18.950 (Differential Geometry) Lecture Notes

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This is MIT's undergraduate 18.950, instructed by Xin Zhou. The formal name for this class is "Differential Geometry".

The permanent URL for this document is `http://www.mit.edu/~evanchen/coursework.html`, along with all my other course notes.

Contents

1. September 10, 2015

Here is an intuitive sketch of the results that we will cover.

1.1. Curves

Consider a curve $\mu : [0, \ell] \to \mathbb{R}^3$ in $\mathbb{R}^3$; by adjusting the time scale appropriately, it is kösher to assume $\|\mu'(t)\| = 1$ for every t in the interval. We may also consider the acceleration, which is $\mu''(t)$; the **curvature** is then $\|\mu''(t)\|$ denoted $k(t)$.

Now, since we've assumed $\|\mu'(t)\| = 1$ we have dt is a "length parameter" and we can consider

$$\int_\mu k(t)\, dt.$$

Theorem 1.1 (Fenchel)
For any closed curve we have the inequality

$$\int_\mu k(t) \ge 2\pi.$$

with equality if and only if μ is a convex plane curve.

Here the assumption that $\|\mu'(t)\| = 1$ is necessary. (Ed note: in fact, this appeared as the last problem on the final exam, where it was assumed that μ bounded a region with mean curvature zero everywhere that was also homeomorphic to a disk.)

Remark 1.2. The quantity $\int_\mu k(t)$ is invariant under obvious scalings; one can easily check this by computation.

Some curves are more complicated, like a trefoil. In fact, geometry can detect topology, e.g.

Theorem 1.3 (Fory-Milnor)
If μ is knotted then $\int_\mu k(t)\, dt > 4\pi$.

1.2. Surfaces

We now discuss the topology of surfaces.

Roughly, the *genus* of a surface is the number of handles. For example, the sphere S^2 has genus 0.

In $\mathbb{R}^3$ it turns out that genus is the only topological invariant: two connected surfaces are homeomorphic if and only if they have the same genus. So we want to see if we can find a geometric quantity to detect this genus.

We define the **Euler characteristic** $\chi(\Sigma) = 2(1 - g(\Sigma))$ for a surface Σ. It turns out that this becomes equal to $\chi = V - E + F$ given a triangulation of the surface.

Define the **Gauss curvature** to a point $p \in \Sigma$: for a normal vector v, we consider B_p the tangent plane, and we consider

$$k(P) = \lim_{r \to 0} \frac{\text{Area}_{S^2}(\mu(\Sigma \cap B_p))}{\text{Area}_\Sigma(\Sigma \cap B_p)}.$$

where μ is the Gauss map; areas are oriented.

Theorem 1.4 (Gauss-Bonnet Theorem)
$\int_\Sigma k(p)dp = 2\pi\chi(\Sigma)$.

This relates a geometric quantity (left) to a topological quantity (right).

Example 1.5
The Gauss map is the identity map for S^2, so $k(p) = 1$ for every p on S^2.

1.3. Parametrized Curves

Let's do some actual calculus now.

Definition 1.6. A **parametrized curve** is a differentiable map $\alpha : I \to \mathbb{R}^3$ where $I = [a, b] \subset \mathbb{R}$.

2. September 15, 2015

Recall last time we considered a differential curve in space, i.e. a differentiable $\alpha : [a, b] \to \mathbb{R}^3$. Given $t \in I$, if $\alpha'(t) \neq 0$ the **tangent line** is the set

$$\left\{\alpha(t) + s\alpha'(t) \mid s \in \mathbb{R}\right\}.$$

2.1. Regular curves

If $t \in [a, b]$ has $\alpha'(t) = 0$, then we say t is a **singular point** of α. In this class we will focus on **regular curves** which do not contain any singularities. In this case,

Given a regular curve $\alpha : I \to \mathbb{R}^3$ we define the **arc length** from a fixed parameter $t_0 \in I$ is

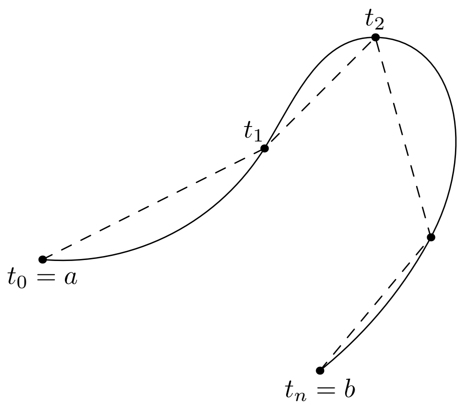
$$s(t) \overset{\text{def}}{=} \int_{t_0}^{t} \left|\alpha'(t)\right| \, dt.$$

Example 2.1 (Actual Coordinates)

Suppose we equip $\mathbb{R}^3$ with its standard inner product and put $\alpha(t) = (x(t), y(t), z(t))$. Then

$$s(t) = \int_{t_0}^{t} \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, dt.$$

The integral captures the ideal that we can approximate a curve by "close" straight line segments..



To be more precise, we consider a partition

$$P : a = t_0 < t_1 < \cdots < t_n = b$$

and consider

$$\ell(\alpha, P) = \sum_i \|\alpha(t_i) - \alpha(t_{i-1})\|$$

and as the **mesh** of P (defined as $\max_i(t_i - t_{i-1})$ approaches zero).

Proposition 2.2 (Riemann Integral for Arc Length)

As the mesh of the partition approaches zero,

$$\ell(a, P) \to \int_a^b \left\|\alpha'(t)\right\| \, dt.$$

Proof. Because $[a, b]$ is compact, there is a uniform C_1 such that $\|\alpha''(t)\| < C$. In what follows all big O estimates depend only on α.

Then we can write for any $t > t_{i-1}$ the inequality

$$\left\|\alpha'(t) - \alpha'(t_{i-1})\right\| = \left\| \int_{t_{i-1}}^{t} \alpha''(r)\, dr \right\| \le \int_{t_{i-1}}^{t} \left\|\alpha''(r)\right\|\, dr \le C\left[t - t_{i-1}\right].$$

Thus, we conclude that

$$\int_{t_{i-1}}^{t_i} \left\|\alpha'(t) - \alpha'(t_{i-1})\right\| = O\left([t_i - t_{i-1}]^2\right). \tag{1}$$

Therefore, we can use (1) to obtain

$$\begin{aligned} \alpha(t_i) - \alpha(t_{i-1}) &= \int_{t_{i-1}}^{t_i} \left[\alpha'(t_{i-1}) + \alpha'(t) - \alpha'(t_{i-1})\right]\, dt \\ &= (t_i - t_{i-1})\alpha'(t_{i-1}) + \int_{t_{i-1}}^{t_i} \left[\alpha'(t) - \alpha'(t_{i-1})\right]\, dt \\ \implies (t_i - t_{i-1}) \left\|\alpha'(t_{i-1})\right\| &= \|\alpha(t_i) - \alpha(t_{i-1})\| + O\left([t_i - t_{i-1}]^2\right). \end{aligned}$$

Thus,

$$\begin{aligned} \int_{t_{i-1}}^{t_i} \left\|\alpha'(t)\right\|\, dt &= \int_{t_{i-1}}^{t_i} \left\|\alpha'(t) - \alpha'(t_{i-1}) + \alpha'(t_{i-1})\right\|\, dt \\ &\le \int_{t_{i-1}}^{t_i} \left\|\alpha'(t) - \alpha'(t_{i-1})\right\|\, dt + (t_i - t_{i-1}) \left\|\alpha'(t_{i-1})\right\| \end{aligned}$$

And applying the above result with (1) gives

$$\begin{aligned} &= O\left([t_i - t_{i-1}]^2\right) + \left[\|\alpha(t_i) - \alpha(t_{i-1})\| + O\left([t_i - t_{i-1}]^2\right)\right] \\ &= \|\alpha(t_i) - \alpha(t_{i-1})\| + O\left([t_i - t_{i-1}]^2\right). \end{aligned}$$

Summing up gives the conclusion. □

2.2. Arc Length Parametrization

Suppose $\alpha : [a, b] \to \mathbb{R}^3$ is regular. Then we have

$$s(t) = \int_a^t \left\|\alpha'(t)\right\|\, dt.$$

Observe that

$$\frac{ds(t)}{dt} = \left\|\alpha'(t)\right\| \ne 0.$$

Thus the Inverse Function Theorem implies we can write $t = t(s)$ as a function of s; thus

$$\widetilde{\alpha}(s) \overset{\text{def}}{=} \alpha\left(t(s)\right)$$

is the **arc length parametrization** of s.

Hence, in what follows we can always assume that our curves have been re-parametrized in this way.

2.3. Change of Orientation

Given $\alpha : (a, b) \to \mathbb{R}^3$ we can reverse its orientation by setting $\beta : (-b, -a) \to \mathbb{R}^3$ by $\beta(s) = \alpha(-s)$.

2.4. Vector Product in $\mathbb{R}^3$

Given two sets of ordered bases $\{e_i\}$ and $\{f_i\}$, we can find a matrix A taking the first to the second. We say that these bases are *equivalent* if $\det(A) > 0$. Each of these two equivalence classes is called an **orientation** of $\mathbb{R}^n$.

Example 2.3 (Orientations when $n = 1$ and $n = 3$)

(a) Given $\mathbb{R}$ and a single basis $0 \neq e \in \mathbb{R}$, then the orientation is just whether $e > 0$.

(b) Given $\mathbb{R}^3$ the orientation can be determined by the right-hand rule.

2.5. OH GOD NO

We define the cross product in $\mathbb{R}^3$ using the identification with the Hodge star:

$$\vec{u} \times \vec{v} = \det \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}.$$

(The lecturer uses $\times$, but this is misleading since it is not an exterior power.) This is the vector perpendicular to $\vec{u}$ and $\vec{v}$ satisfying

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \, \|\vec{v}\| \sin\theta.$$

This is anticommutative, bilinear, and obeys

$$(\vec{u} \times \vec{v}) \cdot (\vec{w}) = \det \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}.$$

Also,

$$(\vec{u} \times \vec{v}) \cdot (\vec{x} \times \vec{y}) = \det \begin{pmatrix} \vec{u} \cdot \vec{x} & \vec{v} \cdot \vec{x} \\ \vec{u} \cdot \vec{y} & \vec{v} \cdot \vec{y} \end{pmatrix}$$

which one can check by using linearity.

More properties:

- $\vec{u} \times \vec{v} = 0 \iff \vec{u} \parallel \vec{v}$.
- $\frac{d}{dt}(\vec{u} \times \vec{v})(t) = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$.

2.6. Concrete examples

Consider the helix

$$\alpha(t) = (a\cos(t), a\cos(t), b).$$

Note that

$$\left\|\alpha'(t)\right\| = \sqrt{a^2 + b^2} \overset{\text{def}}{=} r.$$

Hence the arc length parametrization is just $\widetilde{\alpha}(s) = \alpha(s/r)$.

3. September 17, 2015

3.1. Always Cauchy-Schwarz

Pointing out explicitly from last lecture:

Proposition 3.1 (Triangle Inequality in the Spirit of L_1)
For any $f : I \to \mathbb{R}^n$ we have

$$\left\| \int_0^\ell f(t)\, dt \right\| \le \int_0^\ell \|f(t)\|\, dt.$$

Proof. By components. □

3.2. Curvature

Assume we have a curve $\alpha : I \to \mathbb{R}^3$ with $\|\alpha'(s)\| = 1$ meaning that $\alpha'(s)$ actually interprets an angle. Then we can consider $\alpha''(s)$, which measures the *change* in the angle.

Definition 3.2. The **curvature** $k(s)$ is $\|\alpha''(s)\|$.

Remark 3.3. Note that $k(s) \equiv 0$ if and only if α is a straight line.

Remark 3.4. The curvature remains unchanged by change of orientation.

Recall now that $\|\alpha'(s)\| = 1 \iff \alpha''(s) \perp \alpha'(s)$. Thus, we can define

$$n(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}.$$

and call this the **normal vector**. At this point we define $t(s)$ as the **tangent vector** $\alpha'(s)$. The plane spanned by $\alpha'(s)$ and $\alpha''(s)$ is called the **osculating plane**.

Definition 3.5. If $k(s) = 0$ at some point we say s is a **singular point of order 1**.

3.3. Torsion

Definition 3.6. The **binormal vector** $b(s)$ is defined by $t(s) \times n(s)$.

Proposition 3.7
In $\mathbb{R}^3$ we have that $b'(s)$ is parallel to $n(s)$.

Proof. By expanding with Leibniz rule we can deduce

$$b'(s) = t(s) \times n'(s).$$

Now, $\|b(s)\| = 1 \implies b'(s) \perp b(s)$. Also, $b'(s) \perp t(s)$. This is enough to imply the conclusion. □

Definition 3.8. In light of this, we can define $b'(s) = \tau(s)n(s)$, and we call $\tau(s)$ the **torsion**.

Proposition 3.9 (Plane Curve $\iff$ Zero Torsion)
The curve α is a plane curve only if it has zero torsion (meaning $\tau(s) \equiv 0$ everywhere).

Proof. $\tau(s) = 0 \iff b'(s) = 0 \iff b(s) = b_0$ for a fixed vector b_0. For one direction, note that if $b(s) = b_0$ we have

$$\frac{d}{ds}\left[\alpha(s) \cdot b_0\right] = \alpha'(s) \cdot b_0 = t(s) \cdot b_0 = 0$$

the last step following from the fact that $b(s)$ is defined to be normal to the tangent vector. Thus $\alpha(s) \cdot b_0$ is constant, so we can take s_0 such that

$$b_0 \cdot (\alpha(s) - \alpha(s_0)) = 0$$

holds identically in s. The other direction is similar (take the derivative of the lpane equation). □

Remark 3.10. The binormal vector remains unchanged by change of orientation.

Example 3.11 (Helix)
Consider again the helix

$$\alpha(s) = \left(a\cos\frac{s}{r}, a\sin\frac{s}{r}, b\frac{s}{r}\right).$$

We leave to the reader to compute

$$b(s) = \left(\frac{b}{r}\sin(s/r), -\frac{b}{r}\cos(s/r), a/r\right).$$

Thus

$$b'(s) = \left(\frac{b}{r^2}\cos(s/r), \frac{b}{r^2}\sin(s/r), 0\right).$$

One can also compute

$$n(s) = \frac{\alpha''(s)}{k(s)} = \left(-\cos(s/r), -\sin(s/r), 0\right).$$

Thus one can obtain the torsion $\tau(s) = -b/r^2$.

3.4. Frenet formulas

Let $\alpha : I \to \mathbb{R}^3$ be parametrized by arc length. So far, we've associated three orthogonal unit vectors

$$\begin{aligned} t(s) &= \alpha'(s) \\ n(s) &= \frac{\alpha''(s)}{\|\alpha''(s)\|} \\ b(s) &= t(s) \times n(s) \end{aligned}$$

which induce a positive orientation of $\mathbb{R}^3$. We now relate them by:

Theorem 3.12 (Frenet Formulas)

We always have

$$\begin{aligned} t'(s) &= k(s)n(s) \\ n'(s) &= -k(s)t(s) - \tau(s)b(s) \\ b'(s) &= \tau(s)n(s) \end{aligned}$$

Proof. The first and third equation follow by definition. For the third, write $n = b \times t$ and differentiate both sides, applying the product rule. $\square$

So it seems like we can recover t, n, b from k, τ by solving differential equations. In fact, this is true:

Theorem 3.13 (Fundamental Theorem of the Local Theory of Curves)

Given differentiable functions $k(s) > 0$ and $\tau(s)$ on an interval $I = (a, b)$, there exists a regular parametrized curve $\alpha : I \to \mathbb{R}^3$ with that curvature and torsion.

Moreover, this curve is unique up to a rigid motion, in the sense that if α and $\overline{\alpha}$ both satisfy the condition, then there is an orthogonal linear map $\rho : \mathbb{R}^3 \to \mathbb{R}^3$ with $\det \rho > 0$ such that $\overline{\alpha} = p \circ \alpha + c$. Intuitively, this means "uniqueness up to rotation and translation".

The uniqueness part is not difficult; the existence is an ordinary differential equation.

3.5. Convenient Formulas

The curvature of an α not parametrized by arc length is

$$\kappa = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3}.$$

The torsion is

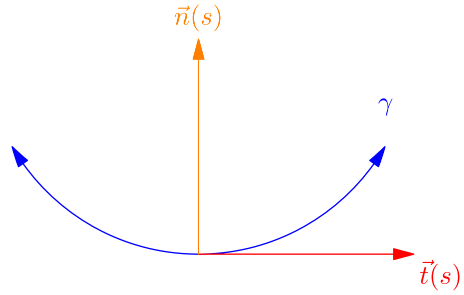
$$\tau = -\frac{(\alpha' \times \alpha'') \times \alpha'''}{|\alpha' \times \alpha''|^2}.$$

4. September 22, 2015

4.1. Signed Curvature

We wish to describe signed curvature for a plane curve.

Assume that $\alpha : I \to \mathbb{R}^2$, so we have a tangent vector $t(s)$. Then we can give a unique normal vector $n(s)$ such that (t, n) forms a positive basis.



Thus we can define the **signed curvature** according to

$$t'(s) = k(s)n(s)$$

which may now be either positive or negative.

4.2. Evolute of a Curve

Given a plane curve $\alpha : I \to \mathbb{R}^2$, we define the **evolute** of α to be the curve

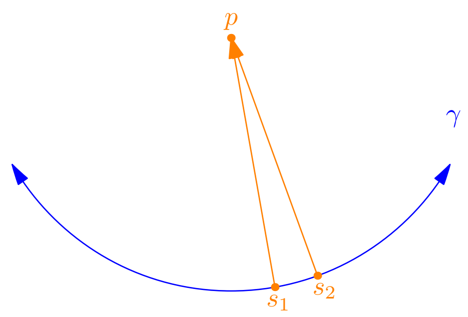
$$\beta(s) = \alpha(s) + \frac{1}{k(s)}n(s).$$

Proposition 4.1 (Properties of the Evolute)
Let $\alpha(s)$ be a curve parametrized by arc length and β its evolute. Then

(a) β is normal to α, and

(b) Let s_1 and s_2 be points, and consider the intersection p of their normal vectors to α. As $s_1, s_2 \to s$, $p \to \beta(s)$.

We call $\beta(s)$ the **center of curvature** of $\alpha(s)$.



Proof. Part (a) is easy, since the Frenet equations imply $n'(s) = -kt$ (as the torsion vanishes), and hence

$$\beta'(s) = \alpha'(s) - \frac{1}{k(s)^2}k'(s)n(s) + \frac{1}{k}n'(s) = t - \frac{k'}{k^2}n + \frac{1}{k}(-kt) = -\frac{k'}{k^2}n(s).$$

Part (b) involves more arithmetic. Let $\alpha(s) = (x(s), y(s))$ so $t(s) = (x'(s), y'(s))$. Then $n(s) = (-y'(s), x'(s))$. Now, given s_1 and s_2 we seek the point

$$p = \alpha(s_1) + \lambda_1 n(s_1) = \alpha(s_2) + \lambda_2 n(s_2)$$

for some λ_1 and λ_2. One can check after work that

$$\lambda_1 = \frac{x'(s_2)(x(s_2) - x(s_1)) + y'(s_2)(y(s_2) - y(s_1))}{y'(s_2)x'(s_1) - x'(s_2)y'(s_1)}.$$

In the limit, one can check that this becomes

$$\lim_{s_1, s_2 \to s} \lambda_1 = \frac{x'(s)^2 + y'(s)^2}{y''(s)x'(s) - x''(s)y'(s)} = \frac{1}{y''(s)x'(s) - x''(s)y'(s)} = \frac{1}{k(s)}.$$

Thus, $\alpha(s) + \lambda(s)n(s) = \beta(s)$ by definition. □

4.3. Isoperimetric Inequality

Problem 4.2. Of all simple closed curves in the plane with a given length ℓ, which one bounds the largest area?

In this situation we have $\alpha : [a, b] \to \mathbb{R}^3$ a regular curve. It is **closed** if $\alpha(a) = \alpha(b)$ and $\alpha^{(k)}a = \alpha^{(k)}b$ for all $k \in \mathbb{N}$. It is **simple** if it is not self-intersecting. We will quote without proof that

Theorem 4.3 (Jordan Curve Theorem)
A closed simple regular curve bounds a region in $\mathbb{R}^2$.

Thus one can assign the curve a positive orientation, namely the counterclockwise direction (formally, $n(s)$ points into the bounded region D described by the Jordan curve theorem).

We will now give:

Theorem 4.4 (Isoperimetric Inequality)
Let α be a simple closed plane curve of length ℓ and let A be the area of the bounded region. Then

$$\ell^2 \geq 4\pi A$$

with equality if and only if α gives a circle.

To bring areas into the picture, we also quote:

Theorem 4.5 (Special Case of Green's Theorem)
Given closed simple regular $\alpha(t) = (x(t), y(t))$ the area bounded is

$$A = -\int_a^b yx' = \int_a^b xy' = \frac{1}{2}\int_a^b (xy' - yx')\, dt$$

which can be found e.g. by using Green's Theorem.

Proof of Isometric Inequality. Parametrize the original curve $\alpha = (x, y)$ by arc length. Thus $\alpha : [a, b] \to \mathbb{R}^2$.

Suppose that x runs from a maximal right side coordinate x_{max} and a minimal x_{min}; we let $2r = x_{\text{max}} - x_{\text{min}}$, and now draw a circle S with radius r, centered at the origin. Now we parametrize S by β but not by arc length: rather, we *parametrize β so that the x-coordinates of β and α always coincide.* So $\beta : [a, b] \to S$. Let $\beta = (x, \overline{y})$.

Let A be the area of the bounded region. Then by our area form

$$\begin{aligned} A + \pi r^2 &= \int_a^b (xy' - \overline{y}x') \, ds \\ &\le \int_a^b \sqrt{(x^2 + \overline{y}^2)(x'^2 + y'^2)} \, ds \\ &= \int_a^b \sqrt{x^2 + \overline{y}^2} \, ds \\ &= \int_a^b r \, ds \\ &= \ell r \end{aligned}$$

Now by AM-GM we have

$$\ell r = A + \pi r^2 \ge 2r\sqrt{A\pi}$$

and this gives the desired bound on A. $\square$

5. September 24, 2015

We now begin the study of surfaces.

5.1. Regular Surfaces

Definition 5.1. A **regular surface** is a subset $S \subseteq \mathbb{R}^3$ such that for every point p, there is a neighborhood $p \in V \subseteq \mathbb{R}^3$ which is *diffeomorphic* to an open subset of $\mathbb{R}^2$.

By diffeomorophic, we mean that there is a map

$$\overline{x} : \mathbb{R}^2 \supset U \to V \cap S \subset \mathbb{R}^3$$

such that $\overline{x}$ is differentiable and bijective, with *continuous inverse*, and moreover the differential $d\overline{x} : \mathbb{R}^2 \to \mathbb{R}^3$ is *injective* (i.e. it has full rank). We call the $\overline{x}$ a **parametrization** at x and the ordered triple $(x, U, V \cap S)$ a **coordinate chart**.

(We don't yet require the inverse to be differentiable because at the moment our set S only has a topological structure, and don't yet have a differential structure on S.)

In practice to check that the differential is injective one simply computes the associated matrix and evaluates the rank (say by finding a 2×2 minor with full rank).

Example 5.2 (Projection of Sphere)

For a sphere $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$, we can take a projection onto the xy, yz, zx planes for a total of six charts. For example on of the charts is

$$\mathbb{R}^2 \supset \{u^2 + v^2 < 1\} \to \mathbb{R}^3 \quad \text{by} \quad (u, v) \mapsto (u, v, \sqrt{1 - u^2 - v^2}).$$

Let's verify this actually works. First, it is clear the map is differentiable. To see that the differential has full rank, we simply use the first two coordinates and put

$$\frac{\partial(u, v)}{\partial(u, v)} = \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1.$$

Finally, we need to construct a continuous inverse. We just take the projection $(x, y, z) \mapsto (x, y)$. This is clearly continuous.

5.2. Special Cases of Manifolds

As the above example shows it does take some time to verify the (rather long) definition of a manifold. So we would like some way to characterize regular surfaces more rapidly.

One obvious way to do this is:

Theorem 5.3 (Graphs are Regular Surfaces)

Let $U \subset \mathbb{R}^2$ and $f : U \to \mathbb{R}$ be differentiable. Then the **graph** of f, i.e. the set

$$\{(x, y, f(x, y)) \mid (x, y) \in U\}$$

is a regular surface.

Proof. Repeat the work we did when we checked the sphere formed a regular surface. Note that in this case we actually have a global parametrization, namely $U \to \mathbb{R}^3$ by $(x, y) \mapsto (x, y, f(x, y))$. $\square$

Here is a second way to do this. First, recall that

Definition 5.4. Given a differential map $F : U \to \mathbb{R}^m$ (where $U \subset \mathbb{R}^n$ is open) a **critical point** $p \in U$ of F is one such that $(dF)_p : \mathbb{R}^n \to \mathbb{R}^m$ is not surjective. The value of $F(p)$ is called **critical value**.

Values in the image $\mathbb{R}^m$ which are not critical are called **regular values**.

In particular, if $F : U \to \mathbb{R}$ then p is a critical value if and only if $(DF)_p$ vanishes, i.e. the gradient is zero.

Now we claim that

Theorem 5.5 (Level Images are Surfaces)
Let $f : \mathbb{R}^3 \supset U \to \mathbb{R}$ is differentiable and c be a regular value. Then the pre-image

$$\left\{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = c\right\}$$

is a regular surface.

The proof of this is essentially the inverse function theorem.

6. September 29, 2015 and October 1, 2015

I was absent this week, so these notes may not be a faithful representation of the lecture.

6.1. Local Parametrizations

Regular surfaces are nice because we have local coordinates at every point.

Proposition 6.1 (Local Parametrization as Graph)
Let $S \subset \mathbb{R}^3$ and $p \in S$. Then there exists a neighborhood V of $p \in S$ such that V is the graph of a differentiable function in one of the following forms: $x = f(y,z)$, $y = f(z,x)$, $z = f(x,y)$.

Sketch of Proof. By definition of regular surface, we can take a parametrization at p. One of the Jacobian determinants is nonzero, by hypothesis. $\square$

Finally, we state a lemma that says that if we know S is regular already, and we have a candidate x for a parametrization $x : U \to \mathbb{R}^3$ for $U \subset \mathbb{R}^2$, then we need not check x^{-1} is continuous if x is injective.

Proposition 6.2
Let $p \in S$ for a regular surface S. Let $x : U \to \mathbb{R}^3$ be differentiable with image in p such that the differential dx is injective everywhere. Assuming further that x is injective then x^{-1} is continuous.

6.2. Differentiability

We want a notion of differentiability for a function $f : V \to \mathbb{R}$ where V is an open subset of a surface S. This is the expected:

Definition 6.3. Let $f : V \subset S \to \mathbb{R}$ be a function defined in an open subset V of a regular surface S. Then f is **differentiable** at $p \in V$ if for some parametrization $x : U \subset \mathbb{R}^2 \to S$, the composition

$$U \xrightarrow{x} V \xrightarrow{f} \mathbb{R}.$$

is differentiable at $x^{-1}(p)$ (giving $x^{-1}(p) \mapsto p \mapsto f(p)$).

One can quickly show this is independent of the choice of parametrization, so "for some" can be replaced by "for all".

6.3. The Tangent Map

Given our embedding in to ambient space, we can define the tangent space as follows.

Definition 6.4. A **tangent vector** to $p \in S$ is the tangent vector $\alpha'(0)$ of a smooth curve $\alpha : (-\varepsilon, \varepsilon) \to S$ with $\alpha(0) = p$.

The "type" of the "tangent vector" is actually vector in this case.

Proposition 6.5

Let $x : U \to S$ be a parametrization ($U \subset \mathbb{R}^2$). For $q \in U$, the vector subspace

$$dx_q(\mathbb{R}^2) \subset \mathbb{R}^3$$

coincides with the set of tangent vectors of S at q.

In particular, $dx_q(\mathbb{R}^2)$ doesn't depend on the parametrization x, so we denote it now by $T_p(S)$, the tangent space at S.

This lets us talk about the notion of a differential of a smooth map between surfaces. Specifically, suppose

$$S_1 \supset V \xrightarrow{\phi} S_2.$$

Then given a map $\alpha : (-\varepsilon, \varepsilon) \to V$ we get a composed path β

$$\begin{tikzcd}
(-\varepsilon,\varepsilon) \arrow[d, "\alpha"'] \arrow[dr, "\beta"] & \\
V \arrow[r, "\phi"'] & S
\end{tikzcd}$$

Proposition 6.6

Consider the situation above. The value of $\beta'(0)$ depends only on $\alpha'(0)$, so we obtain a map

$$(d\phi)_p : T_p(S_1) \to T_{\phi p}(S_2).$$

Moreover, this map is linear; actually,

$$\beta'(0) = d\phi_p(w) = \begin{pmatrix} \frac{\partial \phi_1}{\partial u} & \frac{\partial \phi_1}{\partial v} \\ \frac{\partial \phi_2}{\partial u} & \frac{\partial \phi_2}{\partial v} \end{pmatrix} \begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix}$$

where $\phi = (\phi_1, \phi_2)$ in local coordinates.

7. October 6, 2015

7.1. Local Diffeomorphism

Definition 7.1. A map $\phi : U \subset S_1 \to S_2$ is a **local diffeomorphism** at $p \in U$ if there exists a neighborhood $p \in V \subset U$ such that ϕ is a diffeomorphism of V onto its image in S_2.

Example 7.2 (Local $\neq$ Global)
The map $S^1 \to S^2$ by $z \mapsto z^2$ is a local diffeomorphism but not globally (since it is not injective).

We can check whether functions are local diffeomorphisms by just their differentials.

Proposition 7.3 (Inverse Function Theorem)
Let S_1 and S_2 be regular surfaces, $\phi : U \subset S_1 \to S_2$ differentiable. If the differential

$$(d\phi)_p : T_p S_1 \to T_{\phi(p)} S_2$$

is a linear isomorphism (full rank), then ϕ is a local diffeomorphism at p.

Proof. The point is that at p, $\phi(p)$, S_1 and S_2 look locally like Euclidean space. Using the parametrizations x and $\overline{x}$ respectively, we induce a map $U \to \overline{U}$ whose differential is an isomorphism:

$$\begin{tikzcd} S_1 \arrow[r, "\phi"] \arrow[d, leftrightarrow, "x"'] & S_2 \arrow[d, leftrightarrow, "\overline{x}"] \\ U \arrow[r] & \overline{U} \end{tikzcd}$$

The classical inverse function theorem gives us an inverse map, which we then lift to a map of the surfaces. $\square$

7.2. Normals to Surfaces

Let $p \in S$ be a point, and let $x : U \to S$ be a local parametrization of S at p. Then we define the **normal** vector to the surface as follows: observe that the tangent space $T_p(S)$ has a basis x_1, x_2 corresponding to the canonical orientation of $\mathbb{R}^2$ (as inherited by U). Then we define

$$N(p) = \pm \frac{x_1 \times x_2}{|x_1 \times x_2|}.$$

Then, if two surfaces intersect at p, the angle between the surfaces is the angle between the normal vectors.

7.3. The First Fundamental Form

Let $S \hookrightarrow \mathbb{R}^3$ be a regular surface. Then for any $p \in S$, the tangent plane $T_p S$ inherits an inner product from $\mathbb{R}^3$, because it is a subsace of $\mathbb{R}^3$.

Thus, we have a map

$$I_p : T_pS \to \mathbb{R} \quad \text{by} \quad w \mapsto \langle w, w \rangle = \|w\|^2 .$$

Note that by abuse, we use $\langle -, - \rangle$ despite the fact that this form depends on p

Definition 7.4. The map I_p is called the **first fundamental form** of S at $p \in S$.

The power of I_p is that it lets us forget about the ambient space; we can talk about the geometry of the surface by looking just at I_p. In fact, it was shown very recently by John Nash that the intrinsic and extrinsic notion of geometry coincide: roughly, any such surface with a fundamental form of this sort can be embedded in $\mathbb{R}^n$.

To repeat: we will see that **the first fundamental form allows us to make measurements on the surface without referring back to the ambient space $\mathbb{R}^3$.**

One can compute this explicitly given local coordinates. Suppose we parametrize $\overline{x} : U \subset \mathbb{R}^2 \to S$ and take $p \in U$. Then if we denote $\mathbb{R}^2 = u\mathbb{R} \oplus v\mathbb{R}$, then T_pS has a basis $\{x_u, x_v\}$, where $x_u = \frac{\partial x}{\partial u}(p)$ and $x_u = \frac{\partial x}{\partial v}(p)$.

Now, let $w \in T_p(S)$. Let $\alpha : (-\varepsilon, \varepsilon) \to S$ such that $\alpha(t) = x(u(t), v(t))$ and

$$w = \alpha'(0) = u'(0)x_u + v'(0)x_v.$$

Then by expanding, we obtain

$$\begin{aligned} I_p(w) &= \langle w, w \rangle_p \\ &= \left\langle u'(0)x_u + v'(0)x_v, u'(0)x_u + v'(0)x_v \right\rangle \\ &= \|x_u\|^2 u'(0)^2 + 2 \langle x_u, x_v \rangle u'(0)v'(0) + \|x_v\|^2 v'(0)^2. \end{aligned}$$

We write this as

$$= E \cdot u'(0)^2 + 2F \cdot u'(0)v'(0) + G \cdot v'(0)^2.$$

where

$$\begin{aligned} E &= \langle x_u, x_u \rangle \\ F &= \langle x_u, x_v \rangle \\ G &= \langle x_u, x_v \rangle . \end{aligned}$$

Example 7.5 (Plane)

Let $P \subset \mathbb{R}^3$ be the plane parametrized globally by a single chart

$$x(u, v) = p_0 + uw_1 + vw_2$$

for some orthonormal vectors $w_!$ and w_2. Then, $(E, F, G) = (1, 0, 1)$.

Then given any tangent vector $w = aw_1 + bw_2$ at an arbitrary point p we get $I_p(w) = a^2 + b^2$; we just get the length of the vector.

Example 7.6 (Cylinder)
Consider the cylinder
$$C = \left\{ (x,y,z) \mid x^2 + y^2 = 1 \right\}.$$
We now take the chart
$$x(u,v) = (\cos u, \sin u, v)$$
where the domain is $U = (0, 2\pi) \times \mathbb{R}$; this covers most of the cylinder C (other than a straight line). We notice that the tangent plane is exhibited by the basis $x_u = (-\sin u, \cos u, 0)$ and $x_v = (0,0,1)$. Thus $E = \sin^2 u + \cos^2 u = 1$, $F = 0$, $G = 1$.

So the cylinder and the plane have the same first fundamental form. This is the geometric manifestation of the fact that if we cut a cylinder along an edge, then we get a plane.

Example 7.7 (Stuff Depends on Parametrization)
Suppose now we use the parametrization on the cylinder
$$x(u,v) = (\cos u, \sin u, 5v)$$
Then the tangent plane is exhibited by the basis $x_u = (-\sin u, \cos u, 0)$ and $x_v = (0,0,5)$. Thus $E = \sin^2 u + \cos^2 u = 1$, $F = 0$, $G = 5$.

7.4. Relating curves by the first fundamental form

Let $\alpha : I \to S$ be a regular curve. We already knew that the arc length is given by
$$s(t) = \int_0^t \left\| \alpha'(t) \right\| \, dt.$$
If the curve is contained entirely within a chart $\alpha(t) = x(u(t), v(t))$ then we may rewrite this as
$$\int_0^t \sqrt{I_{\alpha(t)} \alpha'(t)} \, dt = \int_0^t \sqrt{E(u')^2 + 2Fu'v' + G(v')^2} \, dt.$$
Colloquially, we can write this as $ds^2 = E(du)^2 + 2F du dv + G(dv)^2$.

Given $\alpha, \beta : I \to S$ now with $\alpha(t_0) = \beta(t_0)$, the angle θ between the two tangent curves is
$$\cos\theta = \frac{\langle \alpha'(t_0), \beta'(t_0) \rangle}{\left\| \alpha'(t_0) \right\| \left\| \beta'(t_0) \right\|}. =$$
As a special case, suppose α and β are coordinate curves, meaning $\alpha'(t) = x_u$ and $\beta'(t) = x_v$ for all α, β. (Think rulings on a surface.) The angle between them x_u and x_v is thus given by
$$\cos\phi = \frac{\langle x_u, x_v \rangle}{\left\| x_u \right\| \left\| x_v \right\|} = \frac{F}{\sqrt{EG}}.$$
Example application: loxodromes.

7.5. Areas

Let $R \subset S$ be a closed and bounded region of a regular surface contained in the neighborhood of a parametrization $x : U \subset \mathbb{R}^2 \to S$. Let $Q = x^{-1}(R)$. The positive number

$$\int_Q \|x_u \times x_v\| \, du \, dv$$

does not depend on the choice of x and is defined as the **area** of R.

In local coordinates, $\|x_u \times x_v\| = \sqrt{EG - F^2}$.

8. October 8, 2015 and October 15, 2015

I am now attending roughly only every third class or so. Here are my notes from my own reading from the content I missed.

8.1. Differentiable Field of Normal Unit Vectors

Let $N : S \to S^2 \subseteq \mathbb{R}^3$ be a map which gives a normal unit vector at each point.

For example, if $x : U \to S$ (where $U \subset \mathbb{R}^2$) is a parametrization, then we define for each $q \in x(U)$ by

$$N(q) = \frac{x_u \times x_v}{|x_u \times x_v|} \text{ at } q.$$

By abuse, we denote this N_q.

More generally, if $V \subset S$ is open, $N : V \to S^2 \subset \mathbb{R}^3$ is a differentiable map, then we say that N is a **differentiable field of normal unit vectors**

Remark 8.1. Not all surfaces admit such a field of unit vectors (Möbius strip, for example). Of course, most things (like spheres) do.

This N is called the **Gauss map**.

Example 8.2 (Cylinder)

Let S be the cylinder $\{(x, y, z) \mid x^2 + y^2 = 1\}$. We claim that $N = (-x, -y, 0)$ are unit normal vectors. A tangent vector is given by $\alpha'(0) = (x'(0), y'(0), z'(0))$ where $\alpha = (x, y, z)$ has $\alpha(0) = p$. Since $x(t)^2 + y(t)^2$, we see that along this curve, we see that $N \cdot \alpha' = 0$, as desired.

8.2. Second Fundamental Form

In this way, N provides an orientation on S. We see in this case, for any point p we have a map

$$dN_p : T_pS \to T_{N(p)}S^2 \xrightarrow{\simeq} T_pS$$

Example 8.3

Let S be the cylinder $\{(x, y, z) \mid x^2 + y^2 = 1\}$. Let us see what dN_p does to tangent vectors. Using the setup with α as before (so $\alpha(0) = p$), the tangent $N(t)$ to $\alpha(t)$ is $(-x(t), -y(t), z(t))$ and thus

$$dN_p\left(x'(0), y'(0), z'(0)\right) = \left(-x'(0), -y'(0), 0\right).$$

In other words, dN_p is projection onto the xy-plane. In particular, the tangents to the cylinder parallel to the xy-plane are -1 eigenvectors while the tangents to the cylinder parallel to the z-axis are all mapped to zero (in the kernel).

Lemma 8.4 (dN_p is self-adjoint)

As defined, $dN_p : T_pS \to T_pS$ is self-adjoint, in the sense that

$$\langle dN_p(v), w\rangle = \langle v, dN_p(w)\rangle.$$

Proof. Let $\alpha(0) = p$, $\alpha(t) = x(u(t), v(t))$, $\alpha'(0) = u'(0)x_u + v'(0)x_v$. Write

$$dN_p(\alpha'(0)) = \left[\frac{d}{dt}(N(u(t), v(t)))\right]_{t=0} = N_u u'(0) + N_v' v'(0).$$

So, it suffices to show that

$$\langle N_u, x_v \rangle = \langle x_v, N_u \rangle .$$

Take the derivatives of $\langle N, x_u \rangle = 0$ and $\langle N, x_v \rangle = 0$ relative to u and v to see that both are equal to $- \langle N, x_{uv} \rangle = - \langle N, x_{vu} \rangle$. □

Definition 8.5. Let $\mathbb{I}_p$ be the quadratic form defined in $T_p(S)$ by

$$\mathbb{I}_p(v) = - \langle dN_p(v), v \rangle = - \langle v, dN_p(v) \rangle .$$

This is called the **second fundamental form** of S at p.

Now, let us think about curves for a moment.

Definition 8.6. Let C be a regular curve in S, and $p \in S$ a point on it. Let k be the curvature of C at p, and set $\cos\theta = \langle n, N \rangle$, where n is the normal to C and N is the normal vector to S at p. The number $k_n = k \cos\theta$ is then called the **normal curvature** of $C \subset S$ at p.

In other words, k_n is the length of the projection of the vector $kn = \alpha'$ over the normal to the surface at p, with sign given by orientation.

Theorem 8.7 (Meusnier)
The normal curvature of a curve C at p depends only on the *tangent line* at the curve.

Proof. □

Thus, it make sense to talk about the curvature at a *direction* to the surface.

9. October 20, 2015

9.1. Self-Adjoint Maps

Since dN_p is self-adjoint, there exists an orthonormal basis $\{e_1, e_2\}$ of T_pS such that

$$dN_p(e_1) = -k_1 e_1 \quad \text{and} \quad dN_p(e_2) = -k_2 e_2.$$

Definition 9.1. The constant k_1 and k_2 are called the **principal curvature** and the basis e_1 and e_2 are called the **principal directions**.

Given $w \in T_pS$, say $w = \cos\theta e_1 + \sin\theta e_2$, then the value of the normal curvature to k_n along w is

$$\begin{aligned} -\langle dN_p(\cos\theta e_1 + \sin\theta e_2), \cos\theta e_1 + \sin\theta e_2\rangle &= \langle k_1 \cos\theta e_1 + k_2 \sin\theta e_2, \cos\theta e_1 + \sin\theta e_2\rangle \\ &= k_1 \cos^2\theta + k_2 \sin^2\theta. \end{aligned}$$

So we can express any normal curvature in terms of the principal ones.

9.2. Line of Curvatures

Definition 9.2. If a regular curve C on S has the property that for all $p \in C$, the tangent line of C at p is a principal direction at p, then C is called a **line of curvature** of S.

Example 9.3 (Examples of Lines of Curvatures)

(a) In a plane, all curves are lines of curvature (trivially). ($dN_p = 0$, so everything is an eigenvalue.)

(b) All curves on a sphere are also lines of curvature. ($dN_p = \mathrm{id}$, so everything is an eigenvalue.)

(c) On a cylinder, the vertical lines and the meridians are lines of curvatures of S.

Proposition 9.4 (Testing for Line of Curvature)

Let C be a regular curve. Then C is a line of curvature if and only if

$$dN_p(\alpha'(t)) = \lambda(t)\alpha'(t)$$

where $\alpha(t)$ is the arc length parametrization, and $\lambda(t) \in \mathbb{R}$.

Proof. "Harmless exercise". (A better word might be "trivial".) □

9.3. Gauss curvature and Mean Curvature

In the orthonormal basis of p, we have $dN_p = -\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$. Of course, we define

Definition 9.5. The quantity $\det dN_p = k_1 k_2$ is called the **Gauss curvature**. The quantity $\operatorname{Tr}(dN_p) = -(k_1 + k_2)$ is more finicky. We instead define $H = \frac{1}{2}(k_1 + k_2) = -\frac{1}{2}\operatorname{Tr}(dN_p)$ the **mean curvature**.

Remark 9.6. One can view H as the "average" of all curvatures of curves passing through H.

Definition 9.7. A point $p \in S$ is called

- **elliptic** if $\det(dN_p) > 0$, meaning k_1 and k_2 have the same sign.
- **hyperbolic** if $\det(dN_p) < 0$,
- **parabolic** if $\det(dN_p) = 0$, but $dN_p \neq 0$, and
- **planar** if dN_p is the zero matrix.

It's worth noting that planes are not the only example where planar points. For example, if $z = x^4$, and we wrote this around the z-axis to get a surface (i.e. $z = (x^2 + y^2)^2$) then the origin is also a planar point.

Definition 9.8. An **umbilical point** of S is a point $p \in S$ at which $k_1 = k_2$, meaning dN_p is a constant times the identity.

Theorem 9.9

All connected surfaces S which consist entirely of umbilical points are subsets of the sphere or a plane.

Proof. Prove it locally first. Then for any fixed p, for other $q \in S$ take a path from p to q, and use compactness. □

9.4. Gauss Map under Local Coordinates

Let $p \in S$ where S is an regular surface with an orientation $N : S \to S^2 \subset \mathbb{R}^3$. For $p \in S$ and $x : U \to S$ a parametrization near p, we can set

$$\alpha(t) = x(u(t), v(t)) \in S$$

such that $\alpha(0) = p$. So,

$$\alpha' = u' x_u + v' x_v$$

where x_u and x_v are the usual basis of $T_p S$ (i.e. the partial derivatives of x with respect to u and v). Thus, it follows that

$$dN_p(\alpha') = \frac{d}{dt} N(u(t), v(t)) = u' N_u + v' N_v.$$

So, we simply need to compute N_u and N_v. If we sidestep this for now and say $N_u = a_{11} x_u + a_{21} x_v$, and $N_v = a_{12} x_u + a_{22} x_v$, then this reads

$$dN_p \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} a_{11} u' + a_{12} v' \\ a_{21} u' + a_{22} v' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}.$$

Note that the matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ need not be symmetric unless x_u and x_v are orthogonal. (Self-adjoint implies symmetric only in this case.)

One can compute the a_{ij}, but since I really don't want to. The result is called the **equations of Weingarten**.

10. October 22, 2015

10.1. Completing Equations of Weingarten

Let me finish the computation from last time. Take $\alpha(t) = x(u(t), v(t))$, $\alpha(0) = p$, Recall the we have written

$$N_u = a_{11}x_u + a_{21}x_v$$
$$N_v = a_{11}x_u + a_{21}x_v$$

(again these subscripts are partial derivatives at the point p). The point is that

$$dN_p : T_pS \to T_pS \quad \text{by} \quad u'x_u + v'x_v \mapsto u'N_u + v'N_v$$

where $N_u, N_v \in T_pS$ as well. Thus, We have that

$$\mathbb{I}_p(\alpha') = \left\langle N_u u' + N_v v', x_u u' + x_v v' \right\rangle \overset{\text{def}}{=} e(u')^2 + 2fu'v' + g(v')^2$$

for $e = -\langle N_u, x_u \rangle = \langle N, x_{uu} \rangle$ and similarly for others. If we use the first equation, then putting in the a_{ij} we derive

$$-\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{dN_p} \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

In particular, we have

$$k_1 k_2 = \det dN_p = \frac{eg - f^2}{EG - F^2}.$$

Through pain and suffering, we also can compute

$$H = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2}.$$

10.2. An Example Calculation

We consider a torus, parametrized by

$$x(u, v) = ((a + r\cos u)\cos v, (a + r\cos u)\sin v, r\sin u).$$

The partial derivatives are

$$x_u = (-r\sin u\cos v, -r\sin u\sin v, r\cos u)$$
$$x_v = (-(a + r\cos u)\sin v, (a + r\cos u)\cos v, 0).$$

Thus, we can extract

$$E = \langle x, x_u \rangle = r^2$$
$$F = \langle x_u, x_v \rangle = 0$$
$$G = \langle x_v, x_v \rangle = (a + r\cos u)^2.$$

Let $N = \frac{x_u \times x_v}{|x_u \times x_v|}$, noting that the denominator is $\sqrt{EG - F^2}$. We define the constants

$$\begin{aligned}
e &= \langle N_u, x_u \rangle \\
&= \langle N, x_{uu} \rangle \\
&= \frac{\det \begin{pmatrix} -r \sin u \cos v & -r \sin u \sin v & r \cos u \\ -(a + r\cos u) \sin v & (a + r \cos u) \cos v & 0 \\ -r \cos u \cos v & -r \cos u \sin v & -r \sin u \end{pmatrix}}{\sqrt{EG - F^2}} \\
&= \frac{r^2(a + r \cos u)}{r(a + r\cos u)} = r.
\end{aligned}$$

Similarly,

$$\begin{aligned}
f &= \langle N, x_{uv} \rangle = \dots = 0 \\
g &= \langle N, x_{vv} \rangle = \dots = \cos u (a + r \cos u).
\end{aligned}$$

Thus, we have

$$K = \frac{\cos u}{r(a + r\cos u)}.$$

10.3. Elliptic and Hyperbolic Points

Proposition 10.1

Let $p \in S$ with S a regular surface.

- If $p \in S$ is elliptic, then all points in some neighborhood of S are on one side of the tangent plane T_pS
- If $p \in S$ is hyperbolic, for any neighborhood of p there is a point on both sides of T_pS.

Proof. Let $x(u, v)$ be a parametrization near p, with $x(0, 0) = p$. We're interested in the signs of the function

$$f(u, v) = \langle x(u, v) - x(0, 0), N \rangle$$

with N_p the normal at p. Locally, we have

$$x(u, v) = x(0, 0) + x_u(0, 0)u + x_v(0, 0)v + \frac{1}{2}\left[x_{uu}u^2 + 2x_{uv}uv + x_{vv}v^2\right] + R$$

with R an error term with $\frac{R}{u^2+v^2} \to 0$ as $(u, v) \to 0$. Now, $\langle x_u(0, 0), N \rangle = \langle x_v(0, 0), N \rangle = 0$. If we expand and go through the computation, denoting $\overline{R} = \langle R, N \rangle$ one can compute

$$f(u, v) = \frac{1}{2}\mathbb{I}_p(ux_u + vx_v) + \overline{R} = \frac{u^2 + v^2}{2}\left(\mathbb{I}_p\left(\frac{(u, v)}{\sqrt{u^2 + v^2}}\right) + \frac{\overline{R}}{u^2 + v^2}\right).$$

Suppose first that $k_1 \ge k_2 > 0$. Then $\mathbb{I}_p\left(\frac{(u,v)}{\sqrt{u^2+v^2}}\right) = \frac{1}{2}(k_1 \cos^2 \theta + k_2 \sin^2 \theta) \ge \frac{1}{2}\min\{k_1, k_2\}$ is bounded away from zero. So for $u^2 + v^2$ small enough, $f(u, v) \ge 0$. An analogous argument works if $k_1 > 0 > k_2$. $\square$

10.4. Principal Directions

Theorem 10.2 (Criteria for Principal Directions)

Fix a point $p \in S$. Then $u'x_u + v'x_v$ is an eigenvector of $(dN)_p$ if and only if

$$0 = \det \begin{pmatrix} (v')^2 & -u'v' & (u')^2 \\ E & F & G \\ e & f & g \end{pmatrix}.$$

11. October 29, 2015 and November 3, 2015

11.1. Isometries

Let S_1 and S_2 be surfaces.

Definition 11.1. A diffeomorphism $\phi : S \to S'$ is an **isometry** if for all w_1 and w_2 in $T_p(S)$, we have

$$\langle w_1, w_2 \rangle_p = \langle (d\phi)_p(w_1), (d\phi)_p(w_2) \rangle_{\phi(p)} .$$

In that case, we say S_1 and S_2 are **isometric**.

Naturally, this means that the first fundamental form I_p is preserved. In fact, by considering $\langle v+w, v+w \rangle - \langle v, v \rangle - \langle w, w \rangle$ we see that isometries are equivalent to preserving just the first fundamental form.

Definition 11.2. Let $p \in V \subset S'$, where S' is a surface. A map $\phi : V \to S$, where V is a neighborhood of some surface, is a **local isometry** at p if it is an isometry onto a neighborhood of $\phi(p)$

We say S **locally isometric** to S' if one can find a local isometry into S' at every point $p \in S$. If the other direction holds too, we say the surfaces are **locally isometric**.

Properties that are preserved under local isometry are said to be **intrinsic**. for example, the first fundamental form is intrinsic.

Example 11.3 (Cylinder and Plane are Locally Isometric)

The cylinder is locally isometric to a plane $(0, 2\pi) \times \mathbb{R}$, but not globally (since the spaces are not homeomorphic, their fundamental groups differ).

Example 11.4 (Half-Cylinder and Plane are Globally Isometric)

Half a cylinder is globally isometric to the plane $(0, \pi) \times \mathbb{R}$. Note that the second fundamental form of a cylinder is nonzero (unlike the plane),

Thus, second fundamental form is not intrinsic. In fact, we will later see that the mean curvature is not intrinsic, but the global curvature is.

For aid with local coordinates:

Proposition 11.5

Suppose $x : U \to S$ and $x' : U \to S'$ are parametrizations such that we have an equality $E = E'$, $F = F'$ and $G = G'$ of the coefficients of the first fundamental form. Then $x' \circ x^{-1}$ is a local isometry.

11.2. The Christoffel Symbols

This is some very long calculations. The idea is that x_u, x_v, N form a triple of orthogonal vectors at every point p of the surface. Then, we can let

$$\begin{aligned}
x_{uu} &= \Gamma^1_{11} x_u + \Gamma^2_{11} x_v + eN \\
x_{uv} &= \Gamma^1_{12} x_u + \Gamma^2_{12} x_v + fN \\
x_{vu} &= \Gamma^1_{21} x_u + \Gamma^2_{21} x_v + fN \\
x_{vv} &= \Gamma^1_{22} x_u + \Gamma^2_{22} x_v + gN.
\end{aligned}$$

Here the e, f, g coefficients were realized by taking the inner form of each equation with N. Also, note that $x_{uv} = x_{vu}$, so $\Gamma^i_{12} = \Gamma^i_{21}$ for $i = 1, 2$. We say the Γ^i_{jk} are the **Christoffel symbols**.

11.3. Gauss Equations and Mainardi-Codazzi

If we take dot products of all relations with x_u and x_v, we obtain a system of equations that lets us solve for everything in terms of E, F, G and their derivatives with respect to u, v. (Note e, f, g are not present!)

There are some more relations we can derive. For example, note that

$$(x_{uu})_v = (x_{uv})_u.$$

It turns out that by expanding this and equating the x_u coefficients, we obtain that

$$-KE = (\Gamma^2_{12})_u - (\Gamma^2_{11})_v + \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{11}\Gamma^2_{12}$$

where K is the Gauss curvature; this is the **Gauss equation** (Equating the x_v coefficients turns out to give the same equation.) The philosophical point is that it implies:

Theorem 11.6 (Theorema Egregium, due to Gauss)
The Gaussian curvature K is intrinsic, i.e. it is invariant under local isometries.

On the other hand, equating the N coefficients gives

$$e_u - f_u = e\Gamma^1_{12} + f(\Gamma^2_{12} - \Gamma^1_{11}) - g\Gamma^2_{11}.$$

Doing a similar thing gives

$$f_u - g_u = e\Gamma^1_{22} + f(\Gamma^2_{22} - \Gamma^1_{12}) - g\Gamma^2_{12}.$$

These two relations are called the **Mainardi-Codazzi** equations.

11.4. Bonnet Theorem

We may wonder if we can get more relations. The answer is no.

Theorem 11.7 (Bonnet)
Let E, F, G, e, f, g be differentiable functions $V \to \mathbb{R}$, with $V \subset \mathbb{R}^2$. Assume $E, G > 0$, that Gauss and Codazzi holds, and moreover $EG - F^2 > 0$. Then:

(a) for every $q \in V$ there is a neighborhood U of V such that a diffeomorphism $x : U \to \mathbb{R}^3$ such that $x(U)$ is a surface having the above coefficients.

(b) this x is unique up to translation and orthogonal rotation.

12. November 5, 10, 12, 17, 19 2015

In what follows, a vector field w on a surface S will mean a differentiable vector field of *tangents*; i.e. $w(p) \in T_pS$ for every $p \in S$.

12.1. The Covariant Derivative

Definition 12.1. Let w be a vector field, and let $y \in T_pS$ be a tangent vector, realized by $y = \alpha'(0)$ for $\alpha : (-\varepsilon, \varepsilon) \to S$ parametrized. The **covariant derivative** is the projection of $(w \circ \alpha)'(0)$ onto the tangent plane, and denoted $(D_y w)(p)$. We also denoted it by $(Dw/dt)(0)$ if α is given.

This of course does not depend on α. We can apply an analogous definition if w is only defined along points of a given path α.

In particular, this is zero if and only if $(w \circ \alpha)'$ is parallel to the normal vector of $T_p(S)$.

One standard example is the case $w = \alpha'$.

12.2. Parallel Transports

Definition 12.2. Let α be a parametrized curve and w a vector field defined on it. We say α is **parallel** if Dw/dt is zero everywhere on α.

In other words, this occurs if for every $\alpha(t) = p$ and $y = \alpha'(t)$ we have $(D_y w)(p)$ is parallel to the normal vector of $T_p(S)$. In particular if S is the plane this is equivalent to w being constant.

Proposition 12.3

Let w and v be parallel vector fields along α. Then $\langle w(t), v(t) \rangle$ is fixed.

Thus parallel vector fields preserve angles and have constant lengths.

By theory of differential equations, we have

Theorem 12.4 (Unique Parallel Vector Field)

Let α be a parametrized curve in S and pick a fixed initial vector $w_0 \in T_{\alpha(0)}S$. Then there exists a unique parallel vector field w along α such that $w(0) = w_0$.

So in this way we can define the **parallel transport** along a parametrized curve: given α joining p to q, for any vector $w \in T_p(S)$ we can follow along the parallel vector field q to get a unique tangent vector in $T_q(S)$.

12.3. Geodesics

Most important special case of parallelism:

Definition 12.5. If $\alpha(s)$ is a regular parametrized curve and α' is parallel, then α is said to be a **geodesic**.

In this case, it means that the normal vectors $\vec{n}$ for α coincide with normal vectors to the surface itself.

Theorem 12.6 (Unique Geodesic in Any Direction)

Let p be a point of a regular surface S and $0 \neq w \in T_p(S)$. There is a geodesic $\alpha : (-\varepsilon, \varepsilon) \to S$ with $\alpha(0) = p$, and $\alpha'(0) = w$, with the usual uniqueness up to domain.

Example 12.7 (Examples of Geodesics)

(a) Straight lines are always geodesics, because their normal vector vanishes altogether.

(b) In particular, on a plane, the straight lines are the only geodesics.

(c) On a sphere, the great circles are the unique geodesics.

(d) On a right cylinder, geodesics are helixes.

More generally, let α be a regular parametrized curve on an *oriented* surface S. At a point p, consider the tangent vector $\alpha' = \vec{t}$ and then the vector α''. We can project it onto the $T_p(S)$ as before, arriving at $D\alpha'/ds$; its signed length (with respect to $N \times \vec{t}$) is the **geodesic curvature** k_g. Thus geodesics are those curves with $k_g = 0$.

Remark 12.8. Since k_n is the other projection, $k^2 = k_n^2 + k_g^2$.

13. November 24 and 26, 2015

The main point of this section is to present the Gauss-Bonnet Theorem.

13.1. Area Differential Form

This is technically review, but anyways. Let $x : U \to \mathbb{R}^3$ be a parametrization. We can perform integrals of the form

$$\int_R f(u,v)\, |x_u \wedge x_v| = \int_{x^{-1}(R)} f(u,v) \sqrt{EG - F^2}\, du \wedge dv$$

which is the integral of f over the region U. This is the pullback of the differential form $x_u \wedge x_v$ on the surface. For convenience, we will abbreviate this to

$$\int_R f \, d\sigma.$$

13.2. Global Gauss-Bonnet Theorem

We then have the following.

Theorem 13.1 (Gauss-Bonnet)
Let $R \subset S$ be a regular region of an oriented surface S and let $C_1, \dots, C_n$ be positively oriented curves forming the boundary of R. Let $\theta_1, \dots, \theta_p$ be the $p = \binom{n}{2}$ external angles formed by the C_i. Then

$$\sum_{i=1}^{n} \int_{C_i} k_g(s)\, ds + \int_R K \, d\sigma + \sum_{i=1}^{p} \theta_i = 2\pi\chi(R)$$

where χ is the Euler characteristic.

Corollaries:

Corollary 13.2 (Gauss-Bonnet for Simple Regions)
If R is a simple region, then $\chi(R) = 1$ and thus

$$\sum_{i=1}^{n} \int_{C_i} k_g(s)\, ds + \int_R K \, d\sigma + \sum_{i=1}^{p} \theta_i = 2\pi.$$

Corollary 13.3 (Gauss-Bonnet for Orientable Compact Surfaces)
If S is a orientable compact surface then

$$\int_S K \, d\sigma = 2\pi\chi(S).$$

Proof. Take $R = S$, and use an empty boundary. □

14. December 1 and December 3, 2015

14.1. The Exponential Map

Let S be a regular surface and $p \in S$ a point. We consider the tangent plane $T_p(S)$. For a vector $v \in T_p(S)$, we can consider the geodesic γ_v with $\gamma_v'(0) = v$. In the event that $\gamma(1)$ is defined, we define the exponential map by

$$\exp_p(v) = \gamma_v(1)$$

and set $\exp_p(0) = p$.

Example 14.1 (Examples of the Exponential Map)
Let $S = S^2$ be a sphere, and $p \in S$.

(a) $\exp_p(v)$ is defined for each $v \in T_p(S)$. If $|v| = (2k+1)\pi$, then $\exp_p(v) = -p$, while if $|v| = 2k\pi$ then $\exp_p(v) = p$.

(b) If we delete $-p$ from S then $\exp_p$ is only defined in an open disk of radius π.

Proposition 14.2 ($\exp_p$ Defined Locally)
There exists a neigborhood U of $0 \in T_p(S)$ such that $\exp_p$ is defined, meaning we have a map

$$\exp_p : U \to S,$$

and moreover $\exp_p$ is a diffeomorphism.

We say that $V \subset S$ is a **normal neighborhood of p** if V is the diffeomorphic image under $\exp_p$ of some neighborhood U of $T_p(S)$.

14.2. Local Coordinates

We can impose coordinates on normal neighborhoods of a point via the exponential map, since it's a diffeomorphism.

If we put rectangular coordinates (u, v), then the geodesics of S correspond to straight lines through the origin.

Now consider a polar coordinate system with $0 < \theta < 2\pi$ and radius ρ, so pairs (ρ, θ). Note this omits the half-line ℓ corresponding to $\theta = 0$. The images of circles at U (centered at zero) in S are then called the geodesic circles of V while the images of lines through 0 are called the radial geodesics; these correspond to setting ρ and θ constant.

Theorem 14.3 (First Fundamental Form in Geodesic Polar Coordinates)
Let $x : U \setminus \ell \to V \setminus L$ be a system of geodesic polar coordinates (ρ, θ). Then the first fundamental form obeys

$$E = 1, \quad F = 0, \quad G(\rho, \theta) = \rho - \frac{K(p)}{6}\rho^3 + o(\rho^3).$$

The fact that $F = 0$ reflects the so-called *Gauss lemma*: radial geodesics are orthogonal to the geodesic circles.

15. December 8 and December 10, 2015

Our goal is to prove the Bonnet Theorem:

Theorem 15.1 (Bonnet's Theorme)
Let S be a complete surface. Suppose there exists a $\delta > 0$ such that the Gaussian curvature K is bounded below by δ. Then S is compact and any two points are joined by a geodesic with length at most $\frac{\pi}{\sqrt{\delta}}$.

This is "optimal" by taking a sphere. (Definition of "complete" in next section.)

15.1. Completeness

Definition 15.2. A regular connected surface S is **complete** if for any point $p \in S$ the exponential map $\exp_p : T_p(S) \to S$ is defined for every $v \in T_p(S)$.

Theorem 15.3 (Complete Surfaces are Nonextendible)
A complete surface S is **nonextendible**, i.e. it is not a proper subset of any regular connected surface.

Theorem 15.4 (Closed Surfaces Are Complete)
Any closed surface $S \subseteq \mathbb{R}^3$ is complete. In particular, any compact surface is complete.

Complete surfaces have a nice notion of "distance".

Theorem 15.5 (Hopf-Rinow)
If S is a complete surface then for any $p, q \in S$ there is a minimal geodesic joining p to q.

Thus we may define the distance between two points on a complete surface by considering the minimal geodesic.

15.2. Variations

In what follows, S is regular and connected but not necessarily complete.

To do this, consider a curve $\gamma : [0, \ell] \to S$ parametrized by arc length. A **variation** of γ is a differentiable map

$$h : [0, \ell] \times (-\varepsilon, \varepsilon) \to S.$$

such that $h(s, 0) = \gamma(s)$ for each s. We assume also that our variation h is **proper**, meaning $h(0, t) = \gamma(0)$ and $h(\ell, t) = \gamma(1)$. (In other words, h is a path homotopy.)

This then gives rise to a vector field

$$V(s) = \left[\frac{\partial}{\partial t} h(s, t) \right]_{t=0}.$$

which we call the **variational vector field** of h. Since h is proper it follows that $V(0) = V(\ell) = 0$.

Define $L : (-\varepsilon, \varepsilon) \to \mathbb{R}$ by letting $L(t)$ be the arc length of $h(-,t)$. Formally,

$$L(t) = \int_9^\ell \left\| \frac{\partial h}{\partial s}(s,t) \right\| \, ds.$$

Thus $L(t)$ measures the arc length of the "nearby" curves specified by the variation h.

Theorem 15.6 (The First Variation)

We have

$$L'(0) = -\int_0^\ell \langle A(s), V(s) \rangle \, ds$$

where

$$A(s) = \frac{D}{\partial s}\left[\frac{\partial h}{\partial s}(s,0) \right].$$

Proof. Some direct computation. You need the fact that $V(\ell) = V(0) = 0$. □

Remark 15.7. The vector $A(s)$ is the **acceleration vector** of α and $\|A(S)\|$ is the geodesic curvature of α.

Now we can formally write down that geodesics are "locally minimal".

Theorem 15.8 (Geodesics Are Minimal For Any Variation)

Given $\alpha : [0, \ell] \to S$ parametrized by arc length we have α is a geodesic if and only if for any proper variation h of α, we have $L'(0) = 0$.

Proof. If α is a geodesic, then $A(s) \equiv 0$ and the result is immediate.

Otherwise, support $V(s) = f(s)A(s)$ for some nonnegative "weight" function f supported on $(0, \ell)$ where $f(0) = f(\ell) = 0$. By differential equations, we can construct h with this variation. Now according to the previous result,

$$L'(0) = -\int_0^\ell \langle A(s), A(s)f(s) \rangle \, ds = -\int_0^\ell f(s) \|A(s)\|^2 \, ds \le 0$$

which can only be zero if $A(s) \equiv 0$. □

15.3. The Second Variation

From now on, assume $\gamma : [0, \ell] \to S$ is arc length, $V(s)$ is proper, and $V(0) = V(\ell) = 0$, and moreover that we have an orthogonality relation

$$\left\langle V(s), \gamma'(s) \right\rangle = 0.$$

In other words we assume V is an **orthogonal variation**.

We aim to compute $L''(0)$.

Lemma 15.9 (Partials Don't Commute)

Let $x : U \to S$ be a parametrization at $p \in S$. Let K be the Gaussian curvature at p. Then

$$\frac{D}{\partial v}\frac{D}{\partial u}x_u - \frac{D}{\partial u}\frac{D}{\partial v}x_u = K(x_u \times x_v) \times x_u.$$

Proof. More direct computation with Christoffel symbols. □

Remark 15.10. Philosophically, "curvature is something you can see by changing the orders of derivatives".

Lemma 15.11

Let $h : [0, \ell] \times (-\varepsilon, \varepsilon) \to S$, and let $W(s,t) = \frac{\partial h}{\partial t}(s,t)$ be a differential vector field along h. Then we have

$$\frac{D}{\partial t}\frac{D}{\partial s}W - \frac{D}{\partial s}\frac{D}{\partial t}W = K(s,t)\left(\frac{\partial h}{\partial s} \times \frac{\partial h}{\partial t}\right) \times W.$$

Theorem 15.12 (The Second Variation)

Suppose γ is a geodesic parametrized by arc length, and $h : [0, \ell] \times (-\varepsilon, \varepsilon) \to S$ be a proper orthogonal variation of of γ. Letting V be its variational vector field,

$$L''(0) = \int_0^\ell \left(\left\| \frac{D}{\partial s}V(s) \right\|^2 - K(s) \left\| V(s) \right\|^2 \right) \, ds$$

where $K(s)$ is the Gaussian curvature at $h(s,0) = \gamma(s)$.

Proof. We take the derivative of the formula

$$L'(t) = \int_0^\ell \frac{\left\langle \frac{D}{\partial s}\frac{\partial h}{\partial t}(s,t), \frac{\partial h}{\partial s}(s,t) \right\rangle}{\left\langle \frac{\partial h}{\partial s}(s,t), \frac{\partial h}{\partial s}(s,t), \right\rangle^{\frac{1}{2}}}$$

that we previously obtained. Move the differential operator through the integral sign. This gives us, by repeatedly applying Product Rule,

$$L''(0) = \int_0^\ell \left[\left\langle \frac{D}{\partial t}\frac{D}{\partial s}\frac{\partial h}{\partial t}(s,0), \frac{\partial h}{\partial s}(s,0) \right\rangle + \left\| \frac{D}{\partial s}V(s) \right\|^2 - \left\langle \frac{D}{\partial s}V(s), \frac{\partial h}{\partial s}(s,0) \right\rangle \right] ds.$$

Do some further computation now. □

15.4. Proof of Bonnet Theorem

Now that we have $L'(0)$ and $L''(0)$ we can prove Bonnet's Theorem.

Assume S is complete, and $K \ge \delta$ at all points of S. Consider two points p and q such that the distance p, q is maximized, and let $\gamma : [0, \ell] \to S$ be the *minimal* geodesic which joins them (permissible by the Hopf-Rinow Theorem). Now assume for contradiction $\ell > \frac{\pi}{\sqrt{\delta}}$ Thus we have $L'(0) = 0$. But if we can show $L''(0) < 0$, then we will have shown

γ is in fact a *local maximum* and this will be a contradiction. (To see this: imagine the long geodesic joining two points on a sphere. While locally the shortest path, globally it is longer than any perturbation.)

Consider proper variations V; we have

$$L''(0) = \int_0^\ell \left[\left\| \frac{D}{\partial s} V(s) \right\|^2 - K(s) \left\| V(s) \right\|^2 \right] \, ds.$$

Pick a tangent vector $w_0 \in T_p(S)$ such that $w_0 \perp \gamma'(0)$ and $\|w_0\| = 1$, and let $w(t)$ be a parallel transport of w_0 along γ. Then we define $V(s)$ again by

$$V(s) = \sin\left(\frac{\pi}{\ell} s\right) w(s).$$

As $V(0) = V(\ell) = 0$, we again can construct an h corresponding to it. Then direct calculation gives

$$\begin{aligned} \frac{D}{\partial s} V(s) &= -\frac{\pi}{\ell} \cos(\frac{\pi}{\ell} s) w(s) + \underbrace{\sin(\frac{\pi}{\ell} s) \frac{D}{\partial s} w(s)}_{=0} \\ \left\| \frac{D}{\partial s} V(s) \right\|^2 &= \frac{\pi^2}{\ell^2} \cos^2(\frac{\pi}{\ell} s) \\ \|V(s)\|^2 &= \sin^2(\frac{\pi}{\ell} s). \end{aligned}$$

If $K \geq \delta > \frac{\pi^2}{\ell^2}$, then we have

$$L''(0) < -\frac{\pi^2}{\ell^2} \int_0^\ell \left(\cos^2(\frac{\pi}{\ell} s) - \sin^2\left(\frac{\pi}{\ell} s\right) \right) \, ds < -\frac{\pi^2}{\ell^2} \int_0^\ell \left(\cos(2\frac{\pi}{\ell} s) \right) \, ds = 0$$

which is the desired contradiction.

The choice of sin is designed for the ending. If we had used an arbitrary function f as before, we would have $\langle f''(s) + k(s) f(s), f(s) \rangle$. So we choose a function f for which this becomes a zero integral.

A. Examples

Here is a collection of facts about some common examples. Recall for convenience that

$$K = EG - F^2 \quad \text{and} \quad H = \frac{1}{2}\frac{eG - 2Ff + gE}{EG - F^2}.$$

A.1. Sphere

Parametrize the unit sphere S^2 in $\mathbb{R}^3$ by $x(u,v) = (R\cos u\cos v, R\cos u\cos v, R\sin v)$.

$$E = R^2, \quad F = 0, \quad G = (R\cos u)^2$$

$$e = R, \quad f = 0, \quad g = R\cos^2 u.$$

The Gauss curvature at all points is the constant $K = \frac{1}{R^2}$. The mean curvature for this parametrization is $H = \frac{1}{R}$. The geodesics are the great circles.

The genus of S^2 is zero, so its Euler characteristic is 2. This surface is a compact oriented manifold without boundary.

A.2. Torus

Parametrize a torus by $x(u,v) = ((a + r\cos u\cos v), (a + r\cos u\sin v), r\sin u)$.

$$E = r^2, \quad F = 0, \quad G = (a + r\cos u)^2$$

$$e = r, \quad f = 0, \quad g = \cos u(a + r\cos u).$$

The Gauss curvature is

$$K = \frac{\cos u}{r(a + r\cos u)}.$$

The mean curvature is

$$H = \frac{a + 2r\cos u}{2r(a + r\cos u)}.$$

The genus of $S^1 \times S^1$ is 1, so its Euler characteristic is 0. This surface is a compact oriented manifold without boundary.

A.3. Cylinder

Parametrize the cylinder by $x(u,v) = (R\cos u, R\sin u, v)$. Then

$$E = R^2, \quad F = 0, \quad G = 1.$$

$$e = -R, \quad f = 0, \quad g = 0.$$

The Gauss curvature is $K = 0$ everywhere. The mean curvature at all points is $-\frac{1}{R}$. The geodesics are helixes, including meridians and vertical lines.

This surface is not compact, but it is orientable, with empty boundary. It has genus 1 and Euler characteristic zero.