must be on every $u - w$ path in G.

Fact 11.3 (Chartrand and Zhang, 5.4)**.** If v lies on every $u - w$ path in G, then v is a cut-vertex.

Fact 11.4 (Chartrand and Zhang, 5.5)**.** Let G be the connected graph. Let v be a vertex such that

$$d(u, v) = \max_{x \in V(G)} d(u, x)$$

Then v is not a cut vertex.

Proof. If v is a cut vertex, $\exists w$ no longer connected to u. But then a geodesic $u - w$ must pass through v, so $d(u, w) > d(u, v) \geq d(u, w)$ which is a contradiction. □

Corollary 11.5 (Chartrand and Zhang, 5.6)**.** *Every nontrivial connected graph has at least 2 non-cut-vertices.*

Proof. Consider a diameter. □

Corollary 11.6. *If G is nonempty, then*

$$k(G) = \min_{v \in V(G)} k(G - v).$$

Proof. Let C be a component of G, and C nonempty. Then $\exists x \in V(C)$ such that $k(C - x) = k(C)$. In that case $k(G - x) = k(G)$. But clearly, $k(G - v) \geq k(G)$ for any v, so we're done. $\square$

12 March 6, 2013

12.1 Today's Topics

Chapter 5:

- Non-separable graphs
- Blocks
- Vertex connectivity
- Edge connectivity

12.2 Non-separable graphs

Definition. A nontrivial connected graph without cut-vertices is called nonseparable.

Example. C_n and K_n are nonseparable.

Fact 12.1 (Theorem 5.7)**.** If $|G| \geq 4$, then G is nonseparable if and only if every pair of distinct vertices shares a common cycle.

Proof. In book. □

Definition. A nontrivial connected graph with at least one cut-vertex is called separable.

12.3 Blocks

The intuition is that separable graphs will consist of "combinations" of nonseparable graphs ("blocks") in the same way that disconnected graphs consist of connected components.

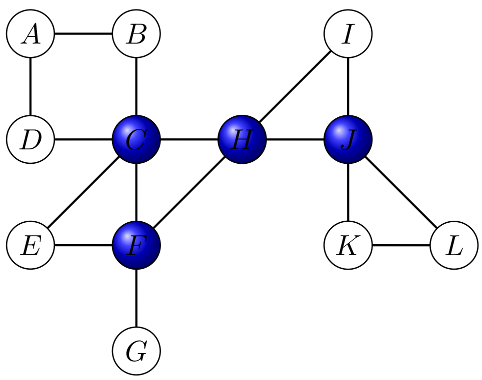


Figure 28: A connected graph. The blocks are $ABCD$, $CEFH$, G, HIJ and JKL. Cut vertices are highlighted.

Definition. Let G be connected. Define a relation $\sim$ on $E(G)$: $e \sim f$ if and only if e and f share a common cycle. For each equivalence class E of $\sim$ over $E(G)$, we get a block $B = G[E]$.

Fact 12.2. Let $B_1, \cdots, B_k$ be the blocks of a connected G. Then,

(i) $E(B_i) \cap E(B_j) = \emptyset$,

(ii) $|V(B_i) \cap V(B_j)| \leq 1$, and

(iii) If $\{v\} = V(B_i) \cap V(B_j)$, then v is a cut-vertex.

Remark. If v is a cut vertex, then $\exists i \neq j : v \in V(B_i) \cap V(B_j)$.

12.4 Vertex connectivity

Definition. $U \subseteq V(G)$ is called a vertex-cut if $G - U$ is disconnected.

Definition. Suppose G is not the complete graph, then $\kappa(G)$, the vertex connectivity of G, is defined by

$$\kappa(G) = \min_{\text{Vertex cut } U} |U|.$$

A vertex cut U_0 which achieves this minimum is called a *minimum vertex-cut*.
Furthermore, $\kappa(K_n) = n - 1$.

Note that a disconnected graph has $\kappa(G) = 0$ since $\emptyset$ works. If G is connected but has a cut-vertex, then $\kappa(G) = 1$. Conversely, if $\kappa(G) = 1$, then G is either K_2 or is connected with at least one cut-vertex.

Definition. G is k-connected if $\kappa(G) \geq k$; i.e. the deletion of strictly less than k vertices will not disconnected the graph.

Example. 0-connected graphs consists of all graphs. 1-connected graphs are connected graphs.

12.5 Edge-connectivity

Definition. $X \subseteq E(G)$ is called an edge-cut if $G - X$ is disconnected.

Definition. $\lambda(G)$ is defined by

$$\lambda(G) \overset{\text{def}}{=} \min_{X \text{ edge-cut}} |X|$$

for $G \neq K_1$. We define $\lambda(K_1) = 0$. An edge-cut which achieves the minimum is called an *minimum edge-cut.*

Example. $\lambda(G) = 0$ if G is disconnected, and $\lambda(G) = 1$ if G is connected and has a bridge. Also, $\lambda(K_n) = n - 1$ (just delete all the edges of a vertex). This is not entirely trivial.

Claim. If $K_n - X$ is disconnected, then $X \geq n - 1$.

Proof. Consider the complement. If $K_n - X$ is disconnected, then $\overline{K_n - X}$ must be connected, so

$$|X| = m\left(\overline{K_n - X}\right) \geq n - 1$$

as desired. □

In fact, more is true:

Theorem 12.3 (Chartrand and Zhang, 5.11)**.** *For any graph G,*

$$\kappa(G) \leq \lambda(G) \leq \min_{v \in V(G)} \deg V.$$

Proof. First, we will show that $\lambda(G) \le \delta(G)$, where δ is the minimum degree function. Assume that $v \in V(G)$ is such that $\deg v = \delta(G)$. Then just cut all the edges of v.

Next, to show $\kappa(G) \le \lambda(G)$, consider a minimum edge-cut X_0. Let us assume that G is connected and is not K_n (otherwise check the conclusion); evidently $n \ge 3$. Remark that $G - X_0$ has exactly 2 components; otherwise, we could find a smaller edge-cut X_0', contradiction. Let $G - X_0 = G_1 \cup G_2$.

Now, we claim that $\exists u \in V(G_1), v \in V(G_2)$ such that uv is not an edge. Note that $G \ne K_n$, so $\delta(G) \le n-2$. If this is false, then

$$\begin{aligned} |X_0| &= |V(G_1)|\,|V(G_2)| \\ &= k(n-k) \end{aligned}$$

Observe $1 \le k \le n-1$ since G_1 and G_2 are nontrivial. Yet $|X_0| \le \delta(G) \le n-2$. It is easy to check that this is impossible.

Now take

$$U = \{x_1 \in V(G) : x_1 \ne u, \exists v \in V(G_2) : x_1 v \in X_0\} \cup \{x_2 \in V(G_2) : ux_2 \in X_0\}.$$

□

13 March 11, 2013

APMO

14 March 13, 2013

14.1 Today's Topics

Sufficient condition for Hamiltonian cycles.

14.2 Necessary Conditions

Definition. A graph is *Hamiltonian* if it contains a Hamiltonian cycle; i.e. a cycle using every vertex once.

The graph G must satisfy

1. Connected
2. $\delta(G) \geq 2$
3. $n \geq 3$
4. $k(G-S) \leq |S|$ for all $S \subseteq V(G)$. In particular, there are no cut vertices.

14.3 Sufficient conditions

Unfortunately, there are no necessary and sufficient conditions. Here are some sufficient conditions.

Fact (Chartrand and Zhang, 6.6)**.** Let G be a graph of order at least 3. If $\deg u + \deg v \geq n$ for any nonadjacent $u, v \in V(G)$ then G is Hamiltonian.

Proof. Fix n. Consider a counterexample H with order n and maximal size; that is, H is non-Hamiltonian. Clearly $H \neq K_n$.

Then $\exists u, v$ such that $H + uv$ becomes Hamiltonian. This implies there exists a path Hamiltonian path from u to v, as If $\exists i$ such that $uw_{i+1}, u_i w \in E(H)$ then this is a

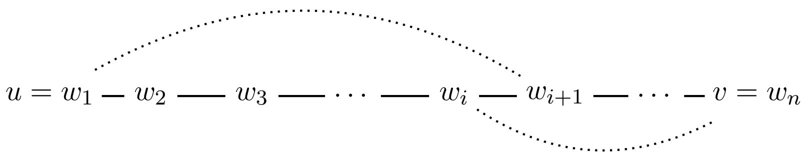


Figure 29: Hamiltonian Paths

contradiction, because then we can get a Hamiltonian cycle. But $\deg u + \deg v \geq n$ so can be shown using Pigeonhole! □

Actually, we can modify this to read

Fact (Chartrand and Zhang, 6.8)**.** If u, v non-adjacent satisfy $\deg u + \deg v \geq n$, the n$H + uv$ is Hamiltonian if and only if H is Hamiltonian.

Exercise. Show that $K_{s,t}$ is Hamiltonian if and only if $s = t \geq 2$.

14.4 Partial Closures

The above theorem motivates the following definition.

Definition. The partial closure G' of G is obtained from G by adding all edges uv where $\deg u + \deg v \geq n$.

Example. The partial closure of C_5 is itself. The closure of $K_{5,5}$ is K_{10}. An additional example is given below.

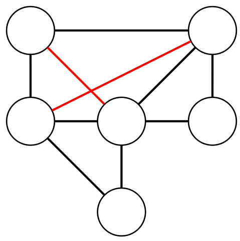


Figure 30: A graph G. Red edges comprise the first partial closure.

Definition. The complete closure of $C(G)$ of G is obtained by taking partial closures until the graph stops changing; i.e.

$$C(G) = \lim_{r \to \infty} G \underbrace{''\cdots'}_{r \text{ times}}.$$

Example. Consider a graph G with degree sequence $9, 6, 6, 6, 5, 5, 5, 5, 4$. Is it Hamiltonian?

So clearly we want to use closures. Anyways here is something that kills it.

14.5 A sufficient criterion with degree sequences only

Theorem 14.1. *Let $|G| \geq 3$. If for every integer $j < \frac{n}{2}$, there are strictly less than j vertices with degree at most j, then H is Hamiltonian. In fact $C(G) = K_n$.*

Proof. Suppose $C(G) \neq K_n$. Then $\exists u, w \in C(G)$ such that $\deg u + \deg w \leq n - 1$. Take the pair such that $\deg u + \deg w$ is **maximal**. Without loss of generality $\deg u \leq \deg w$. Set $k = \deg u$; then $k = \deg u < \frac{n}{2}$, and $\deg w \leq n - 1 - k$.

Claim. If v and w are not adjacent, then $\deg v \leq k$.

Proof of Claim. Consider v such that v and w are not adjacent. Clearly $\deg v + \deg w \leq n - 1$ since this a closure. We claim that $\deg v \leq k$. Otherwise, $\deg v > k$ and $\deg v + \deg w > \deg u + \deg w$, contradicting maximality. □

So every non-neighbor of w has degree at most k. But there are at least $n - \deg w > k$ of them. Since degrees increase under closure, and we have $k \leq \frac{n-1}{2}$, this contradicts the assumption. □

15 March 18, 2013

15.1 Today's Topics

- Digraph
- Oriented graph

15.2 Digraphs

A digraph is a directed graph.

Definition. $D = (V, E)$ is digraph (directed graph) where V is the vertex set and E is the arc set.

As usual, loops are still not permitted.

Definition. The out degree of a vertex v, denoted $\text{od}(v)$, is the number of vertices x for which $vx \in E$. The indegree of a vertex, denoted $\text{id}(v)$ is the number of vertices x for which $xv \in E$.

As usual, we have the provisos $0 \le \text{od}(v) \le n-1$ and $0 \le \text{id}(v) \le n-1$. Now note that in a diagraph it is possible for all the indegrees to be distinct.

Fact (First Theorem for Digraphs)**.** Let D be a digraph with n vertices and m arcs. Then

$$\sum_{v \in V(D)} \text{id}(v) = \sum_{v \in V(D)} \text{od}(v) = m.$$

Proof. Clear. ☐

Finally, remark that a nontrivial (directed) cycle with two edges exists.

15.3 Digraph analogs of concepts in graphs

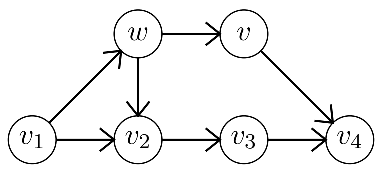


Figure 31: An example of a digraph

As usual, we have a directed walk/trail/path/circuit/cycle.

Fact. Any directed walk can be contracted to a directed path. In particular, any directed circuit can be contracted to a directed cycle

Definition. A digraph is *connected* if the underlying graph is connected.

Definition. A digraph is *strong* (or strongly connected) if there exists a $u - v$ directed path for any $u \neq v$.

Fact. D is strong if and only if D has a spanning directed closed walk.

Proof. If there is a spanning directed closed walk, then D is clearly strong. For the converse just take directed paths $v_1 - v_2 - \cdots - v_n - v_1$. (Note that a cycle is different from a walk!) ☐

15.4 Eulerian Circuits

Definition. A digraph D is Eulerian if there is a directed circuit which contains all edges of D.

Fact. D is Eulerian if and only if $\text{od}(v) = \text{id}(v)$ for all $v \in V(D)$.

Proof. Trivial. □

15.5 Orientable Graphs

Definition. An *orientable graph* is a directed graph with no parallel arcs.

This can be seen as an undirected graph with an orientation imposed on it.

Question. For what G is it possible to orient a graph G such that the resulting digraph is strong?

For example, the following graph, no orientation exists that works:

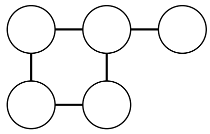


Figure 32: A graph whose orientations are all not strong.

In fact, we have the following theorem.

Theorem. *Given a nontrivial connected G, then there exists a strong orientation for G, if and only if G has no bridges.*

Proof. If we have a bridge in G, then clearly we are doomed.

The converse is to basically keep adding cycles. Suppose every edge is part of a cycle. Take

$$\mathcal{S} \stackrel{\text{def}}{=} \{S \subseteq V(G) : G[S] \text{ has a strong orientation}\}$$

where $G[S]$ denotes the subgraph induced by S.

Note that $\mathcal{S} \neq \emptyset$ because there exists a cycle in G. Not let

$$p \stackrel{\text{def}}{=} \max \{|S| : S \in \mathcal{S}\} \geq n.$$

Suppose that $p \neq n$ and consider a set S with $|S| = p$.

Consider S and $G - S$. There are at least two edges joining them since there are no bridges. Now direct S, and then add direction for any cycle starting/ending in S and having all other vertices in $G - S$. Then we get a bigger S, contradiction. □

16 March 20, 2013

16.1 Today's Topics

- Tournament

16.2 Definitions

Definition. A *tournament* is a complete graph with an orientation.

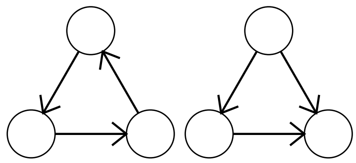


Figure 33: The tournaments on 3 vertices.

There are two tournaments on two vertices and four on four vertices, up to isomorphism.

16.3 Transitive tournaments

Definition. A tournament T is called *transitive* if $ux, xv \in E(T) \implies uv \in E(T)$.

This definition is very rigid, as follows.

Fact. A tournament T is transitive if and only if T has no directed cycles.

The proof is a triviality.

Fact. A tournament T is transitive if and only if the out-degree sequence (or in-degree sequence!) is $\{0, 1, 2, \cdots, n-1\}$.

Remark. There is only one transitive tournament of a given order n, up to isomorphism.

So transitive tournaments are not terribly interesting, because it's just a strict ordering on the vertices. Here is something less trivial.

16.4 Spanning paths and cycles

Fact (Chartrand and Zhang, 7.8)**.** Every tournament has a spanning directed path (i.e. Hamiltonian path).

Before giving the proof, we give a definition.

Definition. Given a directed path in a digraph, the length is 1 less than the number of vertices (analogous to the graph definition.)

Proof. Take the longest directed path $P = v_1 \to v_2 \to \cdots \to v_k$ and consider a vertex v not in P (if it exists; otherwise we win). Clearly $v \to v_k$ otherwise $P' = v_1 \to \cdots \to v_k \to v$ is longer. Then, $v \to v_{k-1}$ since otherwise $v_1 \to \cdots \to v_{k-1} \to v \to v_k$ is longer. Proceeding analogously, we find that $v \to v_i$ for all i.

In particular, $v \to v_1$. But then $v \to v_1 \to \cdots \to v_k$ is longer again, contradiction. □

Fact (Chartrand and Zhang, 7.10)**.** A tournament T has a spanning directed cycle if and only if T is strong.

Proof. If T has a cycle we are done. Conversely, if T is strong, then T is not transitive so there is a cycle. Take the longest cycle C now. If $|C| < n$ then there exists v not in C.

Suppose $C = v_1 \to v_2 \to \cdots \to v_k \to v_1$. If $\exists v \to v_i, v_{i+1} \to v$, where $v \notin C$, then we die because we get a longer cycle. So, for every $x \notin V(C)$, either $V(C) \subseteq N_{\text{out}}(V)$ or $V(C) \subseteq N_{\text{in}}(V)$.

Let X be the set of vertices which all dominate $V(C)$ and Y be the set of vertices which all are dominated by $V(C)$. In other words

$$X \to V(C) \to Y$$

Since it's strong, X and Y must be non-empty. Furthermore, there must be an edge $y \to x$ where $x, y \in X, Y$; this is because T is strong again. But now $x \to v_1 \to v_2 \cdots \to v_k \to y \to x$ is a cycle.

This is a longer cycle, so contradiction! □

16.5 A parting shot

Question. Do there exist two tournaments not isomorphic with the same degree sequence?

17 April 8, 2013

17.1 Today's topics

Chapter 8

- Matching
- Matching in bipartite graphs
- Hall's Theorem

17.2 Matchings

Here is a motivating example! Consider six students A, B, C, D, E, F, with seven books a, c, d, g, h, p, t.

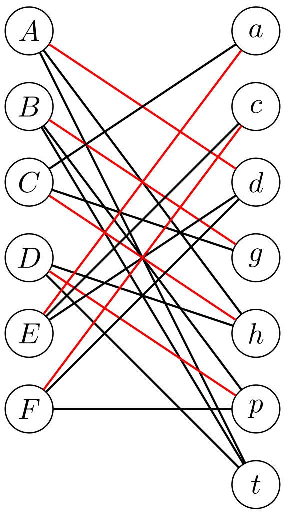


Figure 34: A bipartite graph and a matching on it.

Is it possible for all six students to pick a book without repetition? For this graph the answer is yes; the construction is red.

Definition. Given a graph G, $M \subset E(G)$ is called a *matching* of G if the edges in M share no common end-points.

Definition. A matching M is perfect if $V(M) = V(G)$

Remark. This is of course only possible if G has even order.

Question. How can you tell whether a graph has a perfect matching?

17.3 Hall's Marriage Theorem

Definition. A bipartite graph with left-hand and right-hand sets U and W will be denoted as $B(U, W)$.

Definition (Hall's Condition)**.** A bipartite graph $G = B(U, W)$ satisfies *Hall's Condition* if

$$|N(X)| \geq |X| \text{ for all } X \subseteq U$$

where $N(X) = \cup_{x \in X} N(x)$ is the neighborhood of X.

Note that this is *not* symmetric with respect to U and W.

Theorem 17.1 (Hall's Marriage Lemma)**.** *A bipartite graph $G = B(U, W)$ has a matching of cardinality $|U|$ if and only if it satisfies Hall's condition. In particular, when $|U| = |V|$ this matching is perfect.*

Proof. It is obvious that Hall's condition is necessary. We will prove the other direction using induction on $|U|$. The base case is trivial.

Assume that any bipartite graph $G_1 = B(U_1, W_1)$ with $|U_1| < k$ satisfying Hall's condition has a perfect matching. Consider a bipartite graph $G = B(U, W)$ such that $|U| = k$ and G satisfies Hall's condition.

Let us say that G satisfies the *strong Hall condition* if $|N(X)| \geq |X|$ for any $|X| \geq 1$ and $X \subset U$ (in particular $X \neq U$). We may assume G does not satisfy this property, for otherwise we may simply delete an arbitrary edge e of G, and apply Hall's theorem to $G - e$.

Otherwise let us assume that there exists a nonempty proper subset $X_0 \subset U$ with $|X_0| = |N(X_0)|$. Define

$$F = B(X_0, N(X_0))$$
$$H = B(U - X_0, W - N(X_0))$$

Now, F satisfies Hall's condition because G does as well. But now we claim H satisfies Hall's condition. Consider $S \subseteq U - X_0$. Then, we obtain

$$N_H(S) = N_G(S) - N_G(X_0) = N_G(S \cup X_0) - N_G(X_0)$$

which implies

$$|N_H(S)| = |N_G(S \cup X_0)| - |N_G(X_0)| \geq |S| + |X_0| - |N_G(X_0)| = |S|.$$

So Hall's condition applies, and we may match both F and H. But the right-hand sets of F and H are disjoint. Hence G contains a perfect matching. □

18 April 10, 2013

18.1 Today's Topics

- Consequences of Hall's Theorem
- Tutte's Condition
- Existence of 1-factor

18.2 Assorted Corollaries

Corollary (Chartrand and Zhang, 8.4). *Let $\mathcal{S} = \{S_1, S_2, \cdots, S_n\}$ be a family of finite sets. Then $\mathcal{S}$ has a system of distinct representatives which are pairwise distinct if and only if the union of any k sets S_i has at least k elements.*

Proof. Direct consequence of Hall's Theorem. □

Corollary (Marriage Theorem). *Consider r ladies and r gentlemen. Then every subset of k ladies is collectively acquainted with at least k gentlemen if and only if every subset of k gentlemen is also collectively acquainted with at least k ladies.*

Proof. Both are equivalent to there existing a perfect matching by Hall's Theorem. □

Corollary. *Every r-regular bipartite graph has a perfect matching.*

Proof. Let $G = B(U, W)$. Then the size of G is equal to both $r\,|U|$ and $r\,|W|$, implying $|U| = |W|$.

Then for each $X \subseteq U$, there are $r\,|U|$ edges, and each vertex in V incident to such an edge can accept at most r edges. Hence

$$|N(X)| \geq \frac{r\,|X|}{r} = |X|$$

and we are done by Hall's Theorem. □

18.3 Factors

Definition. A *factor* of a graph G is a spanning subgraph of G. It is called an *r-factor* if it is also r-regular.

For example, any Hamiltonian path/cycle is a factor.
Furthermore, a 1-factor corresponds directly to a perfect matching.

Example. The Peterson graph, $K_2 \times C_5$ and $H_{3,10}$ each contain 1-factors.

Secondly, a 2-factor corresponds to a spanning collection of disjoint cycles.

Example. The Peterson graph, $K_2 \times C_5$, and $H_{3,10}$ also have 2-factors.

18.4 Tutte's Condition

How do we determine whether a graph has a 1-factor? In fact, there is a characterization, as follows. First let us define *Tutte's condition* as follows.

Definition. Let $k_o(H)$ denote the number of odd components of a graph H.

Definition (Tutte's Condition)**.** A graph satisfies *Tutte's condition* if $k_o(G - X) \le |X|$ for every $X \subseteq V(G)$.

Theorem (Chartrand and Zhang, 8.10)**.** *A graph G has a 1-factor if and only if Tutte's condition is satisfied.*

Proof. Deferred to MATH 279. □

As a consequence, we have

Fact. Every connected 3-regular graph without bridges contains a 1-factor.

Proof. We use Tutte's Theorem. Suppose that $G-X$ has odd components $G_1, G_2, \cdots, G_k$.
For each G_i, let E_i be the set of edges joining a vertex of X to a vertex of G_i. Then

$$3\,|G_i| = |E_i| + 2m(G_i).$$

Since G_i is an odd component, we see that $|E_i|$ is odd. Since there are no bridges, it is not the case that $|E_i| = 1$. Therefore, $|E_i| \ge 3$.

Hence, there are at least $3k$ edges joining a vertex of X to some E_i. Since X is 3-regular in G, it follows that $|X| \ge \frac{3k}{3} = k$. Then by applying Tutte's Theorem we are done. □

18.5 More on factors

Definition. A graph G is 1-factorable if $E(G)$ is the disjoint union of edges from 1-factors.

Example. Figure 35 shows a 1-factoring of K_4 by colors.

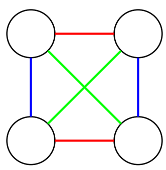


Figure 35: A 1-factoring of K_4. The three matchings each have their own color.

Fact (Chartrand and Zhang, 8.13)**.** The Peterson graph is not 1-factorable.

Proof. Suppose on the contrary that this is the case; that is

$$E(PG) = M_1 \cup M_2 \cup M_3.$$

Then $H \overset{\text{def}}{=} M_2 \cup M_3$ is a 2-factor; that is, a union of disjoint cycles. But PG has girth 5 and does not have a Hamiltonian cycle, so we must have $H = C_5 \cup C_5$. But then M_2 is a subgraph of H which is not possible. □

19 April 15, 2013

19.1 Today's topics

Chapter 8

- F-factorable graph
- Kirkman's Schoolgirl Problem

19.2 F-factorable graph

As stated last time,

Definition. G is 1-factorable if G is a disjoint union of 1-factors.

Example. K_4 is 1-factorable, but the Peterson graph is not.

We generalize this to arbitrary subgraphs.

Definition. Let F be a subgraph of G. Then G is F-factorable if

$$E(G) = E(F_1) \cup E(F_2) \cup \cdots \cup E(F_k)$$

where the union is disjoint, and each F_i is a subgraph of G isomorphic to F.

Example. K_4 is P_4-factorable.

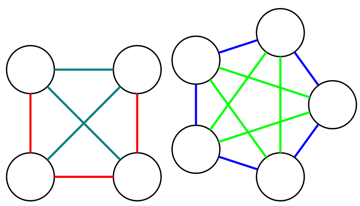


Figure 36: K_4 is P_4-factorable and K_5 is C_5-factorable.

Obviously, $m(F) \mid m(G)$ is necessary but not sufficient.

19.3 Kirkman Schoolgirl Problem

> Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily so that no two shall walk twice abreast.

Let us consider the case instead of nine girls.

Problem (Kirkman with nine girls)**.** A school mistress has 9 schoolgirls whom she wishes to take on a daily walk. The girls are to walk in 3 rows of three each. It is required that no two girls should walk in the same row twice. How many days can it be done?

This is fairly doable for nine girls.

1	2	3
4	5	6
7	8	9

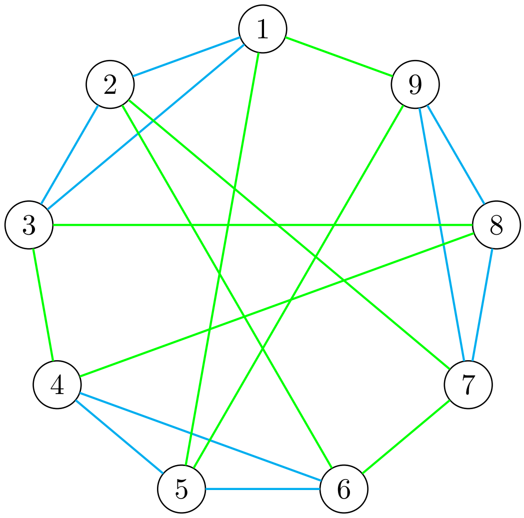


Figure 37: K_9 and some of its $3K_3$ factors.

We can construct by taking the sets $\{1,2,3\}$, $\{1,4,7\}$, $\{1,5,9\}$, $\{1,6,8\}$ for the partners of 1, and then operate cyclically; these correspond to "diagonals", "rows" and "columns".

Now for the graph theory – each day corresponds to a $3K_3$-factor of K_9. In other words, the problem is equivalent to showing that $(3K_3)$-factorable.

To resolve Kirkman's problem,

Theorem 19.1 (Ray-Chaudhuri, Wilson)**.** *K_n is tK_3-factorable if and only if $n = 6k+3$ and $t = 2k+1$.*

Proof. Necessity is trivial. Clearly we must have

$$m(tK_3) \mid m(K_n) \implies 3t \mid \frac{n(n-1)}{2}$$

by focusing on the number of edges. Also, K_n must have $3t$ vertices, hence $n = 3t$. Then $2 \mid n-1$ forces n odd and the conclusion is immediate. □

19.4 Test 2

There will be six questions to be answered in 90 minutes.

1. Give example
2. Chapter 5: vertex/edge connectivity
3. Chapter 6: Eulerian and Hamiltonian circuits
4. Chapter 7: Digraphs, tournaments
5. Chapter 8: Matching in a bipartite graph
6. Proof

20 Test 2 Aftermath

Score: 93/100. OOPS!

Max:	93
Mean:	70
Min:	22

20.1 Notable Things

- Do not screw up the first question.
- For 1d, $H_{3,10}$ is a much cleaner solution.
- The latter parts of exercise 2 are indeed casework.

21 April 24, 2013

21.1 Today's Topics

Chapter 9

- Planar graph
- Characterization

21.2 Definitions

Definition. A *plane graph* is a graph drawn in the plane without edge crossing.

Definition. A *planar graph* is graph which can be drawn as a plane graph.

Example. K_3 is a plane graph; K_4 is a planar graph. However, K_5 is *not* planar. Also, $K_{2,3}$ is planar, but $K_{3,3}$ is not.

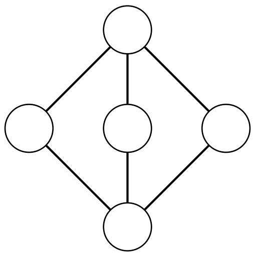


Figure 38: $K_{2,3}$ drawn planar

Surprisingly, $K_{3,3}$ and K_5 are the in fact the "root of all evil"! We can use them to characterize planar graphs.

Remark. Subgraphs of planar graphs are planar.

21.3 Euler and its consequences

Let us introduce the classic theorem of Euler.

Theorem 21.1 (Euler)**.** *If G is a connected plane graph of order n, size m, and r regions, then*

$$n - m + r = 2.$$

In fact, 2 is called the *Euler characteristic* of S^2.

Proof. By induction on m for fixed n. Since G is connected, $n - 1 \leq m \leq \binom{n}{2}$.

If $m = n - 1$, then G is a tree so there are no cycles and $r = 1$; this verifies the base case. Otherwise, there exists a cycle, so we just delete an edge from that cycle, reducing both m and r by 1 and hence preserving the quantity $n - m + r$, while leaving G connected. □

We can use this to restrict m in a planar graph. Here is one such restriction. Let us first define the girth of a graph:

Definition. The *girth* of a graph G is the length of a smallest cycle if it exists; otherwise, it is $+\infty$.

Theorem 21.2. *If G is a plane graph of order n, size m and girth g, then*

$$m \leq \frac{g(n-2)}{g-2}.$$

Proof. It suffices to consider the case where G is connected; otherwise we can simply add bridges to unite components until it is connected. In this way g is unchanged because no now cycles are introduced.

For each of the r regions, there is a cycle of length at least g; since each edge is a member of exactly two regions we obtain the inequality

$$m \geq \# \text{ edges in some cycle} \geq \frac{1}{2}rg = \frac{1}{2}(2+m-n)g.$$

Solving for m with the knowledge $g = 3$ yields that $m \leq \frac{g(n-2)}{g-2}$ as desired. □

Corollary 21.3. *If G is a plane graph with $n \geq 3$ and size m then $m \leq 3n-6$.*

Proof. If G is a forest, then $m \leq n-1 \leq 3n-6$. Otherwise, note that $g \geq 3 \implies \frac{g}{g-2} \leq 3$. □

Remark. As a result of $m \leq 3n-6$, planar graphs are asymptotically *sparse* since m is bounded by $O(n)$.

Using these results, it is easy to check that K_5 and $K_{3,3}$ are nonplanar.

Remark. In a *maximal planar graph of order n*, all regions must be triangles. In fact these are constructible; here is a maximal graph with $n = 5$.

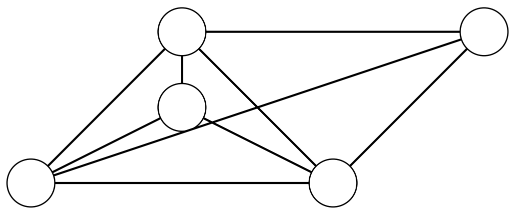


Figure 39: A maximal $n-5$ graph

21.4 Subdivisions

The Peterson Graph is nonplanar. Who's responsible?

Definition. A graph G' is called a *subdivision* of G' can be obtained from G by replacing some edges with path graphs.

This definition is useful because a graph and its subdivision are either both planar and both nonplanar.

Here is a deep result.

Theorem 21.4 (Kuratoski)**.** *A graph G is planar if and only if it contains a subdivision of K_5 and $K_{3,3}$.*

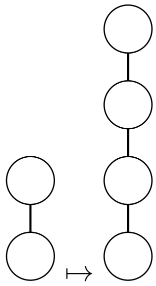


Figure 40: Trading

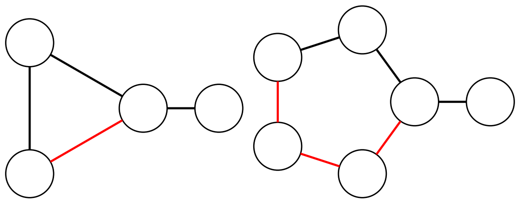


Figure 41: The left graph is a subdivision of the right graph

21.5 Another Question

Question. Is it possible that both G and $\overline{G}$ are nonplanar?

Clearly this cannot hold for large enough n, since $\max\left\{m(G), m(\overline{G})\right\} = O(n^2)$. Working out the details, we have

$$\binom{n}{2} = m(G) + m(\overline{G}) \le 6n - 12 \implies n^2 - 13n + 24 \le 0 \implies n \le 10.$$

So which n are actually constructible?

The cases $n \le 4$ are easy. For $n = 5$, we can take $G = C_5$. For $n = 6$, we may take $G = K_2 \cup K_4$.

It turns out that $n \in \{7, 8\}$ are possible, but $n \in \{9, 10\}$ are not. The cases $n \in \{7, 8\}$ can be achieved by taking G as a maximal planar graph.

22 April 29, 2013

22.1 Today's topics

Graph minors

22.2 Contraction and Minors

Definition. Consider an edge $e = uv \in E(G)$ in a graph G. Then the graph obtained from the contraction of e is $G' \stackrel{\text{def}}{=} G - v$, with the additional edges $\{ux : x \cup N(v)\}$.

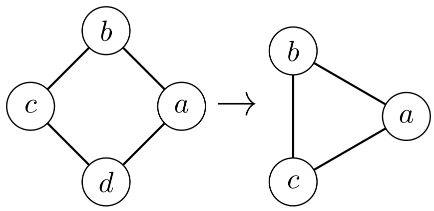


Figure 42: The edge cd is contracted.

Example. K_n with any edge contracted results in K_{n-1}.

Definition. H is a minor of G if H can be obtained from G by a finite sequence of deletion of vertices/edges and/or contraction of edges, in any order.

Example. Here are some obvious examples.

1. K_1 is a minor of any nontrivial graph.
2. Any subgraph of a graph is a minor.
3. K_5 is a minor of K_t if $s \leq t$.

Example. K_5 is a minor of the Peterson Graph.

Example. $K_{3,3}$ is a subdivision of PG. This follows from the fact that PG is a subdivision of $K_{3,3}$.

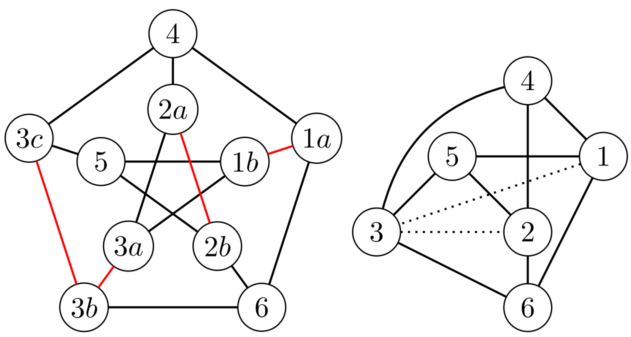


Figure 43: The Peterson Graph. Contract the red edges, then delete the dotted ones.

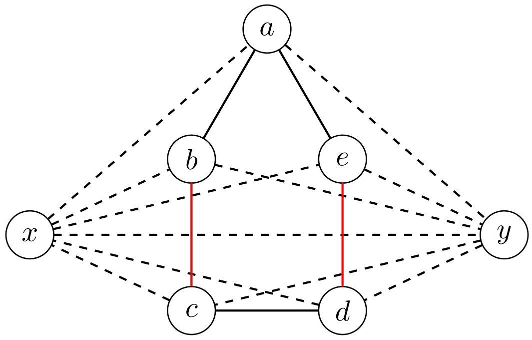


Figure 44: K_5 is a minor of $C_5 + K_2$. Contract the red edges. The edges between C_5 and K_2 are dashed for clarity.

22.3 Properties of Minors

Fact. If G is planar, then any minor of G is also planar.

Fact. If G is a subdivision of H, then H is a minor of G.

Note that the converse of this is *not true.* The operations one can make going from a graph G to its minor is a strict subset of form a subdivision of a graph G to G.

Minors are useful because of the following theorem.

Theorem 22.1 (Wagner)**.** *G is non-planar if and only if K_5 or $K_{3,3}$ is a minor of G.*

Proof. Long, although shortened by invoking Kuratoski. □

22.4 Forbidden Graphs

Definition. A property $\mathcal{P}$ is said to be *minor-hereditary* if G has property $\mathcal{P}$, then any minor of G also has property $\mathcal{P}$.

Example. Planarity is minor-hereditary. Being a forest is also minor-hereditary.

Definition. A graph is said to be *outer-planar* if the graph can be drawn as a plane graph such that every vertex touches the outer region.

Note that K_4 is not outer-planar.

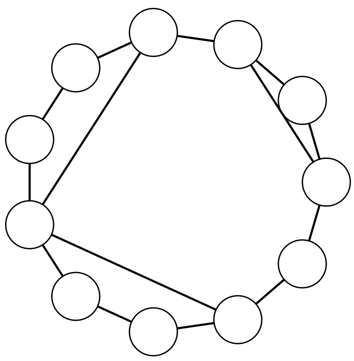


Figure 45: An outer-planar graph

Example. Outer-planarity is minor-hereditary.

Theorem 22.2 (Robertson-Seymour)**.** *Given an infinite sequence of graphs, there are two graphs G and H such that G is a minor of H. In other words, this is a quasi-well ordering.*

Proof. Twenty papers spanning over 500 pages. □

Corollary 22.3. *If $\mathcal{P}$ is a minor-hereditary property, then there exists a finite family of graphs $\{H_1, H_2, \cdots, H_k\}$ for which the following statement holds:*

G has property $\mathcal{P}$ if and only if none of the H_i are a minor of G for any i.

Proof. Consider the set S of graphs *without* property $\mathcal{P}$. Let

$$F = \{G \in S : \nexists H \in S \text{ a minor of } G\}.$$

By Robertson-Seymour, F must be finite. Now, F is the forbidden set – since $\mathcal{P}$ is minor-hereditary, any graph with property $\mathcal{P}$ cannot contain any of the forbidden graphs in F. □

Example.

1. G is planar if and only if K_5 and $K_{3,3}$ are not minors of G.
2. G is a forest if and only if C_3 is not a minor of G.
3. G is outer-planar if and only if K_4 or $K_{2,3}$ is not a minor of G.

There is an analogous result for properties for vertex-induced subgraphs. In this case, the forbidden family may be infinite.

23 May 1, 2013

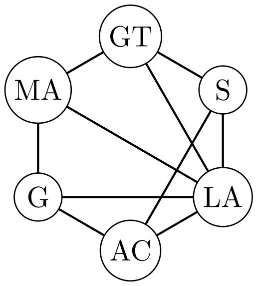
23.1 Today's Topics

Vertex coloring:

- Definition of chromatic number
- Examples
- Lower bounds
- Upper bounds

23.2 Motivation

Here are some classes.



Student	Signups
Alice	LA, S
Bob	MA, LA, G
Carol	MA, G, LA

Figure 46: Graph theory, stats, linear alg, adv calculus, geometry, modern algebra

Question. Given a set of classes and the students taking each class, what is the smallest number of time slots needed?

This is basically a vertex coloring.

23.3 Vertex Coloring

Definition. Given a graph G, a k-coloring of G is an assignment of k colors to the vertices of G such that no two adjacent vertices use the some color.

Definition. The *chromatic number* of G is the minimum k for which a k-coloring of G exists. The chromatic number of G is denoted by $\chi(G)$.

Example. Some obvious chromatic numbers:

- $\chi(K_n) = n$
- $\chi(P_n) = 2$ (where $n \ge 2$)
- $\chi(C_n) = \frac{1}{2}(5 + (-1)^{n+1})$ (where $n \ge 3$). In other words, $\chi(C_{2k}) = 2$, and $\chi(C_{2k+1}) = 3$.

Example. The Harary Graph$H_{3,10}$ has chromatic number 2, because it is bipartite.

Fact. A graph G is bipartite and non-empty if and only if $\chi(G) = 2$.

Proof. This is obvious. □

Example. $\chi(C_5 \times K_2) = 3$. Indeed, the chromatic number is at least 3 since it has on odd cycle in it, and a 3-coloring is not difficult to construct.

We can make this easy observation more explicit as follows.

Fact. If H is a subgraph of G, then $\chi(H) \le \chi(G)$.

So, in general, establishing a chromatic number can be done by find a constructing and getting the lower bound.

Fact. For any graphs G_1 and G_2, we have

$$\chi(G_1 + G_2) = \chi(G_1) + \chi(G_2).$$

Proof. This is obvious. Just check that G_1 and G_2 cannot share any colors. □

Example. The graph in figure 46 is actually K_1+C_5. So $\chi(K_1+C_5) = \chi(K_1)+\chi(C_5) = 4$.

23.4 Lower Bound

Here are some easy lower bounds.

First, let us make the following definition.

Definition. The *independence number* of a graph G, denoted $\alpha(G)$, is the maximum number of vertices one can select from G with no edges among them. In other words, it is the size of the largest empty vertex-induced subgraph.

Example. The independent number for some canonical graphs are as follows:

- $\alpha(K_n) = 1$ for any n.
- $\alpha(C_n) = \left\lfloor \frac{1}{2}n \right\rfloor$ for any $n \ge 3$.
- $\alpha(P_n) = \left\lceil \frac{1}{2}n \right\rceil$ for any $n \ge 2$.

Fact. $\chi(G) \ge \frac{n}{\alpha(G)}$ for any graph G.

Proof. Interpret a k-coloring as a grouping of G into independent sets $S_1, S_2, \cdots, S_k$. It follows that $n = \sum_{i=1}^{k} |S_i| \le k \cdot \alpha(G)$, which implies the conclusion. In particular, if $k = \chi(G)$ then we get the conclusion. □

Definition. The clique number of a graph G, denoted $\omega(G)$, is the largest t for which K_t is a subgraph of G.

Example. $\chi(K_n) = n$. Also,

$$\omega(C_n) = \begin{cases} 2 & n \ge 4 \\ 3 & n = 3 \end{cases}.$$

Additionally, $\omega(P_n) = 2$ and $\omega(\text{PG}) = 2$.

Fact. $\chi(G) \ge \omega(G)$.

Proof. $\chi(G) \ge \chi(K_{\omega(G)}) = \omega(G)$. □

23.5 Upper Bound

Fact. $\chi(G) \leq n$.

Proof. Please tell me you know how to prove this. □

Clearly the best way to get bounds is by construction.
The greedy algorithm give us a bound as follows:

> Fix a labelling of $V(G)$. Then color each vertex in order, adding a new color only if necessary.

Being the greedy algorithm, this may be suboptimal. For instance, consider a labelling of P_4 as below:

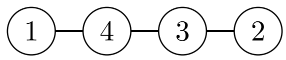


Figure 47: P_4, which has chromatic number 2, but the greedy algorithm produces a 3-coloring.

As a corollary of the algorithm, we may observe

$$\chi(G) \leq \max_{)i=1}^{n} |N(v_i) \cap \{v_1, v_2, \cdots, v_{i-1}\}|.$$

In particular, since $N(v_i) \leq \Delta(G) + 1$, we have the weak bound

Fact. $\chi(G) \leq \Delta(G) + 1$ for all graphs G.

However, this can actually be strengthened!

Theorem 23.1 (Brook's Theorem)**.** *Let G be a connected graph. If G is not K_n nor an odd cycle, then $\lambda(G) \leq \delta(G)$.*

Proof. 279A. □

23.6 More examples

Exercise. Show that $\chi(\text{PG}) = 3$.

Proof. $\chi(\text{PG}) \leq 3$ via Brook's Theorem. On the other hand, $\chi(\text{PG}) \geq 3$ because it contains an odd cycle. Hence, $\chi(\text{PG}) = 3$. □

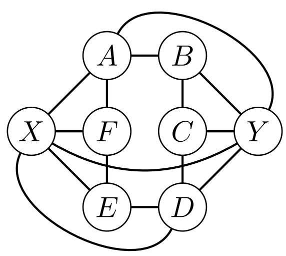


Figure 48: An arbitrary graph.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Table 2: 2×2 Sudoku

Exercise. Find the chromatic number of the graph shown in figure 48.

Proof. $\chi(G) = 4$. Color chase with the bound $\chi(G) \geq 3$ (because of the triangle). $\square$

Exercise (Fake Sudoku)**.** Consider the graph G on vertices $\{1, 2, \cdots, 16\}$, where two vertices are adjacent if they are in the same row, column, or 2×2 box (see figure 2). Find the chromatic number.

Proof. Clearly $\chi(G) \geq 4$ because there is a 4-clique. But a 4-coloring exists, because otherwise Sudoku would be a very boring game. $\square$

24 May 6, 2013

24.1 Today's Topics

- $\chi(G)$ versus $\chi(\overline{G})$
- Coloring maps

24.2 Examples

Question. How are $\chi(G)$ and $\chi(\overline{G})$ related?

Example. When $G = C_4$, we have 2 and 2, respectively. When $G = K_n$, we have n and 1 respectively.

24.3 Relating chromatic numbers

Theorem 24.1. *For any graph G of order n, $\chi(G)\chi(\overline{G}) \ge n$ and $\chi(G) + \chi(\overline{G}) \le n + 1$.*

Notice that both bounds are tight in our example. It is not hard, however, to find instances where these bounds are not tight. For example, take the Peterson graph. Here $n = 10$, $\chi(\text{PG}) = 5$, and $\chi(\overline{\text{PG}}) = 4$

Proof. For the first part, note that

$$\chi(G)\chi(\overline{G}) \ge \frac{n}{\alpha(G)}\omega(\overline{G}) = \frac{n}{\alpha(G)}\alpha(G) = n.$$

The second inequality is via induction on n. The base case is $n = 1$, which is trivial. For the inductive step, consider a graph G of order $n + 1$. Take any $v \in V(G)$; then $G - v$ is a graph of order n. By the induction assumption, we have

$$\chi(G - v) + \chi(\overline{G - v}) \le n + 1.$$

We want to compare $\chi(G - v)$ and $\chi(G)$. Obviously we have $\chi(G) \le \chi(G - v) + 1$, but this is not tight enough. However, if $\deg_G v < \chi(G - v)$, then this inequality must be strict – we simply pick a color that v is not adjacent to.

If either inequality is strict, then we can simply sum the inequalities to yield the desired. Otherwise, we must have $\deg_G v \ge \chi(G - v)$ and $\deg_{\overline{G}}(v) \ge \chi(\overline{G - v})$. Then, taking both the inequalities as equalities gives

$$\begin{aligned} n + 2 &> n + 1 \\ &= 2 + (n - 1) \\ &= 2 + \deg_G v + \deg_{\overline{G}} v \\ &\ge 1 + \chi(G - v) + 1 + \chi(\overline{G - v}) \\ &= \chi(G) + \chi(\overline{G}) \end{aligned}$$

□

24.4 Perfect Graphs

Definition. A graph G is called perfect if for every vertex-induced subgraph H of G, we have $\chi(H) = \omega(H)$.

Theorem 24.2 (Perfect Graph Theorem, Lovász, 1972)**.** *G is perfect if and only if $\overline{G}$ is perfect.*

Theorem 24.3 (Strong Perfect Graph Theorem)**.** *G is perfect if and only if G does not contain C_{2k+1} or $\overline{C_{2k+1}}$ for any $k \ge 2$.*

24.5 Map coloring

Definition. A *map* is a region subdivided into smaller regions. Two subregions are *neighboring* if they share a boundary line.

Remark. Utah and New Mexico are not neighbors because they intersect at only a single point.

Problem (The problem of 5 princes)**.** Divide the kingdom into five pieces such that each prince's region is adjacent to any other region.

Obviously this is not possible because of the four color theorem. For a flavorful description of this,

Problem (The problem of 5 palaces)**.** Do the same, but draw a palace in each region and connect them with non-intersecting roads.

It's easy to see the n-prince problem is equivalent to the n-palace problem, so $n = 5$ fails because K_5 is not planar. Okay.

This idea results in the *dual* of a map.

Definition. The *dual* of a map is the graph whose vertices are the faces are the regions of the map, and edges between any two neighboring/adjacent region.

It is easy to check that duals are planar. So the map coloring problem is simply the coloring of the planar graphs.

24.6 The 6-color problem and 5-color problem

Four colors is too hard. The six-color case, however, is immediate by the following lemma.

Lemma 24.4. *If G is planar, then $\delta(G) \le 5$.*

Proof. $E \ge 3V - 6$, so $\sum_{v \in V} \deg v < 6V$. This implies some vertex has degree less than 6, or at most 5. ☐

Corollary. *If G is planar, then $\chi(G) \le 6$.*

Proof. Induct on $n \ge 6$. For the inductive step, take the vertex v of minimal degree. 6-color all other vertices and then apply a suitable color to v. Since $\deg v \le 5$ this is possible. ☐

We can improve this.

Theorem 24.5. *If G is planar, then $\chi(G) \le 5$.*

Proof. Induct on $n \ge 5$ this time. Take a vertex v with degree at most 5. If $\deg v < 5$, then this is easy, since we can just 5-color $G - v$ as before.

On the other hand, suppose $\deg v = 5$. Since there cannot be a K_5 among these 5, we may assume v_1 and v_3 are non-adjacent. Contracting edges vv_1 and vv_3, we obtain a graph H of order $n - 2$. Now 5-color H. This implies there is a coloring scheme of $G - v$ where v_1 and v_3 are the same color. So, we simply select v as the fifth color. ☐

n	1	2	3	4	5	6	$\cdots$	10	11
#	1	1	2	6	20	99	$\cdots$	1052805	1744929

Table 3: Number of connected planar graphs

25 May 8, 2013

25.1 Today's Topics

Theorem 25.1 (Four Color Theorems)**.** *If G is planar, then $\chi(G) \le 4$.*

25.2 Brute Force

The number of connected planar graphs for n vertices is given as follows.

25.3 Small cases

$n \le 4$ is easy. $n \le 5$ can be knocked out using Brook's Theorem. We can push stronger.

Fact. The four-color theorem holds for $n \le 12$.

Proof. Since $\frac{2E}{n} \le \frac{2(3n-6)}{n} = 6 - \frac{12}{n} \le 5$. If G is not the icosahedron, then either either $n \le 11$ or G is not 5-regular. This is because the icosahedron is the unique edge-maximal planar graph with 12 vertices. Since the icosahedron is 4-colorable, we can ignore this case.

Hence there must be a vertex of degree at most 4. So now we can just mimic the proof of the five-color theorem. Note that in this case, if v is neighbors to $v_1, v_2, \cdots, v_4$, then they cannot for a K_4 because then they would form a K_5 with v. □

This only works for finitely many graphs, however.

25.4 Tightening via girth

If G is a tree/forest, then $\chi(G) = 2$. Now using girth, we can tighten our bound.

Fact. The four-color theorem holds for any graph with girth not equal to 3.

Proof. If G has girth at least four, we have $E \le \frac{g}{g-2}(n-2) \le 2n-4$. Hence,

$$\frac{2E}{n} \le \frac{2(2n-4)}{n} = 4 - \frac{8}{n} < 4.$$

Hence there is a vertex of degree at most 3. Color everything else and induct. □

25.5 General Proof

Construct an unavoidable set of reducible configurations. 633 left.

25.6 Open Problem

The contrapositive of the four color theorem reads

Corollary 25.2. *If $\chi(G) \ge 5$, then G is nonplanar; i.e. K_5 or $K_{3,3}$ is a minor of G.*

It turns out that $\chi(G) \ge 5 \implies K_5$ minor is actually equivalent to the four-color theorem (so in particular it is true).

This generalizes to an open problem.

Problem (Hadwiger Conjecture)**.** For any graph G, if $\chi(G) \ge r$ then K_r is a minor of G.

This is known for $r \le 6$.

26 Pre-Final

26.1 Current Grades

Homework Scores: 48, 50, 50, 46, 50, 50, 50, 50, 50, 50
Project Score: 10/10
Test 1 Score: 100/100
Test 2 Score: 93/100

26.2 Final

Final: 2:45 - 5:00 PM on Wednesday, May 15

1. Graph
2. Degree sequence
3. Reconstruction
4. Tree
5. k-connectivity
6. Hamiltonian
7. Tournament
8. Matching
9. Planar graph (minor)
10. Vertex coloring