18.786 (Number Theory II) Lecture Notes

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This is MIT's graduate course 18.786, instructed by Andrew Sutherland. The formal name for this class is "Number Theory II". The official lecture notes are available at https://math.mit.edu/classes/18.786/lectures.html. These notes cover the part of the course dedicated to modular forms (starting from lecture 6); as always any errors are my responsibility.

The permanent URL for this document is http://web.evanchen.cc/coursework.html, along with all my other course notes.

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§6 February 26, 2018 (Monday)

I actually missed this lecture since I was just getting back from Romania.

§6.1 Setup

Let $\mathbb{P} = \mathbb{C} \cup \{\infty\} = \mathbb{CP}^1$ denote the Riemann sphere, viewed as the complex plane plus an infinity point. Each $\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{C})$ gives an automorphism of $\mathbb{P}$ by

$$\alpha \cdot z = \frac{az+b}{cz+d}$$

the so-called *fractional linear transformation*. By elementary geometry, these maps clines in $\mathbb{P}$ to other clines.

Define

$$\begin{aligned} \mathbb{H} &= \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\} && \text{upper half plane} \\ \mathbb{D} &= \{z \in \mathbb{C} \mid |z| < 1\} && \text{unit disk.} \end{aligned}$$

Lemma 6.1

There is a complex analytic isomorphism $\mathbb{H} \to \mathbb{D}$ given by the automorphism

$$\rho = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}.$$

§6.2 Automorphisms of the upper half plane

Suppose α has *real* entries and $\det \alpha > 0$. For such α we have $z \in \mathbb{H} \implies \alpha z \in \mathbb{H}$, since $\operatorname{Im} \alpha z = \frac{\det \alpha \operatorname{Im} z}{|cz+d|^2}$. Thus, we can define

$$\mathrm{GL}_2^+(\mathbb{R}) = \{\alpha \in \mathrm{GL}_2(\mathbb{R}) \mid \det \alpha > 0\}.$$

This gives a natural map

$$\iota \colon \mathrm{GL}_2^+(\mathbb{R}) \to \mathrm{Aut}(\mathbb{H}).$$

Proposition 6.2

The map ι induces an isomorphism

$$\mathrm{Aut}(\mathbb{H}) \simeq \mathrm{GL}_2^+(\mathbb{R})/\mathbb{R}^\times \simeq \mathrm{SL}_2(\mathbb{R})/\{\pm 1\} = \mathrm{PSL}_2(\mathbb{R}).$$

It follows that

$$\begin{aligned} \mathrm{Aut}(\mathbb{D}) &= \rho \,\mathrm{Aut}(\mathbb{H}) \rho^{-1} \\ &= \rho \,\mathrm{SL}_2(\mathbb{R}) \rho^{-1}/\{\pm 1\} \\ &= \left\{ \alpha = \begin{bmatrix} u & v \\ \overline{v} & \overline{u} \end{bmatrix} \mid u, v \in \mathbb{C}, \det \alpha = 1 \right\} /\{\pm 1\} \\ &= \mathrm{SU}(1,1)/\{\pm 1\} \end{aligned}$$

the *special unitary group* of signature $(1,1)$ over $\mathbb{C}$.

Let $\mathbb{H}^* = \mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ be the compactification of $\mathbb{H}$, and let $\mathbb{D}^* = \{z \in \mathbb{C} \mid |z| \le 1\}$ be the compactification of $\mathbb{D}$ (closed unit disk). The automorphism ρ then extends to $\rho : \mathbb{H}^* \to \mathbb{D}^*$ sending $\infty \mapsto 1$, and $\mathbb{R} \mapsto S^1$.

§6.3 Types of actions

Definition 6.3. Let $\gamma \in \mathrm{GL}_2^+(\mathbb{R})$, with discriminant $d = \mathrm{tr}(\gamma)^2 - 4 \det \gamma$. We say:

- γ is **elliptic** if $d < 0$. Its eigenvalues are complex conjguates and it fixes a unique point of $\mathbb{H}$.
- γ is **parabolic** if $d = 0$; it has a double real eigenvalue and thus fixes a unique point of $x \in \mathbb{R} \cup \{\infty\}$.
- γ is **hyperbolic** if $d > 0$; it has two different real eigenvalues and thus fixes two point of $x, x' \in \mathbb{R} \cup \{\infty\}$.

Definition 6.4. Let Γ be a group acting on a space X. We say Γ acts **properly discontinously** on X if for any points x and y in X, there exist neighorhoods $U \ni x$ and $V \ni y$ such that $\#\{\gamma \in \Gamma \mid \gamma U \cap V \neq \varnothing\} < \infty$.

Theorem 6.5 (Fuchsian group)

For $\Gamma \subseteq \mathrm{SL}_2(\mathbb{R})$, Γ acts properly discontinously on $\mathbb{H}$ if and only if Γ is a *discrete* group. In that case we say, Γ is a **Fuchsian** group.

§6.4 Some important special cases

So we defined elliptic/parabolic/hyperbolic with respect to fixed points of $\mathbb{H}$ and $\mathbb{R} \cup \{\infty\}$. However, in practice, if we want to do coordinate calculations we will basically always conjugate by an element σ such that the fixed points move to either i, ∞, and 0. This way the entries of the matrix are super nice.

So, we first work out which automorphisms fix i:

$$\begin{aligned} \mathrm{SL}_2(\mathbb{R})_i &= \{\alpha \in \mathrm{SL}_2(\mathbb{R}) \mid \alpha i = i\} = \mathrm{SO}_2(\mathbb{R}) \\ \mathrm{GL}_2^+(\mathbb{R})_i &= \left\{\alpha \in \mathrm{GL}_2^+(\mathbb{R}) \mid \alpha i = i\right\} = \mathbb{R}^\times \cdot \mathrm{SO}_2(\mathbb{R}) \end{aligned}$$

where $\mathrm{SO}_2(\mathbb{R})$ is the special orthogonal group

$$\mathrm{SO}_2(\mathbb{R}) = \left\{ \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \right\}.$$

In addition,

$$\mathrm{GL}_2^+(\mathbb{R})_\infty = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mid a, d \in \mathbb{R}^\times,\ b \in \mathbb{R},\ ad > 0 \right\}$$

with the parabolic elements are those with $a = d$.

As for hyperbolic points, we can conjugate so the fixed points are 0 and ∞; in this case

$$\mathrm{GL}_2^+(\mathbb{R})_{0,\infty} = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \mid a, d \in \mathbb{R}^\times,\ ad > 0 \right\}$$

and so hyperbolic points boil down to conjugates of diagonal matrices.

§6.5 Group actions

The space $\mathbb{H}$, acted on by $\mathrm{SL}_2(\mathbb{R})$, is a **homogeneous space** (i.e. a topological space with a transitive group action).

We need the following theorem.

Theorem 6.6 (Homeomorphism given by homogeneous spaces)

Let G be a topological group and X a homogeneous space acted on by G. Assume that G is locally compact with a countable basis, and X is a locally compact Hausdorff space. Let $x \in X$ and let $G_x = \{g \in G \mid gx = x\}$ denote the stabilizer. Then the space of right cosets is homeomorphic to X by the map

$$G/G_x \to X \qquad \text{by} \quad gG_x \mapsto gx.$$

Applying this result to $G = \mathrm{SL}_2(\mathbb{R})$ acting on $X = \mathbb{H}$, where i is the stabilizer, gives

Theorem 6.7

We have an homeomorphism

$$\mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}_2(\mathbb{R}) \to \mathbb{H} \qquad \text{by} \quad \alpha \cdot \mathrm{SO}_2(\mathbb{R}) \mapsto \alpha i.$$

An even stronger condition:

Definition 6.8. A **principal homogeneous space** (known also as "torsor") is a homogeneous space in which the stabilizer is trivial. Then $G \simeq X$.

§7 February 28, 2018 (Wednesday)

§7.1 Geodesic spaces

Let (X, d) be a metric space.

Definition 7.1. An **isometry** of a metric space (X, d) is a distance-preserving bijection $g : X \to X$.

Remark 7.2. Isometries are automatically homeomorphisms in the category of topological spaces, but they need not be holomorphic when X is a Riemann surface (conjugation).

Definition 7.3. Let x and y be two points of a metric space (X, d). A **path** γ from x to y (which we abbreviate $\gamma : x \to y$) is a continuous map $\gamma : [a, b] \to X$, where $x = \gamma(a)$ and $y = \gamma(b)$. The **length** of the path is

$$\ell(\gamma) = \sup_{\alpha=t_1\le\cdots\le t_n=b} \sum_{i=1}^{n-1} d\left(\gamma(t_i), \gamma(t_{i+1})\right).$$

The path γ is **rectifiable** if $\ell(\gamma) < \infty$.

Note that isometries preserve path length.

On the other hand, we can try to recover a metric from a given length function $\ell : \operatorname{Hom}([a, b], X) \to \mathbb{R}_{\ge 0} \cup \{\infty\}$, by defining

$$d(x, y) = \inf_{\gamma:x\to y} \ell(\gamma)$$

which is a metric as long as rectifiable paths exist between any two points. If these form a metric, then X is a **length space**.

Theorem 7.4 (Characterization of metric spaces arising from a length function)

A metric space (X, d) is a length space if and only if X is locally compact, complete, and satisfies **Menger convexity**: for any distinct $x, y \in X$ there is some point $z \in X$ which satisfies $d(x, z) + d(z, y) = d(x, y)$.

Definition 7.5. Let (X, d) be a length space with length ℓ. A rectifiable path $\gamma : x \to y$ is a **geodesic segment** if it achieves the minimum length $d(x, y)$.

As usual, this is preserved by isometries.

Definition 7.6. A **geodesic** is a continuous map $(-\infty, \infty) \to X$ such that restrictions to compact intervals are geodesic segments. (Think "line" and "line segment" in $\mathbb{R}^n$.) A length space X in which every $x, y \in X$ are connected by a geodesic segment are **geodesic spaces**.

Let X be a real manifold and a chart $\psi : U \to V \subseteq \mathbb{R}^n$. Then we can compute lengths of a path $\gamma : [a, b] \to U \subseteq X$ using any metric or length element on $\mathbb{R}^n$. For example, the Euclidean metric $ds = \sqrt{dx_1^2 + \cdots + dx_n^2}$ gives a length function

$$\ell(\gamma) \overset{\text{def}}{=} \ell(\psi \circ \gamma) = \int_a^b \sqrt{\sum \left(\frac{dx_i}{dt}\right)^2}\, dt.$$

for $\gamma : [a, b] \to U$ a path More generally, if $\lambda : V \to \mathbb{R}_{>0}$ is any continuous function then $\lambda(\vec{x})\, ds$ is also a length function.

§7.2 Hyperbolic length

We now wish to make $\mathbb{H}$ into a geodesic space, which is certainly a real manifold.

Definition 7.7. The **hyperbolic length element** is

$$ds = \frac{|dz|}{\operatorname{Im} z} = \frac{\sqrt{dx^2 + dy^2}}{y}.$$

This induces a metric $ds^2 = \frac{dx^2+dy^2}{y^2}$ (which turns out to be Riemannian but we won't need that). Then given a path $\gamma : [a, b] \to \mathbb{H}$, writing $\gamma(t) = x(t) + y(t)$ we get

$$\ell(\gamma) = \int_a^b ds(\gamma(t)) = \int_a^b \frac{\sqrt{(dx/dt)^2 + (dy/dt)^2}}{y(t)}\, dt.$$

Proposition 7.8

The hyperbolic length makes $\mathbb{H}$ into a geodesic space.

Remark 7.9. This does not change $\mathbb{H}$ as a topological space from the Euclidean one, but it *is* different as a geodesic space. It should be emphasized that geodesic spaces are more than just topological spaces, because the exact metric matters: the set of geodesics will be different.

Moreover, once we have the length element ds, we get a corresponding form

$$dA = \frac{dxdy}{y^2}$$

which gives us a measure $\mu(S) = \iint_S dA$, invariant under $\operatorname{Isom}(H)$.

The correspondence measures on $\mathbb{D}$ are

$$ds = \frac{2|dw|}{1 - |w|^2}$$
$$dA = \frac{4\, dx\, dy}{(1 - x^2 - y^2)^2}.$$

Theorem 7.10

The map $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ via (orientation preserving) isometries. In fact

$$\mathrm{PSL}_2(\mathbb{R}) \simeq \operatorname{Isom}^+(\mathbb{H}) \simeq \operatorname{Aut}(\mathbb{H})$$

where $\operatorname{Isom}^+(\mathbb{H})$ is the set of orientation preserving isometries.

Proof. Suffices to show $d(\alpha s) = ds$, which is a computation. □

Remark 7.11. Any two points in $\mathbb{H}$ are connected by a unique geodesic, which is a either a semicircle whose diameter lies along $\mathbb{R}$, or (if x and y have the same real part) a vertical line through them (still perpendicular to $\mathbb{R}$).

§7.3 Iwasawa decomposition

Definition 7.12. We have a homeomorphism

$$N \times A \times K \xrightarrow{\simeq} \mathrm{SL}_2(\mathbb{R}) \qquad (n, a, k) \mapsto nak$$

where

$$N = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \mid b \in \mathbb{R} \right\}$$
$$A = \left\{ \begin{bmatrix} a & 0 \\ 0 & 1/a \end{bmatrix} \mid a \in \mathbb{R} \right\}$$
$$K = \mathrm{SO}_2(\mathbb{R}) \simeq U(1) \simeq \mathbb{R}/\mathbb{Z}.$$

Proof. The map is continuous and open, so we need to show it is a bijection.

- Injective: note from definition that $NA \cap K = \{1\} = N \cap A$. If $n_1 a_1 k_1 = n_2 a_2 k_2$ now then then $a_2^{-1} n_2^{-1} n_1 a_1 = k_2 k_1^{-1} \in K$, so both are 1, then $n_2^{-1} n_1 = a_2 a_1^{-1}$, so both are 1.
- Surjective: Here is an outline. For $\alpha \in \mathrm{SL}_2(\mathbb{R})$, let $z = \alpha i$. Let $n = \begin{bmatrix} 1 & -\operatorname{Re} z \\ 0 & 1 \end{bmatrix}$. So $n\alpha = z - \operatorname{Re} z = xi$ for some $x \in \mathbb{R}$. Then $a = \begin{bmatrix} 1/\sqrt{x} & 0 \\ 0 & \sqrt{x} \end{bmatrix} \in A$. Whee. □

This is called the **Iwasawa decomposition** of $\mathrm{SL}_2(\mathbb{R})$ Thus every element of $\mathrm{SL}_2(\mathbb{R})$ is of the form nak. This map is *not* a homomorphism, but it is a homeomorphism since it is given by multiplication in topological groups.

The picture is: "first it rotates (through K), then it scales (through A), then it translates (through N)".

§7.4 Fuchsian group

We return to a Fuchsian group $\Gamma \subseteq \mathrm{SL}_2(\mathbb{R})$. At this point, a minor annoyance is that we might have $-1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \in \Gamma$ but -1 acts as the identity. Let us introduce some notation to get rid of ± 1 signs.

Definition 7.13. We denote $Z(\Gamma) \overset{\text{def}}{=} \Gamma \cap \{\pm 1\}$. Thus $Z/\Gamma(Z)$ is the image of Γ in $\mathrm{PSL}_2(\mathbb{R})$, and we have a map

$$\mathrm{SL}_2(\mathbb{R}) \to \mathrm{Aut}(\mathbb{H}) \qquad \text{by} \quad \Gamma \mapsto \Gamma/Z(\Gamma).$$

Definition 7.14. For $z \in \mathbb{H}^*$, if z is a fixed point of $\alpha \in \Gamma$ with $\alpha \notin Z(\Gamma)$, then we call z elliptic/parabolic/hyperbolic according to what α is.

If z is parabolic, we also say it is a **cusp**.

Definition 7.15. For $z \in \mathbb{H}$, $x \neq x' \in \mathbb{R} \cup \{\infty\}$ we write:

$$\Gamma_z \overset{\text{def}}{=} \Gamma \cap \mathrm{SL}_2(\mathbb{R})_z$$
$$\Gamma_x \overset{\text{def}}{=} \Gamma \cap \mathrm{SL}_2(\mathbb{R})_x$$
$$\Gamma_{x,x'} \overset{\text{def}}{=} \Gamma \cap \mathrm{SL}_2(\mathbb{R})_{x,x'}$$

to denote the elements of Γ fixing them. (Recall we are using G_x for the G-stabilizer of a point x.)

Theorem 7.16 (Stabilizers of points in $\mathbb{H}^*$)

In each of the three cases:

- If $z \in \mathbb{H}$ is elliptic, then Γ_z is a finite cyclic group.
- If $x \in \mathbb{R} \cup \{\infty\}$ is parabolic (a cusp), then Γ_x contains only parabolic elements, and moreover and $\Gamma_x / Z(\Gamma) \simeq \mathbb{Z}$.
- If $\Gamma_{x,x'} \neq Z(\Gamma)$ then $\Gamma_{x,x'}/Z(\Gamma) \simeq \mathbb{Z}$.

Proof.

- As Γ is discrete, so is Γ_z. But we have $\mathrm{SL}_2(\mathbb{R})_z$ is conjugate to $\mathrm{SL}_2(\mathbb{R})_i \cong U(1)$ which is a compact abelian group. This implies finite and cyclic.
- WLOG $x = \infty$ by conjugation via a translation. By hypothesis, since $x = \infty$ is parabolic, Γ_∞ contains some nontrivial parabolic matrix, say $\begin{bmatrix} 1 & e \\ 0 & 1 \end{bmatrix}$.

  We now prove that Γ_x contains only parabolic elements. Assume for contradiction α is not parabolic, then write $\alpha = \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}$ where $a \neq \pm 1$. If WLOG $|a| < 1$, then

  $$\alpha^n \begin{bmatrix} 1 & e \\ 0 & 1 \end{bmatrix} \alpha^{-n} = \begin{bmatrix} 1 & a^{2n}e \\ 0 & 1 \end{bmatrix}$$

  which contradicts the fact that Γ was supposed to be discrete.

  Hence Γ_∞ contains only parabolic elements, and it is a discrete subgroup of $N = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \mid b \in \mathbb{R} \right\}$. Since $N \cong \mathbb{R}$ as an additive topological group, any discrete subgroup must be isomorphic to $\mathbb{Z}$, as desired.
- Again by conjugation, $\Gamma_{x,x'}$ is conjugate to a *discrete* subgroup of

  $$\mathrm{SL}_2(\mathbb{R})_{0,\infty} \subseteq A = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \right\} \mathbb{R}^\times / \{\pm 1\}.$$

  By taking log's we get the same conclusion as before. $\square$

Definition 7.17. For $z \in \mathbb{H}$, let $e_z \overset{\text{def}}{=} \#\Gamma_z / Z(\Gamma)$ denote the order of $z \in \mathbb{H}$.

Remark 7.18. From here it follows that:

- A point $z \in \mathbb{H}$ is elliptic (as opposed to nothing) if and only if $e_z > 1$. This order is finite.
- For $x \in \mathbb{R}$,
  - If x is a cusp or hyperbolic, it has infinite order.
  - These possibliites are mutually exclusive (since the second part of the theorem implies that no parabolic point is hyperbolic).

Corollary 7.19

If Γ' is a finite index subgroup of the Fuchsian group Γ then then each cusp of Γ' is a cusp of Γ.

Proof. If x is a cusp of Γ, then $[\Gamma_x : \Gamma' \cap \Gamma_x] \le [\Gamma : \Gamma_x] < \infty$. So $\Gamma'_x = \Gamma' \cap \Gamma_x \neq Z(\Gamma)$, since $\Gamma_x / Z(\Gamma) \simeq \mathbb{Z}$ is infinite. $\square$

Suppose $-1 \notin \Gamma$, and x is a cusp of Γ. For $\sigma \in \mathrm{SL}_2(\mathbb{R})$ with $\sigma x = \infty$, we have

$$\sigma \Gamma_x \sigma^{-1} = \left\{ \pm \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \mid m \in \mathbb{Z} \right\} \subseteq \mathrm{SL}_2(\mathbb{R})_\infty$$

for some fixed $h > 0$ Then $-1 \notin \Gamma$ means either

- $\begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \in \sigma \Gamma_x \sigma^{-1}$, in which we case x is a **regular** cusp, or
- $\begin{bmatrix} -1 & h \\ 0 & -1 \end{bmatrix} \in \sigma \Gamma_x \sigma^{-1}$, in which case we say it is an **irregular** cusp.

This doesn't depend on the choice of σ.

Remark 7.20. If $-1 \notin \Gamma$ then every elliptic point has odd order. Indeed, if $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ and $\alpha^2 = 1$ then $\alpha = \pm 1$. Thus Γ_z should not have even order.

§8 March 5, 2018 (Monday)

§8.1 Fundamental domains and Dirichlet domains

Definition 8.1. A **fundamental domain** for a Fuchsian Γ is a closed connected set $F \subseteq \mathbb{H}$ with two properties:

- The translates of F cover $\mathbb{H}$ (succinctly, $\mathbb{H} = \Gamma F$).
- Let $\gamma \in \Gamma$ act nontrivially (meaning $\gamma \notin Z(\Gamma)$). Let F° denote the interior of F. Then $\gamma F^\circ \cap F^\circ = \varnothing$.

So these are almost like a choice of representatives, but there is overlap permitted on the boundary.

Definition 8.2. A **Dirichlet domain** for Γ is a set of the form

$$F_{z_0} \overset{\text{def}}{=} \{z \in \mathbb{H} \mid d(z, z_0) \le d(\gamma z, z_0) \; \forall \gamma \in \Gamma\}.$$

(You can think of them as intersections of half-planes.)

It's not obvious this is a domain (or even that it's closed and connected) but we will prove this and more.

Proposition 8.3

A Dirichlet domain is a fundamental domain with the following additional properties:

- If $\gamma \in \Gamma \setminus Z(\Gamma)$, then $F \cap \gamma F$ is a connected subset of a geodesic (i.e. empty, a point, or a geodesic segment or ray).
- Let A be a compact subset of $\mathbb{H}$. Then

$$\#\{\gamma \in \Gamma \mid A \cap \gamma F\} < \infty.$$

In other words, F is **locally finite**.

Proof. Let $F = F_{z_0}$ in what follows. For γ not fixing z_0, define

$$\begin{aligned} F_\gamma &\overset{\text{def}}{=} \{z \in \mathbb{H} : d(z, z_0) \le d(z, \gamma z_0)\} \\ U_\gamma &\overset{\text{def}}{=} \{z \in \mathbb{H} : d(z, z_0) < d(z, \gamma z_0)\} \\ C_\gamma &\overset{\text{def}}{=} \{z \in \mathbb{H} : d(z, z_0) = d(z, \gamma z_0)\}. \end{aligned}$$

Thus $F_{z_0} = \bigcap_\gamma F_\gamma$.

We now prove them any properties.

F is connected On problem set 2 it will be proved that C_γ is a geodesic, and that F_γ is **convex** in the hyperbolic metric (for any x and y in F_γ the geodesic joining x and y lies in F_γ). Also,

$$\mathbb{H} = U_\gamma \sqcup C_\gamma \sqcup \gamma U_{\gamma^{-1}}$$

since $zin\gamma U_{\gamma^{-1}} \iff \gamma^{-1} z \in U_{\gamma^{-1}} \iff d(\gamma^{-1}z, z_0) < d(\gamma^{-1}z, \gamma^{-1}z_0) \iff d(z, \gamma z_0) < d(z, z_0)$. Moreover,

$$F = \bigcap_\gamma F_\gamma$$

is the intersection of convex sets, hence convex, hence connected.

F is closed We now check F is closed. Suppose not and (z_n) is a sequence in F, approaching $z \notin F$. Then there is a $\gamma \in \Gamma$ such that $d(z, \gamma z_0) < d(z, z_0)$. On the other hand, let $r = d(z, z_0) - d(z, \gamma z_0) > 0$. Pick r such that $d(z, z_n) < r/3$, then

$$\begin{aligned} d(z_n, \gamma_0) &\le d(z_n, z) + d(z, \gamma z_0) \\ &= d(z_n, z) + d(z, z_0) - r \\ &\le -\frac{2}{3}r + d(z, z_0) \\ &\le -\frac{2}{3}r + d(z, z_n) + d(z_n, z_0) \\ &\le -\frac{1}{3}r + d(z_n, z_0) < d(z_n, z_0) \end{aligned}$$

contradiction to $z_n \in F$. This shows F is closed.

Translates of F cover $\mathbb{H}$ Next, we show that $\mathbb{H} = \Gamma F$. Given $z_1 \in \mathbb{H}$, let A and B be compact balls about z_0, z_1 of radius $D(z_0, z_1)$. Then

$$0 < \#\{\gamma \in \Gamma : \gamma A \cap B \neq \varnothing\} < \infty$$

and there exists $\gamma_1 \in \Gamma$ such that

$$d(z_1, \gamma_1 z_0) \le d(z_1, \gamma z_0) \qquad \forall \gamma \in \Gamma.$$

This implies $d(\gamma^{-1} z_1, z_0) \le d(\gamma^{-1} z_1, \gamma z_0)$ for every $\gamma \in \Gamma$. Hence $\gamma_1^{-1} z_1 \in F$, and so $z_1 \in \gamma F$ as desired.

Interiors of translates do not overlap Finally, we check $\gamma F^\circ \cap F^\circ = \varnothing$ for $\gamma \notin Z(\Gamma)$. Let $F^\circ = \bigcap_\gamma U_\gamma$. Now, if $z_1 \in F^\circ$, then $d(z_1, z_0) < d(z_1, \gamma z_0) = d(\gamma^{-1} z_1, z_0)$. But if $z_1 \in \gamma F^\circ$ too then $d(\gamma^{-1} z_1, z_0) < d(\gamma^{-1} z_1, \gamma^{-1} z_0) = d(z_1, z_0)$, which is a contradiction.

The set $F \cap \gamma F$ is a subset of a geodesic This is almost the same as the previous proof, with $\le$ replaced by $=$. Recall that $F \cap \gamma F$ is closed and convex, hence connected.

We now repeat the same argument: if $z_1 \in F_\gamma$, then $d(z_1, z_0) \le d(z_1, \gamma z_0) = d(\gamma^{-1} z_1, z_0)$ and if $z_1 \in \gamma F_\gamma$ too then $d(\gamma^{-1} z_1, z_0) \le d(\gamma^{-1} z_1, \gamma^{-1} z_0) = d(z_1, z_0)$. Consequently, $d(z_1, z_0) = d(z_1, \gamma z_0)$. Hence $F \cap \gamma F$ is contained in the geodesic C_γ.

Translates intersect compact sets in finitely many sets Since A is compact, it has at most finitely many connected components. So WLOG let A be compact and connected (by looking at any component). Then

$$\#\{\gamma \in \Gamma \mid A \cap \gamma F \neq \varnothing\} = \#\left\{\gamma \in \Gamma \mid \gamma' A \cap \gamma F \neq \varnothing\right\}$$

for any $\gamma' \in \Gamma$, so WLOG assume $A \cap F \neq \varnothing$.

Assume $A \cap \delta F \neq \varnothing$ for some $\delta \in \Gamma - Z(\Gamma)$. Then A intersect $F \cap \delta F$. Consequently, $A \cap C_\delta \neq \varnothing$. Pick r so that $A \subseteq B_r(z_0)$, so $B_r(z_0) \cap C_\delta neq \varnothing$. Then $d(z_1, z_0) = d(z_1, \delta z_0) \le r$ and

$$\#\{\gamma \in \Gamma : d(z_0, \gamma z_0) \le r\} = \#\{\gamma \in \Gamma_1 : \gamma\{z_0\} \cap B_r(z_0) \neq \varnothing\} < \infty.$$

$\square$

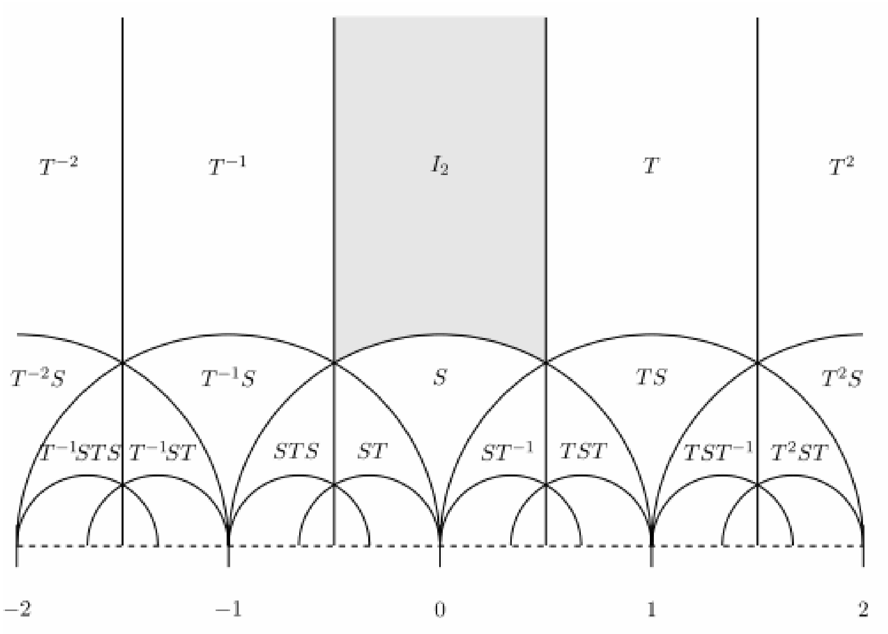
Now for $\gamma \in \Gamma - Z(\Gamma)$, we note that $L_\gamma \overset{\text{def}}{=} F \cap \gamma F$ is a connected subset of a geodesic.

Definition 8.4. If $\#L_\gamma > 1$, then L_γ is infinite and called a **side** of F. Then $\partial F = F - F^\circ$ is a union of sides, and any two sides are disjoint or intersect in a **vertex**.

Thus the connected components of the boundary look likes a sequence of sides L_1, L_2, ..., where $L_i \cap L_{i+1}$ meet in a single vertex, called **endpoints**. We also consider geodesic rays as having **endpoints** at infinity.

Example 8.5

Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, generated by $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then the usual fundamental domain F has three sides with two vertices and an endpoint at infinity (see picture).



http://arkadiusz-jadczyk.eu/blog/2017/04/dedekind-tessellation-circles/

Remark 8.6. If Γ is finitely generated (which will happen in all the cases we care about), there is an algorithm that computes these fundamental domains.

§8.2 Quotient spaces

Our goal is to get a compact Riemann surface by quotient-ing $\mathbb{H}$ by the action of Γ.

The quotient of a Hausdorff space may not be Hausdorff but the following lemma gives a criteria for spaces X with action by a group G that we will use. (I think the converse is not true, but in this course it is the criteria we will always use anyways. The hypothesis is weaker than properly discontinuous though, as we are about to see.)

Lemma 8.7

Let X be a topological space with an action by a group G. Suppose for all $x, y \in X$, there are neighbors $U \ni x$ and $V \ni y$ such that $gU \cap V \neq \varnothing \iff gx = y$ for $g \in G$. Then the quotient space $G \setminus X$ is Hausdorff.

Proof. Easy. If $\pi \colon X \to G \setminus X$ is the projection, it is an open map, and $\pi(U)$ and $\pi(V)$ will be the desired neighborhoods of $\pi(x)$ and $\pi(y)$ whenever x and y are not in the same orbit. $\square$

Definition 8.8. Let $P_\Gamma = \{x \in \mathbb{R} \cup \{\infty\}$ a cusp of $\Gamma\}$. We extend the upper half-plane $\mathbb{H}$ by "just a little bit" and define

$$\mathbb{H}_\Gamma^* = \mathbb{H} \cup P_\Gamma.$$

Then for any $\ell > 0$, we define $U_\ell \overset{\text{def}}{=} \{z \in \mathbb{H} \mid \operatorname{Im} z > \ell\}$ and $U_\ell^* = U_\ell \cup \{\infty\}$. Then we define a topology on H_Γ^* as follows:

- For $z \in \mathbb{H}$, the fundamental system of neighborhoods is the same as in $\mathbb{H}$.
- For $x \in P_\Gamma$, we know $\mathrm{SL}_2(\mathbb{R})$ acts transitively on $\mathbb{R}$ (although Γ might not!) and we thus pick $\sigma \in \mathrm{SL}_2(\mathbb{R})$ with $\sigma x = \infty$. Then, we pick a fundamental system of neighborhoods
$$\left\{\sigma^{-1} U_\ell^* \mid \ell > 0\right\}.$$

Example 8.9

This means that in $\mathbb{H}_\Gamma^*$, the neighborhoods of ∞ are quite large: containing huge swaths of the upper-half plane above some point. In this sense it might be more accurate to write $i\infty$ instead of ∞, but we will not do so.

Then Γ acts on $\mathbb{H}_\Gamma^*$: but it does not act properly discontinuously! Indeed, recall that:

> Let Γ be a group acting on a space X. We say Γ acts **properly discontinuously** on X if for any points x and y in X, there exist neighorhoods $U \ni x$ and $V \ni y$ such that $\#\{\gamma \in \Gamma \mid \gamma U \cap V \neq \varnothing\} < \infty$.

If we let $x = y = \infty$, then we run into issues. Indeed, the neighborhoods of of ∞ are exactly the U_ℓ^*'s, and taking γ to be any translation (of which there are infinitely many) is bad. In fact:

$$\{\pm 1\}\Gamma_\infty = \left\{ \pm \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \mid h \in \mathbb{R} \right\}.$$

So there is some work to do:

Lemma 8.10

Suppose ∞ is a cusp of Γ and write

$$\{\pm 1\}\Gamma_\infty = \left\{ \pm \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}^n \mid n \in \mathbb{Z} \right\}.$$

for some $h > 0$. Then for any $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$, if $|ch| < 1$ then $\gamma \in \Gamma_\infty$.

Lemma 8.11

Let $x_1, x_2 \in P_\Gamma$ and σ_1, σ_2 with $\sigma x_1 = \sigma x_2 = \infty$. Write $\sigma_i \Gamma_{x_i} \sigma_i^{-1}\{\pm 1\} = \left\{ \pm \begin{bmatrix} 1 & h_i \\ 0 & 1 \end{bmatrix}^n \mid n \in \mathbb{Z} \right\}$ for $i = 1, 2$. If ℓ_1 and ℓ_2 satisfy $\ell_1 \ell_2 > |h_1 h_2|$ then

$$\gamma \sigma_1^{-1} U_{\ell_1} \sigma_2^{-1} U_{\ell_2} \neq \varnothing \implies \gamma x_1 = x_2.$$

These two lemmas give:

Corollary 8.12

Let $x \in P_\gamma$, $\sigma x = \infty$. Write $\sigma \Gamma_x \sigma^{-1}\{\pm 1\} = \left\{ \pm \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}^n \mid n \in \mathbb{Z} \right\}$ for $h > 0$. Then:

- For $\ell > h$ we have $\gamma \sigma^{-1} U_\ell \cap \sigma^{-1} U_\ell \neq \varnothing \implies \gamma \in \Gamma_x$.
- For any compact $A \subseteq \mathbb{H}$, there exists ℓ such that $A \cap \gamma \sigma^{-1} U_\ell = \varnothing$ for $\gamma \in \Gamma$.

Consequently,

Corollary 8.13

The action of Γ on $\mathbb{H}_\Gamma^*$ satisfies Lemma 8.7 and thus gives a Hausdorff quotient space $\Gamma \setminus \mathbb{H}_\Gamma^*$.

Theorem 8.14

If the space $\Gamma \setminus \mathbb{H}_\Gamma^*$ is compact, then the number of ellipic points and cusps is finite.

Proof. Let $a = \pi(z) \in \Gamma / \mathbb{H}_\Gamma^*$, and $\pi \colon \mathbb{H}_\Gamma^* \to \Gamma \setminus \mathbb{H}_\Gamma^*$ Choose a neighborhood $U \ni z$ so that $\forall \gamma \in \Gamma$, we have $\gamma U \cap U \neq \varnothing \implies \gamma z = z$. Then $\pi(U) - \{a\}$ has no elliptic points or cusps.

As $\pi(U)$ is open, compactness implies finitely many $\pi(U)$ cover $\Gamma \setminus \mathbb{H}_\Gamma^*$. $\square$

This now gives us the following definition.

Definition 8.15. A **Fuchsian group of the first kind** is a Fuchsian group Γ such that $\Gamma \setminus \mathbb{H}_\Gamma^*$ is compact.

§9 March 7, 2018 (Wednesday)

§9.1 More on Dirichlet domains

Recall the definition of a Fuchsian group of first kind.

Definition 9.1. Let F be a Dirichlet domain for Γ. If $x \in \mathbb{R} \cup \{\infty\}$ is the endpoint of two disjoint sides, we call x a **proper vertex** of F on $\mathbb{R} \cup \{\infty\}$.

Lemma 9.2

Proper vertices are not hyperbolic points.

Proof. Let $x \in \mathbb{R} \cup \{\infty\}$ be a proper vertex. Assume for contradiction x is hyperbolic, i.e. fixed by $\gamma \in \Gamma$. WLOG $x = \infty$, and $\gamma = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ where $a \neq d$ and $a/d < 1$. Then the two sides ending in x lie on the geodesics

$$\operatorname{Re} z = b_1, \qquad \operatorname{Re} z = b_2, \qquad b_1 < b - 2.$$

Let

$$M_\ell = \{z \in \mathbb{H} \mid \operatorname{Im} z \geq \ell, b_1 \leq \operatorname{Re} z \leq b_2\}$$

and note that for large enough ℓ we have $M_\ell \subset F$ due to local compactness around the point ∞.. Let U be an open neighborhood of i with $A = \overline{U}$ compact. Then for all sufficiently large $N \in \mathbb{Z}$, we have $A \cap \gamma^n F \supseteq A \cap \gamma^n M \neq \varnothing$ which contradicts locally finite. $\square$

§9.2 Complex structure on $\mathbb{H}^*_\Gamma$

Let

$$\pi \colon \mathbb{H}^*_\Gamma \to \Gamma \setminus \mathbb{H}^*_\Gamma$$

be the quotient map. Let $z_0 \in \mathbb{H}^*_\Gamma$ and let $p \in \pi(z_0) \in \Gamma \setminus \mathbb{H}^*_\Gamma$. We want to define a chart t_p on $\pi(U)$ for some open neighborhood U of z_0.

- If z is ordinary (not elliptic or a cusp[1]), equivalently $\Gamma_p = Z(\Gamma)$, then we pick any open neighborhood of $U \ni z_0$ such that $\gamma U \cap U \neq \varnothing \iff \gamma \in \Gamma_{z_0} = Z(\Gamma)$ (possible since locally finite). Let $V_p = \pi(U)$, then V is an open neighborhood of p homeomorphic to U via π. Hence we define the chart in this case as

$$t_p \colon V_p \xrightarrow{\pi^{-1}} U \subseteq \mathbb{C}.$$

- If p is elliptic, then we proved that $\Gamma_{z_0}/Z(\Gamma)$ is a finite cyclic group. Pick $\rho \in \mathrm{SL}_2(\mathbb{C})$, viewed as $\rho \colon \mathbb{H} \to \mathbb{D}$, such that $\rho(z_0) = 0$. Let $B_r = \{z \in \mathbb{C} : |z| < r\}$ be a ball of radius r around the origin, and let $U = \rho^{-1}(B_r)$ with $r > 0$ small enough such that $\gamma U \cap U \neq \varnothing \iff \gamma \in \Gamma_{z_0}$.

  Now by Schwartz's lemma, $\rho \Gamma_{z_0} \rho^{-1}$ is a rotation of $\mathbb{D}$ about U by $\frac{2\pi n}{e}$ where $e = \#\Gamma_0/Z(\Gamma)$. Thus $\Gamma_{z_0} U = U$, so we have homeomorphisms

$$\pi(U) \simeq \Gamma_{z_0} \setminus U \simeq \rho \Gamma_{z_0} \rho^{-1} \setminus B_r.$$

[1] Note that $\mathbb{H}^*_\Gamma$ has no hyperbolic points by definition

Let $\phi \colon B_r \to B_r$ defined by $\phi(w) = w^e$. Then $|phi$ is invariant under rotation by $2\pi/e$ and induces a homoeomorphism $\phi_1 : \rho\Gamma_0\rho^{-1} \setminus B_r \xrightarrow{\simeq} B_r$.

Diagram:

$$\begin{tikzcd}
 & U \arrow[r, "\rho"] \arrow[dl, "\pi"'] \arrow[d, "\pi"] & B_r \arrow[d, "\pi"] \arrow[dr, "\phi"] & \\
V_p = \pi(U) \arrow[r, "\sim"] & \Gamma_{z_0} \setminus U \arrow[r, "\sim"] & \rho\Gamma_{z_0}\rho^{-1} \setminus B_r. \arrow[r, "\sim", "\phi_1"'] & B_{r^e}
\end{tikzcd}$$

We then let t_p be the map $V_p \to B_{r^e}$ in the bottom row, meaning it is the unique map satisfying

$$t_p \circ \pi(z) = (\rho z)^e.$$

- Finally, assume $p = \pi(z_0)$ as a cusp. Pick $\sigma \in \mathrm{SL}_2(\mathbb{R})$ so $\sigma(z_0) = \infty$, and $U^* = \sigma^{-1} U_\ell^*$, and finally

  $$U_\ell^* = \{z \in \mathbb{H} \mid \operatorname{Im} z > \ell\} \cup \{\infty\}$$

  where ℓ is large enough so that

  $$\gamma U_\ell^* \cap U_\ell^* \neq \varnothing \iff \gamma \in \Gamma_{z_0}.$$

  Then $V_p = \pi(U^*)$ is homeomorphic to $\Gamma_{z_0} \setminus U_\ell^* \simeq \sigma\Gamma_{z_0}\sigma^{-1} \setminus U_\ell^*$. Let

  $$\sigma\Gamma_{z_0}\sigma^{-1}\{\pm 1\} = \left\{ \pm \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}^n \mid n \in \mathbb{Z} \right\}$$

  for some $h > 0$. Then define

  $$\Psi(z) \overset{\text{def}}{=} \begin{cases} \exp\left(\frac{2\pi i z}{h}\right) & z \in U_\ell \\ 0 & z = \infty \end{cases}$$

  and $\Psi_1 : \sigma\Gamma_{z_0}\sigma^{-1} \setminus U_\ell^* \to B_r$, where $r = \exp(-\frac{2\pi\ell}{h})$.

  Finally, we define t_p according to the diagram

  $$\begin{tikzcd}
   & U^* \arrow[r, "\sigma"] \arrow[dl, "\pi"'] \arrow[d, "\pi"] & U_\ell^* \arrow[d, "\pi"] \arrow[dr, "\psi"] & \\
  V_p = \pi(U) \arrow[r, "\sim"] & \Gamma_{z_0} \setminus U^* \arrow[r, "\sim"] & \rho\Gamma_{z_0}\rho^{-1} \setminus U_\ell^*. \arrow[r, "\sim", "\psi_1"'] & B_{r^e}
  \end{tikzcd}$$

  in other words, satisfying

  $$t_p \circ \pi(z) = \begin{cases} \exp\left(\frac{2\pi i \sigma z}{h}\right) & z \in U \\ 0 & z = z_0. \end{cases}$$

One can verify that the transition maps are holomorphic, and so $X_\Gamma = \Gamma \setminus \mathbb{H}_\Gamma^*$ is a Riemann surface (which is connected, since F is connected).

Definition 9.3. For $p \in \pi(z_0) \in X_\Gamma$, define $e_p = \#\Gamma_{z_0}/Z(\Gamma)$ to be the **ramification index** of p.

This has the propertythat

- $e_p = 1$ iff p is ordinary
- $1 < e_p < \infty$ iff p is elliptic
- $e_p = \infty$ iff p is a cusp.

Theorem 9.4

If $\Gamma \setminus \mathbb{H}$ is compact, then Γ has no cusps.

Proof. Suppose $x \in \mathbb{R} \cup \{\infty\}$ is a cusp. Let $p = \pi(x) \in \Gamma \setminus \mathbb{H}^*_\Gamma$ and choose a neighborhood of V_p of p such that $\overline{V_p} \cap (\Gamma \setminus \mathbb{H}) = \overline{V_p} - \{p\} = \{z \in \mathbb{C} \mid 09 < |z| < r\}$ for some $r > 0$; but this punctured disk is not compact. Yet, $\overline{V_p} \cap \Gamma \setminus \mathbb{H}$ is closed and since $\Gamma \setminus \mathbb{H}$ is compact we are supposed to have $\overline{V_p} \cap \Gamma \setminus \mathbb{H}$ be compact, contradiction. $\square$

§9.3 Comments on Riemann surfaces

Recall that meromorphic functions on the Riemann surface X_Γ should form a field $K(X_\Gamma)$ or $\mathbb{C}(X_\Gamma)$. When X_Γ is compact, holomorphic functions are constant and $\mathbb{C}(X_\Gamma)$ has transcendence degree 1, hence is an algebraic extension of $\mathbb{C}(t)$.

For nonconstant $\phi \in \mathbb{C}(X_\Gamma)$ we define

$$n_0(\sigma) = \sum_{v_p(\phi) > 0} v_p(\phi)$$

to be the number of 0's with multiplicity, while

$$n_\infty(\sigma) = \sum_{v_p(\phi) < 0} -v_p(\phi)$$

is the number of poles with multiplicity. Then

$$[\mathbb{C}(X_\Gamma) : \mathbb{C}(\phi)] = n_0(\phi) = n_\infty(\phi).$$

Also, recall that for compact Riemann surfaces X and Y, any nonconstant morphism $f \colon X \to Y$ is surjective, and hence we say it is a **cover** of Riemann surfaces. This induces a map of function fields

$$f^\times : K(Y) \to K(X) \qquad \phi \mapsto \phi \circ f$$

and hence we can define $\deg f \overset{\text{def}}{=} [K(X) : K(Y)]_f$. Hence, for any $q \in X$ we can define a ramification index $e_{q,f} = v_q(t_\alpha \circ f)$, where t_α is some chart at $f(p)$. Then for any $p \in Y$ we have

$$\deg f = \sum_{q \in f^{-1}(p)} e_{q,f}.$$

Let $\chi(X)$ denote the Euler characteristic of a compact Riemann surface X (defined via a triangulation, for example; this works because we are in the very concrete context of Euler characteristic). Since our manifolds have no boundary $\chi(X)$ is even.

Definition 9.5. We define the genus of a Riemann surface X by

$$g(X) = \frac{2 - \chi(X)}{2} \in \mathbb{Z}_{\ge 0}.$$

Then we have the Riemann-Hurwitz formula: for any cover $f\colon X \to Y$,

$$2g(X) - 2 = (\deg f)\,(2g(Y) - 2) + \sum_{q \in X} (e_{q,f} - 1)\,.$$

Example 9.6

If $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and we take F to be the usual fundamental domain, then upon adding the point at ∞ we get a "triangle"; then $\chi(X_\Gamma) = 2$ and hence $g(X_\Gamma) = 0$.

§9.4 Another criteria for Fuchsian groups

We will show in this section that Fucschian groups being of the first kind is equivalent to it having "finite volume". Of course to do this we need to first define a measure.

Let X be locally compact, and let $\mathcal{C}_{\text{cont}}(X)$ denote the space of continuous functions $X \to \mathbb{C}$ which are compactly supported. If $M\colon \mathcal{C}_{\text{cont}}(X) \to \mathbb{C}$ is a linear functional such that $M(\phi) \ge 0$ for all $\phi \ge 0$ (meaning ϕ is real-valued and nonnegative), then there is a measure μ_M such that

$$M(f) = \int_X f \mu_M \qquad \forall f \in \mathcal{C}_{\text{cont}}(X).$$

Remark 9.7. Drew says: "I think the right way to do measure theory is to just define the integral as a linear functional". (Serre does this, for example).

In any case, we can define the functional/measure explicitly since integrating on $\mathbb{C}$ is not that hard.

aaaaaa

Theorem 9.8 (Siegel)

Let Γ be a Fuchsian group. The corresponding Riemann surface X_Γ if and only if the volume $v(X_\Gamma) = v(\Gamma \setminus \mathbb{H}^*_\Gamma)$ (equivalently, the volume of any fundamental domain) is finite.

§10 March 12, 2018 (Monday)

§10.1 j-function

For $\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$, we let $j(\alpha, z) = cz + d$ and by currying $j(\alpha)\colon z \mapsto cz + d$. In that case, we have the following invariants:

- $j(\alpha\beta, z) = j(\alpha, \beta z) j(\beta, z)$.
- $j(\alpha^{-1}, z) = j(\alpha, \alpha^{-1} z)^{-1}$.
- $\frac{d}{dz}(\alpha z) = \frac{\det \alpha}{j(\alpha,z)^2}$.
- $\operatorname{Im}(\alpha z) = \frac{\det \alpha \operatorname{Im} z}{|j(\alpha,z)|^2}$.

We also have the following equivalences:

$$\begin{aligned}
\exists \lambda \in \mathbb{C}^* \quad & j(\alpha, z) = \lambda j(\beta, z) \\
\iff \exists \lambda \quad & \lambda = j(\alpha\beta^{-1}, \beta z) \\
\iff \exists \lambda \quad & \lambda = c\beta^{-1} z + d \\
\iff & c = 0 \\
\iff (\alpha\beta^{-1})\infty = \infty, & \text{ or equivalently } \alpha\beta^{-1} \in \mathrm{GL}_2^+(\mathbb{R}).
\end{aligned}$$

§10.2 k-slash operator

Let $k \in \mathbb{Z}$. For any $f\colon \mathbb{H} \to \mathbb{C}$ and $\gamma \in \mathrm{GL}_2^+(\mathbb{R})$ we define the **k-slash operator** by

$$(f|_k\gamma)(z) \overset{\text{def}}{=} \det(\gamma)^{k/2} j(\gamma, z)^{-k} f(\gamma z).$$

(Note that $\det \gamma = 1$ for γ in a Fuchsian group.) Note that $\bullet|_k$ is a linear operator on the space of functions $\mathbb{H} \to \mathbb{C}$. Also, a quick calculation gives

$$f|_k(\alpha\beta) = (f|_k\alpha)|_k\beta.$$

Moreover, $f|_k\mathrm{id} = f$, and so we get a right linear group action on the complex vector space of functions $\mathbb{H} \to \mathbb{C}$.

Note also that if $\alpha = a = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ is a scalar, then

$$f|_k\alpha = (a^2)^{k/2} a^{-k} f(z) = (\operatorname{sign} a)^k f(z).$$

Definition 10.1. Let Γ be a Fuchsian group of the first kind. For $k \in \mathbb{Z}$, a function $f\colon \mathbb{H} \to \mathbb{C}$ is an **automorphic form of weight k for Γ** if

- f is meromorphic on $\mathbb{H}$, and
- $f|_k\gamma = f$ for every $\gamma \in \Gamma$.

We let $\Omega_k(\Gamma)$ denote the set of automorphic forms of weight k for Γ, which is a $\mathbb{C}$-vector space.

Remark 10.2. If k is odd and $-1 \in \Gamma$, then $\Omega_k(\Gamma) = \{0\}$. Indeed, $f_k|_k(-1) = (-1)^k f$, so when k is odd this means $f = 0$.

Some properties that follows:

- If $\Gamma' \subseteq \Gamma$ then $\Omega_k(\Gamma) \subseteq \Omega_k(\Gamma')$.
- If $f \in \Omega_k(\Gamma)$ and $\alpha \in \mathrm{GL}_2^+(\mathbb{R})$ then $f|_k\alpha \in \Omega_k(\alpha^{-1}\Gamma\alpha)$.
- Now let $f \in \Omega_k(\Gamma)$ and $g \in \Omega_\ell(\Gamma)$. Then $fg \in \Omega_{k+\ell}(\Gamma)$.

Definition 10.3. We let $\Omega(\Gamma) = \sum \Omega_k(\Gamma)$ denote the vector space generated by the set of automorphic forms for Γ (of any weight).

Lemma 10.4

Ω is a graded ring (actually a commutative graded $\mathbb{C}$-algebra).

Proof. We wish to show the map $\bigoplus_k \Omega_k \to \Omega$ by $(f_k)_k \mapsto \sum f_k$ is injective since it is obviously surjective.

Since the volume $v(X_\Gamma)$ is finite and $\Gamma_\infty \setminus \mathbb{H}$ is infinite, we have Γ/Γ_∞ infinite. Thus we pick $\{\gamma_n\}_{n\ge0}$ in Γ, such that $j(\gamma_m, z) \neq j(\gamma_n, z)$ for all $m \neq n$. Now for all $n \in \mathbb{Z}$ we have

$$\begin{aligned}\sum_{k\in S} j(\gamma_n, z)^k f_k(z) &= \sum_{k\in S} j(\gamma_n, z)^k (f|_k\gamma_n)(z) \\ &= \sum_{k\in S} j(\gamma, z)^k \det(\gamma) j(\gamma_n, z)^{-k} f(\gamma_n, z) \\ &= 0.\end{aligned}$$

Note that S is finite, so WLOG set $S = \{0, 1, \dots, N\}$. Then for each n we now have

$$\sum_{k=M}^{N} j(\gamma_n, z)^k f_k(z) = 0.$$

Let $M \le n \le N$, then this becomes a linear system of equations whose coefficient matrix is

$$\left[j(\gamma_n, z)^k\right]_{\substack{0\le k\le N\\0\le n\le N}} = \begin{bmatrix} j(\gamma_0, z)^0 & j(\gamma_0, z)^1 & \dots & j(\gamma_0, z)^N \\ j(\gamma_1, z)^0 & j(\gamma_1, z)^1 & \dots & j(\gamma_1, z)^N \\ \vdots & \vdots & \ddots & \vdots \\ j(\gamma_N, z)^0 & j(\gamma_N, z)^1 & \dots & j(\gamma_N, z)^N \end{bmatrix}$$

which has full rank, since it is a Vandermonde determinant. □

§10.3 Behavior on cusps

Our automorphic forms are meromorphic functions on $\mathbb{H}$, but we want to also think of their behavior on $\mathbb{H}^*_\Gamma$, i.e. adding in the cusps. Since these functions aren't defined on the cusps yet, we will have to do some work.

Let x be a cusp of Γ, and $\sigma \in \mathrm{SL}_2(\mathbb{R})$ such that $\sigma x = 0$ (again conjugating to zero). Then

$$\sigma\Gamma_x\sigma^{-1}\{\pm1\} = \left\{ \pm \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}^n \mid n \in \mathbb{Z} \right\} \qquad h > 0.$$

Assuming k is even, we then have

$$f|_k\sigma^{-1} \in \Omega_k(\sigma\Gamma\sigma^{-1}) \qquad \forall f \in \Omega_k$$

In other words,

$$\left(f|_k\sigma^{-1}\right)(z+h) = \left(f|_k\sigma^{-1}\right)(z) \qquad \forall z \in \mathbb{H}.$$

Thus we can bring in Fourier analysis: there exists a function $g \colon \mathbb{D} - \{0\} \to \mathbb{C}$ such that

$$\left(f|_k\sigma^{-1}\right)(z) = g\left(\exp\left(2\pi i z/h\right)\right).$$

Since f is meromorphic on $\mathbb{H}$ we have g is meromorphic on $\mathbb{D} \setminus \{0\}$.

Definition 10.5. Let g be as above.

- If k is even we say $f \in \Omega_k(\Gamma)$ is meromorphic/holomorphic/zero at a cusp x if g is meromorphic/holomorphic/zero at 0.
- If k is odd, we replace f with f^2 above.

(One can check this is independent of the choice of σ and g.)

Now let $g(w) = \sum_{n=n_0}^{\infty} a_n w^n$ be the Laurent series for g at $w = 0$ (with $a_{n_0} \neq 0$) on some $U_\ell = \{z \in \mathbb{H} \mid \operatorname{Im} z > \ell\}$. If k is odd, we use f^2 instead and then

$$\left(f|_k\sigma^{-1}\right)(z+h) = \begin{cases} (f|\sigma^{-1})(z) & x \text{ regular cusp} \\ -(f|\sigma^{-1})(z) & x \text{ irregular cusp} \end{cases}$$

Thus, for odd k we have either

$$\left(f|_k\sigma^{-1}\right)(z+h) = \begin{cases} \sum_{n \ge n_0 \text{ even}} a_n \exp(\pi i n z/h) & \text{regular cusp} \\ \sum_{n \ge n_0 \text{ odd}} a_n \exp(\pi i n z/h) & \text{irregular cusp.} \end{cases}$$

The Fourier expansion of f at cusps converges absolutely uniformly on any compact subset of U_ℓ (or all of $\mathbb{H}$ if f is holomorphic on $\mathbb{H}$), and we have f is holomorphic (resp zero) at the cusp x if and only if $n_0 \ge 0$ and (resp $n_0 > 0$).

Definition 10.6. Let Γ be a Fuchsian group of the first kind, and $k \in \mathbb{Z}$. We let:

$$\begin{aligned} A_k(\Gamma) &= \{f \in \Omega_k(\Gamma) \mid f \text{ meromorphic at all cusps of } \Gamma\} \\ M_k(\Gamma) &= \{f \in \Omega_k(\Gamma) \mid f \text{ holomorphic on } \mathbb{H} \text{ and cusps of } \Gamma\} \\ S_k(\Gamma) &= \{f \in \Omega_k(\Gamma) \mid f \text{ holomorphic on } \mathbb{H}, \text{ zero cusps of } \Gamma\}. \end{aligned}$$

These are called "meromorphic automorphic forms", "holomorphic automorphic forms"[2] and "cusp forms" respectively.

Each of these sets is a $\mathbb{C}$-vector space, and can be made into a graded $\mathbb{C}$-algebra by $A(\Gamma) = \sum A_k(\Gamma)$, since $A_k(\Gamma)A_\ell(\Gamma) \subseteq A_{k+\ell}(\Gamma)$ and so on.

Remark 10.7. We obviously have

$$S_k(\Gamma) \subseteq M_k(\Gamma) \subseteq A_k(\Gamma) \subseteq \Omega_k(\Gamma).$$

Also:

- If Γ has no cusps then $\Omega_k(\Gamma) = A_k(\Gamma)$ and $M_k(\Gamma) = S_k(\Gamma)$.
- For all $\alpha \in \mathrm{GL}_2^+(\mathbb{R})$, the map $f \mapsto f|_k\alpha$ gives isomorphisms $A_k(\Gamma) \simeq A_k(\alpha^{-1}\Gamma\alpha)$ and similarly for M_k and S_k.

[2] Miyaki calls these "integral forms", but has nothing do with integer coefficients.

- If $0 \neq f \in A_k(\Gamma)$, then $\frac{1}{f} \in A_{-k}(\Gamma)$.

Definition 10.8. The last bullet of the previous remark implies $A_0(\Gamma)$ is a field, which we say is the field of **automorphic functions**.

For $f \in A_0(\Gamma)$ we have $(f|_0\gamma) = f(\gamma z) = f(z)$, hence $f(z) = \varphi \circ \pi_\Gamma(z)$ for some meromorphic function $\varphi \colon X_\Gamma \to \mathbb{C}$ and the projection $\pi_\Gamma \colon \mathbb{H}^*_\Gamma \to X_\Gamma$. Thus we have

$$A_0(\Gamma) \simeq \mathbb{C}(X_\Gamma).$$

Lemma 10.9

Suppose $\Gamma' \subseteq \Gamma$ has finite index. Then $A_k(\Gamma) = \Omega(\Gamma) \cap A_k(\Gamma')$ and similarly for M_k, S_k. In particular, $A_k(\Gamma) \subseteq A_k(\Gamma')$ as desired.

Proof. We proved earlier that the cusps of Γ' and Γ coincide. $\square$

Theorem 10.10

Let Γ be a Fuchsian group of the first kind, and $f \in \Omega_k(\Gamma)$ holomorphic on $\mathbb{H}$. If there exists $v > 0$ such that $f(z) = O\left((\operatorname{Im} z)^{-v}\right)$ as $\operatorname{Im} z \to 0$ uniformly with respect to $\operatorname{Re} z$, then $f \in M_k(\Gamma)$.

If $v < k$ then $f \in S_k(\Gamma)$ as well.

Proof. WLOG k is even (for odd k use f^2 instead).

If $\Gamma \setminus \mathbb{H}$ is compact (no cusps) we are done.

Suppose x is a cusp of Γ on $\mathbb{R}$ (meaning $x \neq \infty$), and as usual write

$$\sigma \Gamma_x \sigma^{-1}\{\pm 1\} = \left\{ \pm \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}^n \mid n \in \mathbb{Z} \right\}$$

for $h > 0$ and $\sigma x = \infty$. Now write

$$\left(f|_k\sigma^{-1}\right)(z) = \sum_{n \in \mathbb{Z}} a_n \exp\left(\frac{2\pi i n z}{h}\right).$$

Here, we have

$$a_n = \frac{1}{h}\int_{z_0}^{z_0+h} \left(f|_k\sigma^{-1}\right)(z) \exp\left(\frac{2\pi i n z}{h}\right).$$

Let $\sigma^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $c \neq 0$ and hence

$$\operatorname{Im}\left(\sigma^{-1}z\right) = \frac{\operatorname{Im} z}{|cz+d|^2} = O\left(\frac{1}{\operatorname{Im} z}\right)$$

as $\operatorname{Im} z \to \infty$. If we let $z_0 = iy - h/2$, we have

$$|a_n| = O\left(y^{v-k} \exp\left(\frac{2\pi n y}{h}\right)\right)$$

as $y \to \infty$. If $n < 0$ then $a_n = 0$, hence holomorphic at x. If $v < k$, then also $a_0 = 0$, hence zero at x.

If $x = \infty$, since $\Gamma \neq \Gamma_\infty$ (since Γ is Fuschian of the first kind) for $\gamma \in \Gamma \setminus \Gamma_\infty$, we have $\gamma\infty \in \mathbb{R}$, so we can apply the previous argument. $\square$

Theorem 10.11

For $f \in \Omega_k(\Gamma)$, we have $f \in S_k(\Gamma)$ if and only if $f(z)(\operatorname{Im} z)^{k/2}$ bounded on $\mathbb{H}$.

Proof. See Miyaki. $\square$

Corollary 10.12

If $f \in S_k(\Gamma)$, if x_0 is a cusp of Γ, $\sigma x_0 = \infty$, then the Fourier expansion of f at x_0, then write

$$(f|_k \sigma^{-1})(z) = \sum_{n \geq 1} a_n \exp\left(\pi i n z / h\right).$$

This Fourier expansion satisfies $a_n = O(n^{k/2})$.

§11 March 14, 2018 (Wednesday)

As always Γ is a Fuchsian group of the first kind.

§11.1 Automorphic forms with characters

We will generalize our definition of automorphic form slightly to allow characters. Let

$$\chi \colon \Gamma \to U(1)$$

be a character with finite image (hence its values are roots of unity). (Here $U(1)$ is the unit circle.) Then $\Gamma_\chi = \ker \chi$ is a finite index subgroup of χ.

Definition 11.1. We let

$$\Omega_k(\Gamma, \chi) = \{f \in \Omega(\Gamma_\chi) \mid f|_k\chi = \chi(\gamma)f \quad \forall \gamma \in \Gamma\}$$

and call them the **automorphic forms with character**. We then define $A_k(\Gamma, \chi) = \Omega_k(\Gamma, \chi) \cap A_k(\Gamma_\chi)$ and so on similarly, and M_k, S_k in the same way.

Remark 11.2. Note that when $-1 \in \Gamma$, we have $\chi(-1) \neq (-1)^k$ thus $\Omega_k(\Gamma, \chi) = \{0\}$.

Observe that for $\Gamma' \subseteq \Gamma_\chi$ has finite index then

- $A_k(\Gamma, \chi) = \Omega_k(\Gamma, \chi) \cap A_k(\Gamma')$ and similarly.
- χ restricts to a character of Γ', and $A_k(\Gamma, \chi) \subseteq A_k(\Gamma', \chi)$. Similarly for M_k, S_k.
- Let $f \in A_k(\Gamma, \chi)$ and $g \in A_\ell(\Gamma, \psi)$. Then $fg \in A_{k+\ell}(\Gamma, \chi\psi)$.

§11.2 Petersson inner product

Let $f, g \in M_k(\Gamma, \chi)$ and assume at least one is a cusp form. Then $fg \in S_{2k}(\Gamma, \chi^2)$, and so the theorem gives $\left| f(z)\overline{g(z)} \right| (\operatorname{Im} z)^k$ is bounded on $\mathbb{H}$.

Then since $f \in M_k(\Gamma, \chi)$, from $f|_k\gamma = f$ we have $f(\gamma z) = \frac{f(z)}{j(\gamma,z)^{-k}}$ and similarly for g, and since $\operatorname{Im}(\gamma z) = \frac{\operatorname{Im} z}{j(\gamma,z)}^2$, we deduce

$$f(\gamma z)\overline{g(\gamma, z)}(\operatorname{Im} \gamma z)^k = f(z)\overline{g(z)}(\operatorname{Im} z)^k \qquad \forall \gamma \in \Gamma.$$

and hence the left-hand side gives a continuous function on $\Gamma \setminus \mathbb{H}$.

Definition 11.3. The **Petersson inner product** is defined for f, g as above by

$$\langle f, g \rangle \overset{\text{def}}{=} \frac{1}{\mu(\Gamma \setminus \mathbb{H})} \int_{\Gamma \setminus \mathbb{H}} f(z)\overline{g(z)}(\operatorname{Im} z)^k \underbrace{d\mu}_{=\frac{dx\,dy}{y^2}}.$$

It is a bilinear form with $\langle f, \overline{g} \rangle = \langle \overline{f}, g \rangle$. (Note the integral won't converge if f and g are both not cusp forms.)

Now suppose Γ_1, χ_1 and Γ_2, χ_2 are such that χ_1 and χ_2 are "commensurable", meaning there is a finite index Γ' with $\Gamma' \subseteq \ker(\chi_1) \cap \ker(\chi_2)$.

Definition 11.4. We define $N_k(\Gamma, \chi)$ as

$$N_k(\Gamma, \chi) = \{g \in M_k(\Gamma, \chi) \mid \langle f, g \rangle = 0 \quad \forall f \in S_k(\Gamma, \chi)\}.$$

(Again, we rely on $f \in S_k(\Gamma, \chi)$ in order for the inner product to make sense).

Proposition 11.5
- $M_k(\Gamma, \chi) = S_k(\Gamma, \chi) \oplus N_k(\Gamma, \chi)$.
- If $\Gamma' \subset \Gamma$ has finite index, and χ' is the restriction of χ to Γ' then
$$N_k(\Gamma, \chi) = N_k(\Gamma', \chi') \cap M_k(\Gamma, \chi).$$

Anyways, we won't use this for a while, but makes sense to define it now.

§11.3 Meromorphic (and Kähler) differentials

Let $f \in \Omega_k(\Gamma)$. Alas the function $f(\gamma z) = j(\gamma, z) f(z)$ is not Γ-invariant, but it is almost Γ-invariant: for $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we have

$$d\gamma(z) = \frac{d}{dz}\frac{az+b}{cz+d} = \frac{ad-bc}{(cz+d)^2}\, dz = j(\gamma, z)^2\, dz.$$

Consequently, the idea is that "$f(z)(dz)^{k/2}$" is Γ-invariant. We'll now make this precise.

Definition 11.6 (Kähler differentials)**.** Let A be a commutative algebra A over a field F. The A-module $\Omega^1_F(A)$ is the set of symbols $\{df \mid f \in A\}$ subject to relations

$$\begin{aligned} dc &= 0 \qquad \forall c \in F \\ d(fg) &= f dg + g df \\ d(f+g) &= df + dg \\ (df)g &= g\, df. \end{aligned}$$

(These will be functions eventually.)

An equivalent definition: define the map

$$A \otimes_F A \to A \qquad \sum f_i \otimes g_i \mapsto \sum f_i g_i$$

and let I be the kernel (augmentation ideal). Then we can also view $\Omega^1_F(A) = I/I^2$, and $df = 1 \otimes f - f \otimes 1$.

Definition 11.7. A **derivation** d is an F-linear map of A-modules for which F is in the kernel, and the Leibniz rule $d(fg) = f\, dg + g\, df$.

Theorem 11.8 (Universal property of Kähler differentials)
The map $A \to \Omega^1_F(A)$ by $f \mapsto df$ is a derivation satisfying the following universal property: if $d_B \colon AtoB$ is another derivation then there exists a unique A-module homomorphism completing the diagram

$$\begin{tikzcd} A \arrow[r, "d_B"] \arrow[dr, "d"'] & B \\ & \Omega^1_F(A) \arrow[u, "\exists!"'] \end{tikzcd}$$

Definition 11.9. We now define

$$\Omega_F^{\otimes n} \overset{\text{def}}{=} \left(\Omega_1^F(A)\right)^{\otimes n} = \{g\ (dz)^n \mid g \in A\}.$$

Then $\Omega_F(A) = \bigoplus_{n\geq 1} \Omega_F^{\otimes n}$ is a graded ring.

Now we consider $A = \mathbb{C}(U)$, the set of meromorphic functions $U \to \mathbb{C}$. Then $\Omega^1_{\mathbb{C}}(\mathbb{C}(U))$ is the set of differential forms "on U".

Remark 11.10. In any context where we can write $f = \sum_n a_n z^n$ for some symbol z, we then have

$$df = \left(\sum_n n a_n z^{n-1}\right) dz$$

as an identity in the ring of Kähler differentials; of course the sum we think of as $\partial f / \partial z$.

Remark 11.11. Note that if $\omega_1 = f\ (dz)^n$ and $\omega_2 = g\ (dz)^n$ are in $\Omega^{\otimes n}_{\mathbb{C}}(\mathbb{C}(U))$ and

$$\frac{\omega_1}{\omega_2} = \frac{f}{g} \in \Omega^0_{\mathbb{C}}(\mathbb{C}(U)) = \mathbb{C}(U)$$

and so it is actually possible to evaluate the "ratio" of two n-fold Kähler differentials, as points.

Now if $\phi : U \to V$ is a holomorphic map between open sets; then we get a map

$$\phi^* \colon \Omega^{\otimes n}_{\mathbb{C}}(\mathbb{C}(V)) \to \Omega^{\otimes n}_{\mathbb{C}}(\mathbb{C}(U)) \qquad f(z)(dz)^n \mapsto f(\phi(z))\phi(z)^n(dz)^n.$$

Finally, given a Riemann surface X and a function field $\mathbb{C}(X)$, given charts $t_p \colon U_p \to V_p$ (where $U_p \subseteq X$ and $V_p \subseteq \mathbb{C}$ are open), then a meromorphic differential on X is a collection of $\{\omega_p\}_{p\in X}$ with $\omega_p \in \Omega^1_{\mathbb{C}}(\mathbb{C}(V_i))$ satisfying transition maps (not written).

Theorem 11.12

Let Γ be a Fuchsian group of the first kind. Let k be an even integer. There is an isomorphism of $\mathbb{C}$-vector spaces

$$\omega \colon A_k(\Gamma) \to \Omega^{\otimes k/2}_{\mathbb{C}}(\mathbb{C}(X_\Gamma))$$

which we denote $f \mapsto \omega_f$.

Proof. Let $k = 2m$, $f \in A_k(\Gamma)$ and let $\pi \colon \mathbb{H}^*_\Gamma \to X_\Gamma$. For a point $P \in X_\Gamma$, we let $z_0 \in \mathbb{H}^*_\Gamma$ such that $\pi(z_0) = P$. Let $U^*_{z_0}$ be a neighborhood of $z_0 \in \mathbb{H}^*_\Gamma$ such that

1. $\gamma U^*_{z_0} \cap U^*_{z_0} \neq 0 \iff \gamma \in \Gamma_{z_0}$,
2. $\gamma \in \Gamma_{z_0} \implies \gamma U^*_{z_0} = U_{z_0}$.

Then take a chart $t_p \colon \pi(U_p) \to V_p \to \mathbb{C}$.

... $\square$

§11.4 ???

Let $f \in \mathbb{C}(X_\Gamma)$. For $p \in X_\Gamma$, we can associate $\operatorname{ord}_p(f)$ by looking at the Laurent series for f on the neighborhood of p, which gives the order of the pole p (or the negative of multiplicity of zero). This has

$$\sum_{p \in X_\Gamma} \operatorname{ord}_p(f) = 0.$$

Now for $\omega \in \Omega_{\mathbb{C}}^{\otimes n}(\mathbb{C}(X_\Gamma))$ not zero we have $\omega = f(z)(dz)^n$ at $p \in X_\Gamma$ for some chart, and we then define $\operatorname{ord}_p(\omega) = \operatorname{ord}_p f$.

Definition 11.13. Define $\operatorname{Div}(X_\Gamma)$ to be the free $\mathbb{Z}$-module on X_Γ and let the **principal divisor** of f as $\sum_{p \in X_\Gamma} \operatorname{ord}_p(f) \cdot p$. The **degree** of a divisor is $\deg(\sum n_p p) \overset{\text{def}}{=} \sum n_p \in \mathbb{Z}$.

Remark 11.14. Since X_Γ is compact, the degree of any principal divisor is zero.

Definition 11.15. Similarly, define $\operatorname{div}(\omega) \sum_p \operatorname{ord}_p(\omega) \cdot p$. Then $\operatorname{div}(\omega_1 \omega_0) = \operatorname{div}(\omega_1) + \operatorname{div}(\omega_0)$

§15 April 4, 2018 (Wednesday)

Several lectures missing due to mild illness. Sorry.

§15.1 Setup

Let $G = \mathrm{GL}_2^+(\mathbb{R})$. We say two subgroups G_1 of G_2 are **commensurable** if $G_1 \cap G_2$ has finite index in G_1 and G_2. This gives an equivalence relation $\sim$.

Now give a Fuchsian group Γ we define the **commensurator** by

$$\mathrm{Comm}_G(\Gamma) = \left\{ g \in G \mid g\Gamma g^{-1} \sim \Gamma \right\}.$$

Let $\chi \colon \Gamma \to \mathbb{C}^\times$ be a character of finite order, and let $S = \{\Gamma' \subseteq \Gamma \colon [\Gamma : \Gamma'] < \infty\}$. Then given a monoid Δ with $\Gamma \subseteq \Delta \subseteq \mathrm{Comm}_G(\Gamma)$, we can put

$$\begin{aligned} \chi \colon \Delta &\to \mathbb{C}^\times \\ \alpha\gamma\alpha^{-1} &\mapsto \chi(\gamma) \end{aligned}$$

for $\gamma \in \Gamma$, $\alpha \in \Delta$.

Now, let R be a commutative ring. We define $\mathbb{T}_R(\Gamma, \Delta)$ to be the Hecke algebra of Γ over R (for Δ). This is a free R-module on double cosets $\Gamma\alpha\Gamma$, $\alpha \in \Delta$, with multiplication given by

$$\Gamma\alpha\Gamma \cdot \Gamma\beta\Gamma = \sum_{\text{double coset } \Gamma\gamma\Gamma} c_\gamma \Gamma\gamma\Gamma$$

where $c_\gamma = \#\{(i,j) : \Gamma\alpha_i\beta_j = \Gamma\gamma\}$ for any decomposition $\Gamma\alpha\Gamma = \bigsqcup \Gamma\alpha_i$, $\Gamma\beta\Gamma = \bigsqcup \Gamma\beta_j$ (this c_γ is independent of the choice of decomposition).

Let M be an R-module with a right Δ-action. Then M^Γ is a $\mathbb{T}(\Gamma, \Delta)$-module via

$$m \mid \Gamma\alpha\Gamma \overset{\text{def}}{=} \sum m^{\alpha_i}$$

with the sum across cosets $\Gamma\alpha\Gamma = \bigsqcup \Gamma\alpha_i$. The map $m \mapsto m|\phi$ is an R-endomorphism of M^Γ for all $\phi \in \mathbb{T}(\Gamma, \Delta)$.

§15.2 Automorphic forms

Let

$$A_k \overset{\text{def}}{=} \bigcup_{\Gamma' \in S} A_k(\Gamma', \chi).$$

Let $f \in A_k$, and consider the Δ-action

$$f^\alpha \overset{\text{def}}{=} \det(\alpha)^{k/2-1} \overline{\chi(\alpha)} f|_k \alpha.$$

Given $f \in A_K(\Gamma_1, \chi)$ we can then put

$$f|\Gamma_1\alpha\Gamma_2 \overset{\text{def}}{=} \det(\alpha)^{k/2-1} \sum \overline{\chi(\alpha_i)} f|_k \alpha_i.$$

The Γ-invariant subspace of A_k is $A_k(\Gamma, \chi)$.

Note the scalars $\mathbb{R}^\times \subseteq \mathrm{Comm}_G(\Gamma)$ acts trivially on f, so we can assume $\Gamma\alpha\Gamma = \bigsqcup \Gamma\alpha_i$ always has $\det(\alpha_i) = \det(\alpha)$.

A **Hecke operator** is a linear map $A_k(\Gamma_1, \chi) \to A_k(\Gamma_2, \chi)$ induced by $\alpha \in \Delta$.

Theorem 15.1

For $\alpha \in \mathrm{Comm}_G(\Gamma)$, let $\alpha' = \det(\alpha)\alpha^{-1}$, (hence $\det(\alpha') = \det(\alpha)$). Then the Petersson inner products

$$\langle f|\Gamma\alpha\Gamma, g\rangle = \left\langle f, g|\Gamma\alpha'\Gamma\right\rangle$$

are equal for all $f \in S_k(\Gamma)$, $g \in M_k(\Gamma)$.

Corollary 15.2

- If $\chi \neq \psi$ are characters of Γ then $\langle f, g\rangle = 0$ whenever $f \in S_k(\Gamma, \chi)$ and $g \in M_k(\Gamma, \psi)$.
- If $f \in N_k(\Gamma)$, then $f|\Gamma\alpha\Gamma \in N(\Gamma)$, for $\alpha \in \mathrm{Comm}_G(\Gamma)$.

§15.3 Modular groups

Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ denote the **full modular group**. A **modular group** is a finite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

Proposition 15.3 (Basic facts about the modular group)

Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. We have:

- Γ is generated by $\omega \overset{\text{def}}{=} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\tau \overset{\text{def}}{=} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.
- If $k \ge 2$ then
$$\dim S_k(\Gamma) = \begin{cases} 0 & k = 2 \\ \lfloor k/12 \rfloor - 1 & k \equiv 2 \pmod 12 \\ \lfloor k/12 \rfloor & \text{else.} \end{cases}$$
- If $k \ge 2$ then
$$\dim M_k(\Gamma) = \begin{cases} \lfloor k/12 \rfloor & k \equiv 2 \pmod 12 \\ \lfloor k/12 \rfloor + 1 & \text{else.} \end{cases}$$
- The volume is $v(X_\Gamma) = \pi/3$, and genus is $g(\Gamma) = 0$.
- The elliptic points are all Γ-equivalent to $i = \sqrt{-1}$ and $\xi_3 = e^{\frac{2}{3}\pi i}$, which have orders two and three In particular, $\Gamma_i = \langle w\rangle$, $\Gamma_{\xi_3} = \left\langle \left[\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right]\right\rangle$.
- The cusps are $\mathbb{Q} \cup \{\infty\}$, which are all Γ-equivalent to ∞. The stabilizer is $\Gamma_\infty = \langle \tau \rangle$.

We then define a special set of subgroups, called the congruence modular groups:

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod N \right\}$$
$$\Gamma_1(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod N, a \equiv d \equiv 1 \pmod N \right\}$$
$$\Gamma(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid b \equiv c \equiv 0 \pmod N, a \equiv d \equiv 1 \pmod N \right\}$$

We say that N is the level. Note that $\mathrm{SL}_2(\mathbb{Z}) = \Gamma_0(1) = \Gamma_1(1) = \Gamma(1)$ and we'll use this notation.

We say Γ is a **congruence modular subgroup** if $\Gamma(N) \subseteq \Gamma \subseteq \Gamma(1)$ for some N.

§15.4 Hecke algebras for modular groups

Lemma 15.4

Let Γ be a modular group. Then

$$\mathrm{Comm}_G(\Gamma) = \mathbb{R}^\times \, \mathrm{GL}_2^+(\mathbb{Q}).$$

So now that we have computed the commensurator, we want to pick the choice of monoid. There are two choices:

Definition 15.5. Let

$$\Delta_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{Mat}_2(\mathbb{Z}) \mid c \equiv 0 \bmod N, \ \gcd(a, N) = 1, \ ad - bc > 0 \right\}$$

and

$$\Delta_0^*(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{Mat}_2(\mathbb{Z}) \mid c \equiv 0 \bmod N, \ \gcd(d, N) = 1, \ ad - bc > 0 \right\}.$$

Thus

$$\Delta_0^*(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{Mat}_2(\mathbb{Z}) \mid c \equiv 0 \bmod N, \ \gcd(ad - bc) = N, \ ad - bc > 0 \right\}.$$

Then we let $\mathbb{T}(N) \overset{\text{def}}{=} \mathbb{T}(\Gamma_0(N), \Delta_0(N))$, $\mathbb{T}^*(N) \overset{\text{def}}{=} \mathbb{T}(\Gamma_0(n), \Delta_0^*(N))$.

It will be nice to write a set of representatives for the double coset now.

Lemma 15.6

Let $\alpha \in \Delta_0(N)$, then there exists unique $\ell, m \in \mathbb{Z}_{>0}$ with $\ell \mid m$ and $\gcd(\ell, N)$ such that

$$\Gamma_0(N) \alpha \Gamma_0(N) = \Gamma_0(N) \begin{bmatrix} \ell & 0 \\ 0 & m \end{bmatrix} \Gamma_0(N).$$

Similarly, if $\alpha \in \Delta_0^*(N)$, then we have the same result with $\begin{bmatrix} m & 0 \\ 0 & \ell \end{bmatrix}$ instead.

Remark. If $\gcd(\ell m, N) = 1$ then in fact

$$\Gamma_0(N) \begin{bmatrix} \ell & 0 \\ 0 & m \end{bmatrix} \Gamma_0(N) = \Gamma_0(N) \begin{bmatrix} m & 0 \\ 0 & \ell \end{bmatrix} \Gamma_0(N).$$

This follows by applying the previous theorem.

Theorem 15.7 (1) $\mathbb{T}(N)$ and $\mathbb{T}^*(N)$ are commutative.

(2) For every $\Gamma_0(N)\alpha\Gamma_0(N) \in \mathbb{T}(N) \cup \mathbb{T}^*(N)$, we have common coset decompositions

$$\Gamma_0(N)\alpha\Gamma_0(N) = \bigsqcup \Gamma_0(N)\alpha_i = \bigsqcup \alpha_i\Gamma_0(N).$$

Actually, we can prove this in a more general context.

Lemma 15.8

Let Γ be a Fuchsian group and $\Gamma \subseteq \Delta \subseteq \operatorname{Comm}_G(\Gamma)$. Suppose $\bullet' \colon \Delta \to \Delta$ (denoted $\alpha \mapsto \alpha'$) is an **anti-isomorphism**, meaning $(\alpha\beta)' = \beta'\alpha'$, $\Gamma' = \Gamma$, and $\Gamma\alpha'\Gamma = \Gamma\alpha\Gamma$. Then

$$\Gamma\alpha\Gamma = \bigsqcup \Gamma\alpha_i = \bigsqcup \alpha_i\Gamma$$

and moreover $\mathbb{T}(\Gamma, \Delta)$ is commutative.

Proof. Consider a decomposition

$$\Gamma\alpha\Gamma = \bigsqcup \Gamma\alpha_i.$$

Apply our anti-isomorphism:

$$\Gamma\alpha'\Gamma = \bigsqcup \alpha_i'\Gamma.$$

However, these two decompositions have the same number of left and right cosets. We proved last time that this alone lets us find a common decomposition.

Now to show abelian, suppose $\Gamma\alpha\Gamma = \bigsqcup \Gamma\alpha_i$ $\Gamma\beta\Gamma = \bigsqcup \Gamma\beta_i$, and take the two products

$$\begin{aligned} (\Gamma\alpha\Gamma)(\Gamma\beta\Gamma) &= \sum c_\gamma \Gamma\gamma\Gamma \\ (\Gamma\beta\Gamma)(\Gamma\alpha\Gamma) &= \sum c_\gamma' \Gamma\gamma\Gamma. \end{aligned}$$

Write

$$\begin{aligned} c_\gamma &= \#\left\{(i,j) : \Gamma\alpha_i\beta_j = \Gamma\gamma\right\} \\ &= \#\left\{(i,j) : \Gamma\alpha_i\beta_j\Gamma = \Gamma\gamma\Gamma\right\} / \#\left\{\Gamma \setminus \Gamma\gamma\Gamma\right\} \\ &= \#\left\{(i,j) : \Gamma\beta_j'\alpha_i'\Gamma = \Gamma\gamma\Gamma\right\} / \#\left\{\Gamma \setminus \Gamma\gamma\Gamma\right\} \\ &= \#\left\{(i,j) : \Gamma\beta_j'\alpha_i' = \Gamma\gamma\right\} \\ &= c_\gamma'. \end{aligned}$$

□

Finally, to prove the theorem from the lemma, it suffices to exhibit an anti-isomorphism. Here it is:

$$\begin{aligned} \phi\colon \Delta_0(N) &\to \Delta_0(N) \\ \begin{bmatrix} a & b \\ cN & d \end{bmatrix} &\mapsto \begin{bmatrix} a & c \\ bN & d \end{bmatrix}. \end{aligned}$$

Finally, let $\chi\colon (\mathbb{Z}/N)^\times \to \mathbb{C}$ be a Dirichlet character modulo N. Then we induce a character $\chi\colon \Gamma \to \mathbb{C}^\times$ by

$$\begin{aligned} \chi\colon \Gamma &\to \mathbb{C}^\times \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} &\mapsto \chi(a) \\ \chi^*\colon \Gamma &\to \mathbb{C}^\times \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} &\mapsto \chi(d). \end{aligned}$$

Now,

- given $\alpha \in \Delta_0(N)$, let $\chi(\alpha) = \overline{\chi(a)}$.
- given $\alpha \in \Delta_0^*(N)$, let $\chi^*(\alpha) = \chi^*(d)$.

Then $\mathbb{T}(N)$ acts on $M_k(N,\chi) \overset{\text{def}}{=} M_k(\Gamma_0(N),\chi)$ and $\mathbb{T}^*(N)$ acts on $M_k(N,\chi^*) \overset{\text{def}}{=} M_k(\Gamma_0(N),\chi^*)$.

Lemma 15.9

We have

$$M_k(\Gamma_1(N)) = \bigoplus_\chi M_k(\Gamma_0(N)).$$

where the sum is over Dirichlet characters modulo N. Similar statements holds for S_k, N_k.

Proof. Note $\Gamma_1(N)$ is normal in $\Gamma_0(N)$. Thus $\Gamma_0(N)$ acts on $M_k(\Gamma_1(N))$ via $f \mapsto f|_k\gamma$. This action induces a representation of the finite abelian group $\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N)^\times$ of the complex vector space $M_k(\Gamma_1(N))$.

The irreducible representations of $(\mathbb{Z}/N)^\times$ all arise from Dirichlet characters. So we can decompose $M_k(\Gamma_1(N))$ into irreducible representations corresponding to χ's, which is the sum we wrote above. $\square$

$$\begin{aligned} & \Gamma_0(N)\alpha\Gamma_0(N) = \bigsqcup \Gamma_0(N)\alpha_i \\ \alpha \in \Delta_0(N) \qquad & f|\Gamma_0(N)\alpha\Gamma_0(N) = \det(\alpha)^{k/2-1} \sum \overline{\chi(\alpha_i)} f|_k\alpha_i \\ \alpha \in \Delta_0^*(N) \qquad & f|\Gamma_0(N)\alpha\Gamma_0(N) = \det(\alpha)^{k/2-1} \sum \overline{\chi^*(\alpha_i)} f|_k\alpha_i \\ & \omega_N\colon \omega(N) = \begin{bmatrix} 0 & -1 \\ N & 0 \end{bmatrix} \\ & \omega_n^{-1}\Gamma_0(N)\omega_N = \Gamma_0(N) \end{aligned}$$

Now

$$\Delta_0(N) \xrightarrow{\sim} \Delta_0^*(N)$$
$$\alpha \mapsto \omega_n^{-1}\alpha\omega_N$$
$$\chi^*(\omega_N^{-1}\alpha\omega_N) = \overline{\chi(\alpha)}.$$

Definition 15.10. Let $\ell \mid m$, $\gcd(\ell, N) = 1$. We let

$$\mathbb{T}(\ell, m) \overset{\text{def}}{=} \Gamma_0(N) \begin{bmatrix} \ell & 0 \\ 0 & m \end{bmatrix} \Gamma_0(N) \in \mathbb{T}(N).$$
$$T(n) \overset{\text{def}}{=} \sum_{\det \alpha = n} \Gamma_0(N)\alpha\Gamma_0(N) \in \mathbb{T}(N).$$

Here, the sum $\det(\alpha) = n$ is over cosets, so this is a finite sum.

Similarly,

$$\mathbb{T}^*(m, \ell) \overset{\text{def}}{=} \Gamma_0(N) \begin{bmatrix} m & 0 \\ 0 & \ell \end{bmatrix} \Gamma_0(N) \in \mathbb{T}^*(N).$$
$$T^*(n) \overset{\text{def}}{=} \sum_{\det \alpha = n} \Gamma_0(N)\alpha\Gamma_0(N) \in \mathbb{T}^*(N).$$

Remark 15.11. $T(n) = \sum_{\ell m = n} T(\ell, m)$, $T^*(n) = \sum_{\ell m = n} T(\ell, m)$. $T(p) = T(1, p)$ and $T^*(p) = T(p, 1)$. If $\gcd(n, N) = 1$ then $T(n, n) = \Gamma_0(N)[\begin{smallmatrix} n & 0 \\ 0 & n \end{smallmatrix}]\Gamma_0(N) = \Gamma_0(N)[\begin{smallmatrix} n & 0 \\ 0 & n \end{smallmatrix}]$.

We have $T(n, n)T(\ell, m) = T(n\ell, nm)$, $T^*(n, n)T^*(m, \ell) = T^*(nm, n\ell)$.

Theorem 15.12

Let $f \in M_k(N, \chi)$ by a modular form of level N and weight k.

(1) $f|T^*(m, \ell) = \overline{\chi}(\ell m) f|T(\ell, m)$, where $\gcd(\ell m, N) = 1$. Moreover, $f|T^*(n) = \overline{\chi}(n) f|T(n)$

(2) $T(\ell, m)$ and $T^*(m, \ell)$ give adjoint linear operations with respect to the Petersson inner product on $S_k(N,\ ci)$. So do $T(n)$ and $T^*(n)$.

(3) $S_k(N, \chi)$ has a basis of simultaneous eignefunctions of all $T(n)$ and $T(\ell, m)$, for $\gcd(n, N) = 1$ and $\gcd(\ell m, N) = 1$.