18.757 (Representation of Lie Algebras) Lecture Notes

Massachusetts Institute of Technology

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This is MIT's graduate course 18.757, instructed by Laura Rider. The formal name for this class is "Representations of Lie Algebras".

The permanent URL for this document is `http://web.evanchen.cc/coursework.html`, along with all my other course notes.

Contents

§1 February 2, 2016

Following Kirillov, chapters 4 and 8. Office is 2-246a, office hours are Tu 11-12 and W 11:30-12:30.

Exercises: 2.1-2.4, 2.12, 2.13[K]; choose 3 to submit.

§§1.1 Representations of groups

Definition 1.1. Let G be a group. A **complex representation** of G is a complex vector space V with group homomorphism $\rho : G \to \mathrm{Aut}(V) = \mathrm{GL}(V)$.

Example 1.2 (Examples of Representations)

(a) For any G, $V = \mathbb{C}$, we have a trivial representation $G \mapsto 1 \in \mathbb{C}^\times$.

(b) Let $G = S_2 = \{e, \tau\}$. Then the sign representation $S_2 \to \mathbb{C}^\times$ is $e \mapsto 1$ and $\tau \mapsto -1$.

(c) Let S_n be the symmetric group on n letters. Let $V = \mathbb{C}^n$ with basis $b_1, \dots, b_n$. Then $S_n \to \mathrm{GL}(V)$ by permuting the basis and extending linearly.

Definition 1.3. A **morphism** of representations $f : V \to W$ is a linear map commuting with the action of G: $f\rho_V(g) = \rho_W(g)f$. We denote by $\mathrm{Hom}_G(V, W)$ the set of such morphisms. We let $\mathrm{Rep}\, G$ denote finite dimensional complex representations.

Definition 1.4. A **subrepresentation** W of V is a linear subspace such that $\rho(g)W \subseteq W$ for all g.

Definition 1.5. Define the **quotient representation** V/W in the obvious way.

Thus, short exact sequence of representations

$$0 \to W \to V \to V/W \to 0.$$

If $f : V \to W$, $\operatorname{im} f$ is a subrepresentation, so is $V/\ker f$, and these are isomorphic as G-representations.

Remark 1.6. What we're getting at is that $\mathrm{Rep}\, G$ is an abelian category. To do this we'd need to define $V \oplus W$ (obvious way)

§§1.2 Digression

A student wants to make cohomology happen. In order to do this, we need a functor (derived functors). One example would be

$$F : \mathrm{Rep}(G) \to \mathrm{Vect}\, \mathbb{C}$$

but this isn't a good example since it's exact. More functors:

- $\mathrm{Hom}_G(V, -)$
- $\mathrm{End}_G(V)$
- $\mathrm{Hom}_G(\mathbb{C}, -)$
- $(-)^G$ (G-invariants).

§§1.3 More on representations

Definition 1.7. A representation V is **irreducible** if its only subrepresentations are 0 and V.

In categorical language this is always called "simple".

> **Example 1.8**
> The group S_3 acting on $\mathrm{GL}(\mathbb{C}^3)$ is not irreducible because the diagonal (t,t,t) is a subrep. The set of (a,b,c) with $a+b+c=0$ is also a subrep; it is two-dimensional and one can check its irreducible.

§§1.4 Recovering G

Can we reconstruct G from its category of representations? If so, the structure of G has multiplication, id and inversion, so we want to have a representation analogues of these operations.

Definition 1.9. If $V, W \in \mathrm{Rep}(G)$ then define $V \otimes W$ in the obvious way: $v \otimes w \mapsto \rho_V(g)v \otimes \rho_W(g)w$.

Observe that the trivial representation $\mathbb{C}$ is now the identity; we have $\mathbb{C} \otimes V \cong V$.
Inversion is less straightforward; if $\dim V \ge 2$ then $\dim V \otimes W \ne 1$. On the other hand,

Definition 1.10. If $V \in \mathrm{Rep}(G)$ we define the dual $V^\vee$ in the obvious way.

Then we at least have a map $V \otimes V^\vee \in \mathbb{C}$.

Remark 1.11. If G is finite, then we can get $\mathbb{C}[G]$ by $\bigoplus_{V \in \mathrm{Rep}(G)} V \otimes V^\vee$ But one can't actually read off G from $\mathbb{C}[G]$ easily.

Remark 1.12. Mark Sellke points out that D_8 and quaternions Q_8 have the same character table, so the answer to the question is actually negative.

There is a natural faithful forgetful functor $\mathrm{Rep}\, G \to \mathrm{Vect}\, \mathbb{C}$. Adjoint?
Suppose we have a fully general $\mathscr{C}$, abelian category with a symmetric $\otimes$. We'd like to factor it through a diagram

$$\begin{tikzcd}
\mathscr{C}, \otimes \arrow[r, "?"] \arrow[d, "\text{Forget}"'] & \mathrm{Rep}\, G \arrow[dl] \\
\mathrm{Vect}(\mathbb{C}) &
\end{tikzcd}$$

Here the forgetful functor is a fiber functor (exact, faithful to $\mathrm{Vect}(\mathbb{C})$).

§§1.5 Schur lemma

Lemma 1.13 (Schur's lemma)

Let V and W be irreducible representations of G. Then

$$\operatorname{Hom}_G(V, W) = \begin{cases} \mathbb{C} \cdot \mathrm{id} & V \cong W \\ 0 & \text{else.} \end{cases}$$

In particular, over $\mathbb{C}$ any morphism $V \to V$ is constant, when V is irreducible since it has an eigenspace.

Goal for course: understand $\operatorname{Rep} G$, its irreducible representations, and when understanding the irreducibles is enough to get all of $\operatorname{Rep} G$. (Answer to "enough": *semisimple* or *completely reducible*.)

Next class, we answer this question for finite groups, and see for Lie groups G an example of an indecomposable but reducible rep.

§§1.6 Lie groups

Let G be a group. It's a real **Lie group** if G is also a smooth manifold such that multiplication $G \times G \to G$ and inversion $G \to G$ are smooth.

A representation of a Lie group G is one such that $\rho : G \to \mathrm{GL}(V)$ is also a morphism of Lie groups.

§2 February 4, 2016

First homework due February 11, 2016

Added: 4.3-4.5, 4.11, 4.13, choose 2 from Tuesday and 2 from these

§§2.1 Representations for Lie groups

Example 2.1 (Examples of Lie groups)

(a) $\mathbb{R}, +$

(b) $\mathbb{R}^\times, \times$, disconnected, with identity component $\mathbb{R}_{>0}, \times$

(c) S^1

(d) $SU(2)$, matrices A with $A \in \mathrm{GL}_2 \mathbb{C}$, $\det A = 1$.

Exercise 2.2. Check that $SU(2)$ can be identified with S^3. Thus compact, simply connected.

The complex Lie group $\mathrm{GL}_n \mathbb{C}$ acts on $\mathbb{C}^n$. This representation is irreducible (why?). If $f : \mathbb{C}^n \to \mathbb{C}^n$ is a G-map then $f = \lambda \mathrm{id}$ by Schur. So the center of $\mathrm{GL}_n \mathbb{C}$ is the scalars $\mathbb{C}^\times$.

Similarly the center of $\mathrm{SL}_n \mathbb{C}$ is the nth roots of unity.

Similarly, the unitary group $U(n)$ has S^1 as its center.

Proposition 2.3

Now let G be an abelian group. Then all irreps of G are one-dimensional.

Proof. Napkin problem 45B. □

Here is an indecomposable but not irreducible representation over characteristic zero. Let $\rho : (\mathbb{R}, +) \to \mathrm{GL}_2 \mathbb{C}$ by

$$\rho(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

The span of $(1, 0)$ is a subrepresentation. So there is an exact sequence

$$0 \to \mathbb{C} \hookrightarrow \mathbb{C}^{\oplus 2} \twoheadrightarrow \mathbb{C} \to 0.$$

But it turns out this doesn't split.

Goal:

- How to decompose representations?
- What are sufficient conditions on G such that we can always split into direct sum of irreducibles?

Lemma 2.4

Let (V, ρ) be a representation of G. If $A : V \to V$ is intertwining, then V_λ is a subrepresentation for every λ.

In particular, if A is diagonalizable, then $V = \bigoplus V_\lambda$ as G-reps.

Corollary 2.5

If $g \in Z(G)$, and $\rho(g)$ is diagonalizable then $V = \bigoplus V_\lambda$.

Definition 2.6. A complex rep V is **unitary** if there exists an inner product which is G-invariant.

Theorem 2.7

Finite-dimensional unitary representations are completely reducible.

Proof. If $W \subseteq V$ is a subrepresenattion, then so is the orthogonal complement $W^\perp$. $\square$

Theorem 2.8

A representation of a finite group is unitary.

Proof. If B is a linear form then

$$\widetilde{B}(v, w) = \sum_{g \in G} B(gv, gw)$$

is a G-invariant positive definite inner form. $\square$

In order to get this to work with a Lie group, we want to replace $\sum_{g \in G}$ with $\int_g$.

§§2.2 Measure theory

Let X be a topological space, locally compact and Hausdorff. Then a σ-algebra $\Sigma \subseteq \mathcal{P}(X)$ is a family of sets which are closed under complements, countable unions, and intersections.

The measure μ should satisfy the obvious additive axioms and also $\mu(C) < \infty$ for any compact C. It should also be inner and outer regular, meaning

$$\mu(U) = \sup \{\mu(K) \mid K \subseteq U \text{ compact}\}$$

and similarly for $U \subseteq K$.

Definition 2.9. If G is a topological group a **left Haar measure** on G is a Borel measure μ on G such that $\mu(gA) = A$ i.e. it is translation invariant. (A Borel measure is a measure which also contains every open set.)

For example, the Lebesgue measure on $\mathbb{R}$ which sends $\mu([a, b]) = b - a$ is translation invariant on $\mathbb{R}$, but of course not on $\mathbb{R}^\times$.

Anyways, we want to define a Borel measure on G. The idea is that if G is compact and we have a "small" compact subset K of G, then we can use the G-action to translate K around and then use it to define the measure on all of G. This is what we'll do.

Let K be compact, and let $V \subseteq G$ be a set with nonempty interior $\mathring{V}$ (for example an open set). Then K is covered by $\{g\mathring{V} \mid g \in G\}$. Now K is compact, so it has a finite subcover, and we define $(K : V)$ to be the size of the smallest such cover.

Now, for any U we define

$$\mu_u(K) = \frac{(K : U)}{(K_0 : U)}$$

where K_0 is a particular fixed compact subset with nonempty interior. It's easy to see $0 \le \mu_U(K) \le (K : K_0)$.

Finally, define

$$X = \prod_{G \supseteq K \text{ compact}} [0, (K : K_0)].$$

The space X is compact under the product topology by Tychonoff's theorem. Also, for any U we can regard μ_U as a point in X, namely the point $(\mu_U(K))_{K \subseteq G}$.

Now if V is open we can define the subset of the space

$$C(V) = \overline{\{\mu_U \mid V \supseteq U \text{ open}\}} \subseteq X$$

a closed subset of X. This has the finite intersection property for $V_i \subseteq G$, because

$$\mu_{\bigcap V_i} \in \bigcap_i C(V_i)$$

across $i = 1, \dots, n$. Since X is compact, this now implies that

$$\bigcap_{G \supseteq V \text{ open}} C(V) \neq \varnothing$$

and we take any μ in this set. This μ is a *Radon measure* on G.

One can check that this μ satisfies all the properties we want, but we at least get translation invariance for free, since $(K : U) = (gK : U)$.

What we have just constructed is:

Theorem 2.10 (Haar, Weil, Cartan)

If G is a (Hausdorff) locally compact topological group then there exists a left Haar measure, unique up to some positive scalar.

Remark 2.11. There is a precise way to get a right Haar measure from a left Haar measure and vice-versa. If G is compact, we can declare $\mu(G) = 1$ which will uniquely identify the measure we want.

Remark 2.12. In a compact group, in fact the left and right invariant Haar measures coincide.

Armed with the Haar measure, we have

Theorem 2.13

Any finite dimensional representation of a compact Lie group is unitary and hence is completely reducible.

Proof. Same as in the group case: if $B : V \times V \to \mathbb{C}$ is any inner product we define the averaged inner form by

$$\widetilde{B}(v, w) = \int_G B(gv, gw)\, dg$$

where dg is the right-invariant Haar measure. Then the G-invariance of $\widetilde{B}$ follows. $\square$

Remark 2.14. Assume the special case that G is a compact Lie group. It is a manifold, and one can show its orientable. Thus there is volume form on G, and then we can show that this volume form is G-invariant.

§3 February 9, 2016

Last time: representations of compact Lie groups are unitary, and hence semisimple.

Example 3.1

Let $S^1 = \mathbb{R}/\mathbb{Z}$ be the circle group. We compute its irreducible representations. Let V_k be the irreducible representation of S^1 on $\mathbb{C}$ by $z \mapsto z^k \in \mathbb{C}^\times$.

For the rest of today, assume G is a compact Lie group.

§§3.1 Matrix coefficients

Assume G compact, and write $\rho(g) \in \mathrm{GL}(V)$ as a matrix in some basis of V. Thus we have maps $G \to \mathbb{C}$ for each i, j.

It's more elegant to do this with an inner form.

Definition 3.2. Let $\phi : G \to \mathrm{GL}(V)$ be a group representation, and define the **matrix coefficient** by

$$\phi_{u,v} : G \to \mathbb{C} \quad \text{by} \quad g \mapsto \langle \phi(g)u, v\rangle$$

for every $u, v \in V$. (Here V has a specific G-invariant $\langle -, -\rangle$.)

Classically, we wanted to study $C^\infty(G, \mathbb{C})$, so we endow it with an inner product

$$\int_G f(x)\overline{g(x)}\, dx.$$

Our goal is to get inner product relations along matrix coefficients.

Lemma 3.3

Let $\ell : V' \to V$ be a map of representations (V', ϕ') and (V, ϕ). Define the operator

$$L = \int_G \phi(x)\ell\phi'(x^{-1})\, dx.$$

Then $L : V' \to V$ is intertwining In particular,

- $L = 0$ if $V \ncong V'$ is irreducible.
- $L = \lambda\mathrm{id}$ if $V \cong V'$ is irreducible.

Proof. Schur's lemma. □

Theorem 3.4

Suppose V and V' are non-isomorphic irreps of G.

(a) The matrix coefficients are pairwise orthogonal in $C^\infty(G)$.

(b) Let $\phi : G \to \mathrm{GL}\, V$ be irreducible, then

$$\langle \phi_{u_1 v_1}, \phi_{u_2 v_2}\rangle = \frac{\langle u_1, v_1\rangle \overline{\langle u_2, v_2\rangle}}{\dim V}.$$

Proof. (a) Consider $u, v \in V$ and $u', v' \in V'$ arbitrary. We will show that ϕ_{uv} and $\phi_{u'v'}$ are orthogonal.

If $\ell : V' \to V$ by $w' \mapsto \langle w', u' \rangle u$, then we know that $L = 0$, meaning

$$\begin{aligned} 0 &= \left\langle Lv', v \right\rangle \\ &= \int_G \left\langle \phi(x)\ell(\phi'(x^{-1})v'), v \right\rangle \, dx \\ &= \int_G \left\langle \phi(x) \left\langle \phi'(x^{-1})v', u' \right\rangle u, v \right\rangle \, dx \\ &= \int_G \left\langle \phi'(x^{-1})v', u' \right\rangle \left\langle \phi(x)u, v \right\rangle \, dx \\ &= \int_G \left\langle \phi(x)u, v \right\rangle \overline{\left\langle u', \phi'(x^{-1})v' \right\rangle} \, dx \\ &= \int_G \left\langle \phi(x)u, v \right\rangle \overline{\left\langle \phi'(x)u', v' \right\rangle} \, dx \end{aligned}$$

the last step since the inner form is G-invariant.

(b) Again let $\ell : V \to V$ by $w \mapsto \langle w, u_2 \rangle u_1$. Note that u_1 is the only nonzero eigenvector of ℓ, and its eigenvalue is $\langle u_1, u_2 \rangle$. So $\operatorname{Tr} \ell = \langle u_1, u_2 \rangle$.

As $L : V \to V$ is a constant map $L = \lambda \mathrm{id}$, and we have $\operatorname{Tr} L = \operatorname{Tr} \ell = \langle u_1, u_2 \rangle$, it follows that L is multiplication by

$$\lambda = \frac{\langle u_1, u_2 \rangle}{\dim V}.$$

By the same argument in part (a) we have

$$\int_G \left\langle \phi(x)u_1, v_1 \right\rangle \overline{\left\langle \phi(x)u_2, v_2 \right\rangle} \, dx = \left\langle Lv_2, v_1 \right\rangle.$$

By the work above, we deduce

$$\left\langle Lv_2, v_1 \right\rangle = \frac{\langle u_1, u_2 \rangle}{\dim V} \left\langle v_2, v_1 \right\rangle = \frac{\langle u_1, u_2 \rangle}{\dim V} \overline{\langle v_1, v_2 \rangle}. \qquad \square$$

§§3.2 Bimodule

We defined a map $V \times V \to C^\infty(G)$ by $(v, w) \mapsto (g \mapsto (\phi(g)v, w)$. We can generalize this to a map

$$m : V^\vee \otimes V \to C^\infty(G) \qquad \text{by} \qquad \xi \otimes v \mapsto \xi\left(\rho(g)v\right).$$

Equivalently, define an "inner product" on $V^\vee \otimes V$ by $\langle \xi, v \rangle = \xi(v)$. Hence above map is $\xi \otimes v \mapsto \langle \xi, \rho(g)v \rangle$.

Then, we equip $V^\vee \otimes V$ with the inner product

$$\langle \xi \otimes v, \eta \otimes w \rangle = \frac{1}{\dim V} \langle \xi, \eta \rangle \langle v, w \rangle.$$

We can now endow it with the following G-bimodule structure:

- Left action: $\xi \otimes (\rho(g)v)$.
- Right action: $(\xi \circ \rho(g)) \otimes v$.

Meanwhile, we endow $C^\infty(G, \mathbb{C})$ with the following action:

- Left action: $(g \cdot f)(h) = f(g^{-1}h)$.
- Right action: $(f \cdot g)(h) = f(gh)$.

Definition 3.5. Let $\hat{G}$ be the set of irreps of G up to isomorphism.

Theorem 3.6

(1) m is a morphism of G-bimodules.

(2) m preserves inner products.

(3) m is injective.

Proof. (1) is computation, (2) follows from earlier theorem with some adjustment, and (3) follows by (2). □

Thus, m defines a mapp

$$m : \bigoplus_{V \in \hat{G}} V^\vee \otimes V \to L^2(G, dx).$$

The image of this map is dense, and in fact if we take the *completed direct sum* this map is an isomorphism (Peter-Weyl).

§§3.3 Characters

Let (V, ϕ) be a rep of G. Then the **character** of ϕ is

$$\chi_\phi(x) = \operatorname{Tr} \phi(x).$$

Properties:

1. The trivial representation has character identically 1.
2. $\chi_{V \oplus W} = \chi_V + \chi_W$.
3. $\chi_{V \otimes W} = \chi_V \chi_W$.
4. $\chi_V(ghg^{-1}) = \chi_V(h)$.
5. $\chi_{V^\vee} = \overline{\chi_V}$.

We can also use the inner product in $C^\infty(G)$.

Theorem 3.7 (Orthogonality of characters)

Let V and W be irreps of G. Then

$$\langle \chi_V, \chi_W \rangle = \begin{cases} 1 & V \cong W \\ 0 & V \not\cong W. \end{cases}$$

Thus $\{\chi_V \mid V \in \hat{G}\}$ is an orthonormal basis in the Hilbert space $L^2(G, dx)^G$.

Corollary 3.8

If V is a finite-dimensional representation of G, define

$$n_i = \langle \chi_V, \chi_{V_i} \rangle \quad V_i \in \hat{G}$$

then

$$V \cong \bigoplus_i V_i^{\oplus n_i}.$$

§4 February 11, 2016

Exercise:

- Chapter 3: 3.3, 3.6, 3.8, 3.13, 3.18
- Chapter 4: 4.11, 4.2, 4.12

§§4.1 Left/right action

The issue last time is that $R_g R_h = R_{gh}$.

- Left action: $L_g(\xi \otimes v) = g\xi \otimes v$.
- Right action: $R_g(\xi \otimes v) = \xi \otimes gv$.
- Left action: $L_g f(x) = f(g^{-1}x)$.
- Right action: $R_g f(x) = f(xg)$.

§§4.2 Lie algebras

A **Lie algebra** $\mathfrak{g}$ is a K-vector space ($K \in \mathbb{R}, \mathbb{C}$) equipped with a Lie bracket $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.

Remark 4.1. We can make every associate algebra A with unit and construct the bracket $[,] : A \times A \to A$ on it.

A **representation of a Lie algebra** $\mathfrak{g}$ is a vector space V and a map of Lie algebras $\mathfrak{g} \to \mathrm{End}(V)$. (Thus V is a $\mathfrak{g}$-module.)

§§4.3 Complexification

Given real $\mathfrak{g}$ we can build a **complexification** $\mathfrak{g}_{\mathbb{C}} \overset{\text{def}}{=} \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. Building this same complexification on groups is harder.

The beauty is that

Theorem 4.2

For G a simply connected Lie group, the category of finite dimensional representations of G coincides with the category of finite dimensional $\mathfrak{g}$-modules.

Example 4.3

(a) If $G = S^1$, then $\mathfrak{g} = \mathbb{R}$ with bracket zero.

(b) Classical groups have Lie algebras. For example, $\mathrm{GL}_n(\mathbb{C})$ gives $\mathfrak{gl}_n(\mathbb{C})$.

(c) $\mathfrak{sl}_n(\mathbb{C})$ is the traceless $n \times n$ matrices.

Let

$$\mathrm{U}(n) = \left\{ M \in \mathrm{GL}_n \mathbb{C} \mid MM^\dagger = \mathrm{id} \right\}$$

be the unitary group. Then

$$\mathfrak{u}(n) = \left\{ M \in \mathfrak{gl}_n \mathbb{C} \mid M + M^\dagger = 0 \right\}.$$

Now, using the fact that any matrix can be written as a sum of a Skew-Hermitian and Hermitian matrix, we get

$$\mathfrak{u}(n)_\mathbb{C} = \mathfrak{u}(n) \oplus i\,\mathfrak{u}(n) \simeq \mathfrak{gl}_n \mathbb{C}.$$

Similarly, if $\mathrm{SU}(n)$ the special unitary group, then $\mathfrak{su}(n)$ is a real Lie algebra, and its complexification is $\mathfrak{sl}_n(\mathbb{C})$.

Lemma 4.4

Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}_\mathbb{C}$ its complexification. Then the categories of $\mathfrak{g}$-modules and $\mathfrak{g}_\mathbb{C}$-modules coincide.

Since $\mathrm{SU}(2)$ and $\mathrm{SL}_2\,\mathbb{C}$ are simply connected, they have the same category of representations as their Lie algebras by the theorem. Since we saw above that $\mathfrak{su}(2)$ has complexification $\mathfrak{sl}_2\,\mathbb{C}$, it follows that

> The categories of representations of each of $\mathrm{SU}(2)$, $\mathfrak{su}(2)$, $\mathfrak{sl}_2\,\mathbb{C}$, $\mathrm{SL}_2(\mathbb{C})$ are all the same.

In particular, since $\mathrm{SU}(2)$ is compact each of these four groups/algebras is completely reducible.

§§4.4 Representations of $\mathfrak{sl}_2\,\mathbb{C}$

Holy cow I am *not* doing this a second time. (Etingof's problem 1.55.) Let $\mathfrak{sl}_2(\mathbb{C})$ have basis E, F, H as usual.

Theorem 4.5 (All Irreps of $\mathfrak{sl}_2\,\mathbb{C}$)

For $N \geq 0$, there is a unique $(N+1)$ dimensional irrep written as

$$E = \begin{pmatrix} 0 & N & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & N-1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & N-2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 2 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 3 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & N-1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & N & 0 \end{pmatrix}$$

$$H = \begin{pmatrix} N & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & N-2 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & N-4 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -N+4 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & -N+2 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & -N \end{pmatrix}$$

So these V_n are pairwise isomorphic and the only irreducible representations. Moreover, by comments above $\mathfrak{sl}_2 \mathbb{C}$ is semisimple, hence all finite-dimensional representations of $\mathfrak{sl}_2 \mathbb{C}$ are sums of these guys.

The notation used by Kirillov is: let $V[\lambda]$ be the λ-eigenspace of V. In particular, as vector spaces any representation V decomposes as $\bigoplus_n V[n]$ (i.e. one can pick a basis of integer-valued eigenvectors).

We now extend this by formally defining a vector space M_λ with infinite basis v^0, v^1, ... We view it as an $\mathfrak{sl}_2$-module with action

$$\begin{aligned}
hv^k &= (\lambda - 2k)v^k \\
fv^k &= (k+1)v^{k+1} \\
ev^k &= (\lambda - k + 1)v^{k-1} \quad k > 0 \\
ev^0 &= 0
\end{aligned}$$

Lemma 4.6

Suppose V is a finite-dimensional irreducible $\mathfrak{sl}_2$-module with nonzero highest weight vector of weight λ. Then there is a surjective map

$$M_\lambda \twoheadrightarrow V.$$

Thus $V \simeq M_\lambda / W$.

Question 4.7. What can you say about the weights of the kernel?

§5 February 18, 2016

Let $\mathfrak{g}$ be a Lie algebra over K, Our goal is to lift $\mathfrak{g}$-representations to representations of a certain associative algebra.

Definition 5.1. The **universal enveloping algebra** of $\mathfrak{g}$, denoted $U\mathfrak{g}$, is generated by the tensor algebra

$$T\mathfrak{g} = \bigoplus_{n \ge 0} g^{\otimes n}$$

(whose algebra multiplication is $\otimes$) modulo the relations (a two-sided ideal)

$$x \otimes y - y \otimes x - [x, y].$$

As we all know taking modulo relations makes it hard to describe $U\mathfrak{g}$ concretely, but we want $U\mathfrak{g}$ to be interesting anyways. In particular, we would like the canonical map

$$\mathfrak{g} \hookrightarrow U\mathfrak{g}$$

to be injective.

§§5.1 An algebraic lemma

Fix an ordered basis $\{X_i\}_{i \in A}$. For a sorted sequence $I = (i_1, \dots, i_p)$ of elements of A, write $i \le I$ if $i \le \min\{i_j\}$.

Consider the symmetric algebra

$$S\mathfrak{g} = K[z_i]$$

where the z_i run through $i \in A$ and commute. We can consider it as filtered by degree in the obvious way, giving

$$S_0 \subseteq S_1 \subseteq S_2 \subseteq \dots$$

where S_p is the span of monomials with $\le p$ terms in it. (This is a little weaker than grading.)

Proposition 5.2 (Hard Part of Poincaré–Birkoff–Witt)

There is a representation of $\mathfrak{g}$ on $S\mathfrak{g}$, say $\pi : \mathfrak{g} \to \mathfrak{gl}(S\mathfrak{g})$, such that

(1) $\pi(x_i) z_I = z_i z_I$ for $i \le I$.

(2) $\pi(x_i) z_I - z_i z_I \in S_p$ for all I with $z_I \in S_p$.

(3) $\pi(x_i) \left(\pi(x_j) z_J\right) = \left(\pi(x_j)\pi(x_i) z_J\right) + \pi[x_i, x_j] z_J$.

Proof. By induction: we define $\pi(x_i)$ on S_p for p inductively, giving π on S.

If $p = 0$, then $\pi(x_i) 1 = z_i$.

Inductively, for $i \le I$ we set

$$\pi(x_i) z_I = z_i z_I$$

and otherwise if $j < i$ with $I = (j, J)$ then

$$\pi(x_i) z_I = \pi(x_i)(z_i z_J) = \pi(x_i)\pi(x_j) z_J = \pi(x_j)\pi(x_i) z_J + \pi[x_i, x_j] z_J.$$

Now, take about four blackboards worth of heavy calculation to verify this works. $\square$

§§5.2 Universal enveloping algebra

Theorem 5.3 (Universality of $U\mathfrak{g}$)

$U\mathfrak{g}$ satisfies the following universal property: for any map $\rho : \mathfrak{g} \to A$ which satisfies $\rho(x)\rho(y) - \rho(y)\rho(x) = \rho([x, y])$ there exists a unique algebra homomorphism $\widetilde{\rho} : U\mathfrak{g} \to A$ which makes the diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\rho"] \arrow[dr, "\widetilde{\rho}"'] & U\mathfrak{g} \arrow[d, hook] \\ & A \end{tikzcd}$$

commute.

Proof. Immediate. □

Theorem 5.4 (Poincaré–Birkoff–Witt)

fix $\{X_i\}_{i \in A}$ an ordered basis of $\mathfrak{g}$. Then the set of all monomials

$$i(X_{i_1}^{j_1}) i(X_{i_2}^{j_2}) \dots i(X_{i_n}^{j_n}) \qquad i_1 < \dots < i_n, \text{ and } j_k \geq 0$$

is a basis of $U\mathfrak{g}$.

To do this we first apply the following lemma.

Lemma 5.5 (Re-ordering indices)

If $z_1, \dots, z_p \in \mathfrak{g}$ and σ is a permutation of $\{1, \dots, p\}$ then $z_1 \dots z_p - z_{\sigma(1)} \dots z_{\sigma(p)} \in U_{p-1}\mathfrak{g}$.

Proof. Easy. Any permutation is a product of simple transpositions. □

Proof of PBW Theorem. It's easy to see that the image is spanning, since All monomials span $T\mathfrak{g}$, so they also span $U\mathfrak{g}$ (the previous lemma shows that "increasing" monomials are sufficient).

Now, we have to show it's a basis. We take our action $\mathfrak{g} \to \mathfrak{gl}(S\mathfrak{g})$ that we worked very hard to prove. Thus, we can lift it to a representation $\pi : U\mathfrak{g} \to \mathrm{GL}(S\mathfrak{g})$, i.e. we now see $S\mathfrak{g}$ is a $U\mathfrak{g}$-module. Now, by condition

$$\pi(x_{i_1} \dots x_{i_p}) \cdot 1_{S\mathfrak{g}} = z_{i_1} \dots z_{i_p} \in S\mathfrak{g}$$

and these are linearly independent in $S\mathfrak{g}$, so they must be linearly independent in $U\mathfrak{g}$ as well. □

§§5.3 Examples

Example 5.6

Let $G = (\mathbb{R}, +)$. This gives the Lie algebra $\mathfrak{g} = (\mathbb{R}, [,] = 0)$. We complexify this to get $\mathfrak{g}_\mathbb{C} = (\mathbb{C}, [,] = 0)$ and then we can put

$$U(\mathbb{C}, [,] = 0) = S(\mathbb{C}) \cong \mathbb{C}[x].$$

Now, consider $\mathbb{C}_{\text{triv}}$. We compute $\operatorname{Ext}^1(\mathbb{C}_{\text{triv}}, \mathbb{C}_{\text{triv}})$. Take the following projective resolution, call it $P^\bullet$, given by the diagram

$$\begin{tikzcd} 0 \arrow[d] & \\ \mathbb{C}[x] \arrow[d, "\cdot x"] & \\ \mathbb{C}[x] \arrow[r] \arrow[d] & \mathbb{C}_{\text{triv}} \cong \mathbb{C}[x]/(x) \\ 0 & \end{tikzcd}$$

and dualize it by $\operatorname{Hom}(-, \mathbb{C}_{\text{triv}})$ to get

$$0 \longrightarrow \operatorname{Hom}(\mathbb{C}[x], \mathbb{C}_{\text{triv}}) \simeq \mathbb{C} \xrightarrow{0} \operatorname{Hom}(\mathbb{C}[x], \mathbb{C}_{\text{triv}}) \simeq \mathbb{C} \longrightarrow 0 \ .$$

Thus, $\operatorname{Ext}^1(\mathbb{C}_{\text{triv}}, \mathbb{C}_{\text{triv}}) = \mathbb{C}$.

Exercise 5.7. Compute $\operatorname{Ext}^1(\mathbb{C}_a, \mathbb{C}_b)$ where $\mathbb{C}_{x=a} \leftrightarrow \mathbb{C}[x]/(x-a)$.

Exercise 5.8. We have $\operatorname{Hom}_{\mathbb{C}[x]}(P^\bullet, P^\bullet)$ is a *dg*-algebra. What can you say about it? Can you write down any *dg*-modules?

Exercise 5.9. We have the cohomology $H^\bullet(\operatorname{Hom}(P^\bullet, P^\bullet)) \cong \operatorname{Ext}^\bullet(\mathbb{C}_{\text{triv}}, \mathbb{C}_{\text{triv}})$. Write down some simple modules.

§6 February 23, 2016

Side remark: it *should* be possible to distinguish the categories of representations of D_8 and Q_8 using $\otimes$-structure.

In this lecture, the group G is finite.

§§6.1 Hopf algebras

Recall that an associative k-algebra is a k-vector space A equipped with a map $m : A \otimes A \to A$ and $i : k \hookrightarrow A$ (unit).

Then a **k-coalgebra** is a map

$$\Delta : A \to A \otimes A \qquad \varepsilon : A \to k$$

called comultiplication and counit respectively. See `https://en.wikipedia.org/wiki/Coalgebra`.

Now a **Hopf algebra** A is a bialgebra A over k equipped with a so-called **antipode** $S : A \to A$. We require that the diagram

$$\begin{tikzcd}
A \otimes A \arrow[rr, "S \otimes \mathrm{id}_A"] & & A \otimes A \arrow[d, "m"] \\
A \arrow[u, "\Delta"] \arrow[d, "\Delta"'] \arrow[r, "\varepsilon"] & k \arrow[r, "i"] & A \\
A \otimes A \arrow[rr, "\mathrm{id}_A \otimes S"] & & A \otimes A \arrow[u, "m"']
\end{tikzcd}$$

commutes.

A **group-like** element in A is an element of

$$G = \{x \in A \mid \Delta(x) = x \otimes x\}.$$

Exercise 6.1. Show that G is a group with multiplication m and inversion S.

Example 6.2 (Group algebra is Hopf algebra)

The group algebra $k[G]$ is a Hopf algebra with

- m, i as expected.
- ε the counit is the trivial representation.
- Δ comes form $g \mapsto g \otimes g$ extended linearly.
- S takes $g \mapsto g^{-1}$ extended linearly.

Theorem 6.3

The group-like elements are precisely the basis elements $1_k \cdot g \in k[g]$.

Proof. Assume $V = \sum_{g\in G} a_g g$ is grouplike. Then by assumption we should have

$$\sum_{g\in G} a_g(g\otimes g) = \Delta(v) = \sum_{g\in G}\sum_{h\in G} a_g a_h (g\otimes h).$$

Comparing each coefficient, we get that

$$a_g a_h = \begin{cases} a_g & g = h \\ 0 & \text{otherwise.} \end{cases}$$

This can only occur if some a_g is 1 and the remaining coefficients are all zero. $\square$

§§6.2 Monoidal functors

A **monoidal category** or tensor category is a category $\mathscr{C}$ equipped with a functor $\otimes : \mathscr{C}\times\mathscr{C}\to\mathscr{C}$ which is associative up to a natural isomorphism and has an element which is a left/right identity, also up to natural isomorphism. Thus for any $A, B, C \in \mathscr{C}$ we have a natural isomorphism

$$A\otimes(B\otimes C) \xrightarrow{a_{A,B,C}} (A\otimes B)\otimes C.$$

Now suppose we have two categories $(\mathscr{C}, \otimes_{\mathscr{C}})$ and $(\mathscr{D}, \otimes_D)$. Then a **monoidal functor** $F : \mathscr{C} \to \mathscr{D}$ is a functor for which we additionally need to select an isomorphism

$$F(A\otimes B) \xrightarrow{t_{A,B}} F(A)\otimes F(B).$$

We then require that the diagram

$$\begin{tikzcd}
F(A\otimes(B\otimes C)) \arrow[r, "t_{A,B\otimes C}"] \arrow[d, "F(a_{A,B,C})"] & F(A)\otimes F(B\otimes C) \arrow[d, "\mathrm{id}\otimes t_{B,C}"] \\
F((A\otimes B)\otimes C) \arrow[d, "t_{A\otimes B,C}"] & F(A)\otimes(FB\otimes FC) \\
F(A\otimes B)\otimes FC \arrow[r, "t_{A,B}\otimes\mathrm{id}"'] & (FA\otimes FB)\otimes FC \arrow[u, "a_{FA,FB,FC}"]
\end{tikzcd}$$

commutes, plus some additional compatibility conditions with the identity of the $\otimes$'s.

We also have a notion of a natural transformation of two functors $t : F \to G$, i.e. making the squares

$$\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, "t_A"] & FB \arrow[d, "t_B"] \\
GA \arrow[r, "Gf"] & GB
\end{tikzcd}$$

commute. Now, suppose $F : \mathscr{C}\to\mathscr{C}$ is a monoidal functor. Then an automorphism is a natural isomorphism $t : F \to F$.

Remark 6.4. Categorical remark: when using abstraction we use equivalence of categories which lets us see similarities between different areas. This is weaker than the very useless notion of an isomorphism of categories.

§§6.3 Application to $k[G]$

Consider the category of $k[G]$ modules endowed with the monoidal $\otimes$ (which is just the $\otimes_k$ representation). We want to reconstruct G from its representations.

Let U be the forgetful functor

$$U : \mathsf{Mod}_{k[G]} \to \mathsf{Vect}_k.$$

It's easy to see this is in fact an monoidal functor.

Now we claim the following:

Theorem 6.5 (G is isomorphic to $\mathrm{Aut}^{\otimes}(U)$)

Consider the map

$$i : G \to \mathrm{Aut}^{\otimes}(U) \quad \text{by} \quad g \mapsto T^g.$$

Here, the natural transformation T^g is defined by the components

$$T^g_{(V,\phi)} : (V,\phi) \to U(V,\phi) = V \quad \text{by} \quad v \mapsto \phi(g)v.$$

Then i is an isomorphism of groups.

In particular, using only $\otimes$ structure this exhibits an isomorphism $G \cong \mathrm{Aut}^{\otimes}(U)$. Consequently this solves the problem proposed at the beginning of the lecture.

Proof. It's easy to see i is a group homomorphism.

To see it's injective, we show $1_G \neq g \in G$ gives T^g isn't the identity automorphism. i.e. we need to find some representation for which g acts nontrivially on V. Now just take the regular representation, which is faithful!

The hard part is showing that it's surjective. For this we want to reduce it to the regular representation.

Lemma 6.6

Any $T \in \mathrm{Aut}^{\otimes}(U)$ is completely determined by $T_{k[G]}(1_{k[G]}) \in k[G]$.

Proof. Let (V,ϕ) be a representation of G. Then for all $v \in V$, we have a unique morphism of representations

$$f_v : k[G] \to (V,\phi) \quad \text{by} \quad 1_{k[G]} \mapsto v.$$

If we apply the forgetful functor to this, we have a diagram

$$\begin{tikzcd} k[G] \arrow[r, "f_V"] \arrow[d, "T_{k[G]}"] & U(V,\phi) \arrow[d, "T_{(V,\phi)}"] \\ k[G] \arrow[r, "f_V"'] & V \end{tikzcd} \qquad \begin{tikzcd} 1_{k[G]} \arrow[r, mapsto, "f_V"] \arrow[d, mapsto, "T_{k[G]}"] & v \arrow[d, mapsto, "T_{(V,\phi)}"] \\ T_{k[G]}(1_{k[G]}) \arrow[r, mapsto, "f_V"'] & T_{(v,\phi)}(v) \end{tikzcd}$$

∎

Next, we claim

Lemma 6.7

$T_{k[G]}(1_{k[G]})$ is a grouplike element of $k[G]$.

Proof. Draw the diagram

$$\begin{tikzcd} k[G] \arrow[r, "\Delta"] \arrow[d, "T_{k[G]}"] & k[G]\otimes k[G] \arrow[r, equal] \arrow[d, "T_{k[G]\otimes k[G]}"] & k[G]\otimes k[G] \arrow[d, "T_{k[G]\otimes k[G]}"] \\ k[G] \arrow[r, "\Delta"] & k[G]\otimes k[G] \arrow[r, equal] & k[G]\otimes k[G] \end{tikzcd}$$

and note that it implies

$$\Delta(T_{k[G]}(1_{k[G]})) = T_{k[G]}(1_{k[G]}) \otimes T_{k[G]}(1_{k[G]}). \qquad \blacksquare$$

This implies surjectivity, by our earlier observation that grouplike elements in $k[G]$ are exactly the elements of G. $\square$

Question: does this generalize to e.g. compact Lie groups?

Exercise 6.8. Make $U\mathfrak{g}$ into a Hopf algebra with comultiplication

$$\Delta(x) = x \otimes 1 + 1 \otimes x \qquad x \in \mathfrak{g}.$$

§§6.4 Centers

Definition 6.9. If M is a $\mathfrak{g}$-module, then the **$\mathfrak{g}$-invariants in M** are the elements of

$$M^{\mathfrak{g}} = \{m \in M \mid g \cdot m = 0 \qquad g \in \mathfrak{g}\}.$$

Define the adjoint action as follows: for $x \in \mathfrak{g}$ and $a \in U\mathfrak{g}$, we define the action

$$\mathrm{ad}_x(a) = xa - ax.$$

Then, we can consider U as a $\mathfrak{g}$-algebra with this action.

Proposition 6.10

$Z(U\mathfrak{g}) = (U\mathfrak{g})^{\mathfrak{g}}$.

Proof. Note $C \in Z(U\mathfrak{g})$ is equivalent to $Cx = xC$ for all $x \in \mathfrak{g}$; equivalently $\mathrm{ad}_x(C) = 0$ for all $x \in \mathfrak{g}$, meaning $C \in (U\mathfrak{g})^{\mathfrak{g}}$. $\square$

Next time: we prove for "semisimple" Lie algebras, the corresponding representations are Lie algebras too. We have already seen a counter-example: if $\mathfrak{g} = (\mathbb{C}, [,] = 0)$ then there is a two-dimensional representation

$$\mathfrak{g} \to \mathfrak{gl}(V) \quad g \mapsto \begin{pmatrix} 0 & g \\ & 0 \end{pmatrix}$$

which is not semisimple. The issue "is" upper triangularity.

On the other hand, consider the Lie subalgebra

$$\mathfrak{b} = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & -\alpha \end{pmatrix} \mid \alpha, \beta \in \mathbb{C} \right\}.$$

It is a Lie subalgebra of $\mathfrak{sl}_2\,\mathbb{C}$, meaning $[\mathfrak{b}, \mathfrak{b}] \subseteq \mathfrak{b}$ and $\mathfrak{b}$ is a vector subspace. This is an example of a solvable Lie algebra:

Definition 6.11. A Lie algebra $\mathfrak{g}$ is **solvable** if the derived series

$$\begin{aligned} D^0\mathfrak{g} &= \mathfrak{g} \\ D^{i+1}\mathfrak{g} &= [D^i\mathfrak{g}, D^i\mathfrak{g}] \qquad i \geq 0 \end{aligned}$$

eventually terminates.

This means that the quotients D^{i+1}/D^i in the derived series are abelian Lie algebras.

§7 February 25, 2016

§§7.1 Brief remark on monoidal categories

We already defined a monoidal category: more fully it is a tuple $(\mathscr{C}, \otimes, a, \mathbf{1}, \iota)$ and a map $\iota : \mathbf{1} \otimes \mathbf{1} \to \mathbf{1}$. These have to satisfy some axioms.

Then a tensor category additionally requires $\mathscr{C}$ abelian, plus the $\otimes$ should be "rigid": every object needs a right dual X^* giving maps

$$X^* \otimes X \xrightarrow{\text{ev}} \mathbf{1} \xrightarrow{\text{coev}} X \otimes X^*$$

and a left dual ${}^*X$.

From this one can deduce that $- \otimes X$ has both a left and right adjoint, hence $\otimes$ has to be exact.

§§7.2 Ideals

Recall $\mathfrak{sl}(2)$ has a Lie subalgebra

$$\mathfrak{b} = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & -\alpha \end{pmatrix} \mid \alpha, \beta \in \mathbb{C} \right\}$$

and we have

$$\mathfrak{b} \supset [\mathfrak{b}, \mathfrak{b}] = \left\{ \begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix} \right\} \overset{\text{def}}{=} \mathfrak{n} \supset 0.$$

We see that $\mathfrak{n}$ is an ideal of $\mathfrak{b}$, in this sense:

Definition 7.1. Let $\mathfrak{h}$ be a sub Lie algebra of $\mathfrak{g}$. We say $\mathfrak{h}$ is an ideal if $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$.

Then, $\mathfrak{g}/\mathfrak{h}$ makes sense as a Lie algebra.

(This is symmetric, so no distinction between left or right ideals.)

We have the "first isomorphism theorem":

Proposition 7.2

If $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a morphism of Lie algebras:

- $\ker f$ is an ideal of $\mathfrak{g}_1$.
- $\operatorname{im} f$ is a subalgebra in $\mathfrak{g}_2$.
- $\operatorname{im} f = \mathfrak{g}_1 / \ker f$.

§§7.3 Solvable and nilpotent algebras

Definition 7.3. The Lie algebra $\mathfrak{g}$ is **solvable** if the **derived series**

$$D^0\mathfrak{g} = \mathfrak{g}, \qquad D^i\mathfrak{g} = [D^{i-1}\mathfrak{g}, D^{i-1}\mathfrak{g}]$$

eventually terminates.

Definition 7.4. The Lie algebra $\mathfrak{g}$ is **nilpotent** if the **lower central series**

$$D_0\mathfrak{g} = \mathfrak{g}, \qquad D_i\mathfrak{g} = [\mathfrak{g}, D_{i-1}\mathfrak{g}]$$

eventually terminates.

Lemma 7.5

Nilpotent implies solvable.

Proof. We observe by induction that $D^i\mathfrak{g} \subseteq D_i\mathfrak{g}$ for all i. Thus if $D_k\mathfrak{g} = 0$ for $k \gg 0$ it follows that $D^k\mathfrak{g} = 0$ for $k \gg 0$. $\square$

Lemma 7.6

$\mathfrak{g}$ is solvable if and only if $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

Proof. We prove only that $[\mathfrak{g}, \mathfrak{g}]$ nilpotent implies $\mathfrak{g}$ solvable. We of course know $[\mathfrak{g}, \mathfrak{g}]$ is solvable, and also $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is abelian (hence solvable).

Then we appeal to the fact that if we have a short exact sequence

$$0 \to I \to \mathfrak{g} \to \mathfrak{g}/I \to 0$$

and I, $\mathfrak{g}/I$ are solvable, so is $\mathfrak{g}$. $\square$

Definition 7.7. A Lie algebra $\mathfrak{g}$ is called

- **semisimple** if there are no nonzero solvable ideals.
- **simple** if $\mathfrak{g}$ is *not* abelian and there are no nontrivial ideals.

Proposition 7.8

$\mathfrak{g}$ simple implies $\mathfrak{g}$ semisimple.

Proof. Consider the ideal $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}$. Since $\mathfrak{g}$ is simple and not abelian, this forces $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

So $\mathfrak{g}$ is not a solvable ideal. Since $\mathfrak{g}$ is also the *only* ideal by assumption, this implies $\mathfrak{g}$ is semisimple. $\square$

Example 7.9

$\mathfrak{sl}(2, \mathbb{C})$ is simple. To see this let e, f, h be the usual basis. Then $\operatorname{ad} h = [h, -] : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{sl}(2, \mathbb{C})$ has the eigenvalues

- 2 with eigenvector e
- -2 with eigenvector f
- 0 with eigenvector h.

In particular, $\operatorname{ad}_h(I)$ is diagonalizable. Then given any ideal I, we have $\operatorname{ad}_h(I) \subseteq I$, by definition.

Thus, any I needs to be spanned by a subset of $\{e, f, h\}$ (invariant subspace of a diagonal matrix).

Then we can check that the presence of *any* of these implies the presence of the other three. (First, $[e, f] = -[f, e] = h$ so if either $e, f \in I$ then $h \in I$. Also, $h \in I$ implies $[h, e] = 2e$ and $[h, f] = -2f$ are in I.)

Exercise 7.10. Convince yourself this works for $\mathfrak{sl}(2,k)$ if k is an algebraically closed field of characteristic other than 2.

On the other hand, show the span of h in $\mathfrak{sl}(2,\overline{\mathbb{F}_2})$ is a proper ideal. So $\mathfrak{sl}(2,\overline{\mathbb{F}_2})$ is not simple.

Example 7.11

Let $\mathfrak{b} = \mathfrak{gl}(n,k)$ be the sub algebra of upper triangular matrices and $\mathfrak{n}$ the strictly upper triangular matrices. We claim that $\mathfrak{b}$ is solvable and $\mathfrak{n}$ is nilpotent.

Definition 7.12. A **flag** on a finite dimensional vector space V is a sequence

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V.$$

Define for a flag $\mathscr{F}$ the submatrices

$$\begin{aligned} \mathfrak{b}\mathscr{F} &= \{x \in \mathfrak{gl}\, V \mid xV_i \subseteq V_i\} \\ \mathfrak{n}\mathscr{F} &= \{x \in \mathfrak{gl}\, V \mid xV_i \subseteq V_{i-1}\} \end{aligned}$$

The **standard flag** is the obvious flag $\mathscr{F}^{\text{std}}$ on the standard basis.

Let $a_k(\mathscr{F}) = \{x \in \mathfrak{gl}\, V \mid xV_i \subseteq V_{i-k}\}$. Hence $\mathfrak{b}(\mathscr{F}) = a_0(\mathscr{F})$ and $\mathfrak{n}(\mathscr{F}) = a_1(\mathscr{F})$. Observe that if $x \in a_k(\mathscr{F})$ and $y \in a_\ell(\mathscr{F})$ then the standard product xy lies in $a_{k+\ell}(\mathscr{F})$. In particular,

$$[a_k, a_\ell] \subseteq a_{k+\ell}.$$

In particular, $D_i\mathfrak{n} \subseteq a_{i+1}$ by induction.

Note also that $a_k(\mathscr{F}) = 0$ for all $k \gg 0$. In particular, $\mathfrak{n}(\mathscr{F})$ is nilpotent, and thus this proves $\mathfrak{n}(\mathscr{F}^{\text{std}}) = \mathfrak{n}$ is nilpotent.

On the other hand, it is not true that $\mathfrak{b}(\mathscr{F})$ is solvable for any flag $\mathscr{F}$. However, it is with the assumption:

Definition 7.13. A flag is **complete** if it has maximal length (i.e. $\dim V_{i+1} - \dim V_i = 1$).

We'll show it just for $\mathfrak{b}(\mathscr{F}^{\text{std}})$.

Claim 7.14. $[\mathfrak{b},\mathfrak{b}] = \mathfrak{n}$. In other words, a matrix M is strictly upper triangular if and only if it can be written as $M_1M_2 - M_2M_1$ for upper triangular matrices M_1, M_2.

Proof. First, we claim $[\mathfrak{b},\mathfrak{b}] \subseteq \mathfrak{n}$. This is because $[x,y] \in \mathfrak{n} = a_1$ for $x, y \in \mathscr{F}^{\text{std}}$, just by looking at the diagonal entries of the multiplication (as $(xy)_{ii} = (yx)_{ii}$).

To show the reverse inclusion, we compute directly with a basis. $\square$

Consequently,

$$[\mathfrak{b},\mathfrak{b}] = \mathfrak{n} = a_1$$

is nilpotent, so $\mathfrak{b}$ is solvable by the lemma.

On the other hand, we would like to show $\mathfrak{b}$ is not nilpotent. Compute

$$D_1\mathfrak{b} = [\mathfrak{b},\mathfrak{b}] = \mathfrak{n} \quad \text{then} \quad D_2\mathfrak{b} = [\mathfrak{b}, D_1\mathfrak{b}] = [\mathfrak{b},\mathfrak{n}]. = \mathfrak{n}$$

and hence $D_i\mathfrak{b} = \mathfrak{n}$ for all $i \geq 2$.

§§7.4 Representation theory of solvable Lie algebras

We will show that a solvable Lie algebra has only one-dimensional irreps.

Proposition 7.15

Let $\mathfrak{g}$ be solvable and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ a representation. Then there exists a common eigenvector of $\rho(x)$ across all $x \in \mathfrak{g}$.

Proof. By induction on $\dim \mathfrak{g}$. Since $\mathfrak{g}$ is solvable, we have

$$\mathfrak{g} \subsetneq [\mathfrak{g}, \mathfrak{g}].$$

Let $\mathfrak{g}'$ be any codimension 1 subspace of $\mathfrak{g}$ containing $[\mathfrak{g}, \mathfrak{g}]$; thus $\mathfrak{g}'$ is an ideal (and in particular a subalgebra); hence solvable.

Then as vector spaces we have

$$\mathfrak{g} = \mathfrak{g}' \oplus \mathbb{C}x$$

for some $0 \neq x \in \mathfrak{g}$. Now by induction hypothesis there is $v \in V$ where $\rho(h) = \lambda(h)v$ for all $v \in \mathfrak{g}'$.

Let W denote the span of $v^0 = v$, $v^1 = \rho(x)v$, $v^2 = \rho(x)\rho(x)v$, ...

Claim 7.16. This is a $\mathfrak{g}$-subrepresentation.

Proof. Just need to check it's preserved by $\mathfrak{g}'$, because preservation under x by construction. We do this by induction on k (with $k = 0$ being by hypothesis). Then

$$hv^k = hxv^{k-1} = xhv^{k-1} + [h, x]v^{k-1}$$

and $[h, x] \in \mathfrak{g}'$ since $\mathfrak{g}'$ is an ideal.

In fact, we actually have

$$hv^k = \lambda(h)v^k + \lambda([h, k])v^{k-1}. \qquad \blacksquare$$

To complete the proof of proposition, consider a minimal n such that v^{n+1} is in the span of v^0, v^1, ..., v^n. Then, $\rho(h)$ acting on $v^0, v^1, \dots, v^n$ is an upper triangular matrix with $\lambda(h)$ on the diagonal, hence

$$\mathrm{Tr}_W(\rho(h)) = (n + 1)\rho(h).$$

But

$$0 = \mathrm{Tr}_W\,[\rho(x), \rho(h)] = \mathrm{Tr}_W\,\rho([x, h]) \implies \lambda[(x, h)] = 0.$$

Now any $w \in W$ is a common eigenvector of $\mathfrak{g}'$, so we can choose a $w \in W$ which is an eigenvector for x (which requires the fact that this is a complex representation for an eigenvector to exist). $\square$

Corollary 7.17

Any irreducible complex representation of a solvable Lie algebra is one-dimensional.

Thus

- Studying all representations of **solvable** Lie algebras is hard, but the irreducible ones are all one-dimensional.
- Studying representations of **semisimple** Lie algebras reduces to the study of just irreps, but the irreps in this case are much more complicated.

§8 March 1, 2016

Last time we saw that if $\mathfrak{g}$ is solvable, then any representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ has a common eigenvector for all of $\mathfrak{g}$. (Hence the irreducibles are one-dimensional, but $\mathfrak{g}$ might be indecomposable.)

§§8.1 Lie's theorem and Engel's theorem

Corollary of this:

Theorem 8.1 (Lie's theorem)

If $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a complex representation of solvable $\mathfrak{g}$ then there exists a basis of V such that $\rho(x)$ is upper triangular in this basis (for all $x \in \mathfrak{g}$).

Proof. There exists a common eigenvector $v \in V$. Consider $V/\mathbb{C}v$, a quotient representation. Induct down. □

Remark 8.2. If $\mathfrak{g}$ is nilpotent, it's solvable. (However, we can't improve this to get zeros on the main diagonal; take the one-dimensional trivial representation.)

Next best thing:

Theorem 8.3

If V is finite-dimensional representation over $\mathbb{C}$ and $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ is a Lie algebra such that $\rho(x)$ is nilpotent on V for all x, then we can find a basis in which each $\rho(x)$ is strictly upper triangular.

We now use this to prove:

Definition 8.4. An element $x \in \mathfrak{g}$ is called **ad-nilpotent** / **ad-semisimple** if $\operatorname{ad}_x : \mathfrak{g} \to \mathfrak{g}$ is nilpotent / semisimple.

Theorem 8.5 (Engel's theorem)

A Lie algebra $\mathfrak{g}$ is nilpotent if and only if all $x \in \mathfrak{g}$ are ad-nilpotent.

Proof. If $\mathfrak{g}$ is nilpotent, then $D_n\mathfrak{g} = 0$ for some n, which certainly implies $(\operatorname{ad}_x)^n = [x, [x, [\dots, [x, y]]]] = 0$.

Conversely, if ad_x is always nilpotent, by preceding theorem we can pick a *single* basis so that ad_x is strictly upper triangular in every basis. Hence we can build a flag

$$0 \subset \mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \dots \subset \mathfrak{g}_n = \mathfrak{g}$$

for which $\operatorname{ad}_x \mathfrak{g}_i \subseteq \mathfrak{g}_{i-1}$ for every x, i.

Thus each $\mathfrak{g}_i$ is an ideal. We claim $D_{n-i}\mathfrak{g} \subseteq \mathfrak{g}_i$. For $i = n$ it's clear; inductively if $n - i > m$ then

$$D_{n-i} = [\mathfrak{g}, D_{n-i-1}\mathfrak{g}] \subseteq [\mathfrak{g}, \mathfrak{g}_{i+1}] \subseteq \mathfrak{g}_i.$$

Hence $D_m\mathfrak{g} = 0$for $M \gg 0$, as desired. □

§§8.2 Jordan decomposition

Definition 8.6. A linear operator $A : V \to V$ is

- **nilpotent** if $A^n = 0$ for $n \gg 0$.
- **semisimple** if for all invariant $W \subseteq V$, there exists an invariant complement.

Let $A : V \to V$, where V is over an algebraically closed field. Then we have the following facts.

(1) The map A is semisimple if and only if A is diagonalizable.

(2) If A is semisimple and W is an A-invariant subspace, then the restrictions of A to W and V/W are semisimple.

(3) The sum of two commuting nilpotent operators is nilpotent.

(4) The sum of two commuting semisimple operators is semisimple.

Theorem 8.7 (Jordan decomposition)

Let V be a vector space over k algebraically closed Any linear operator $A : V \to V$ is uniquely the sum of a nilpotent operator A_n and a semisimple operator A_s which commute with each other. Moreover, $A_s = p(A)$ for some polynomial $p \in k[A]$, hence $A_n = A - p(A)$.

Proof. Generalized eigenspaces. Blah. □

Theorem 8.8

Let V be a vector space over k algebraically closed Let $A : V \to V$ and define

$$\mathrm{ad}_A : \operatorname{End} V \to \operatorname{End} V \qquad \mathrm{ad}_A(B) = AB - BA.$$

Then

$$(\mathrm{ad}_A)_s = \mathrm{ad}_{A_s}$$

and $\mathrm{ad}_{A_s} = P(\mathrm{ad}_A)$ for some $P \in k[t]$, $P(0) = 0$.

Proof. Let $A = A_s + A_n$, and consider $\mathrm{ad}_A = \mathrm{ad}_{A_s} + \mathrm{ad}_{A_n}$. It's easy to see ad_{A_s} and ad_{A_n} commute. Choose a basis of V where A_s is diagonal and A_s is strictly upper triangular.

Let E_{ij} be the corresponding basis of $\operatorname{End} V$. Then we have

$$\mathrm{ad}_{A_s}(E_{ij}) = (\lambda_i - \lambda_j)E_{ij}$$

hence ad_{A_s} is semisimple. On the other hand:

Exercise 8.9 (Kirillov 5.7)**.** ad_{A_n} is nilpotent.

Hence by uniqueness, we get $\mathrm{ad}_{a_s} = (\mathrm{ad}_A)_s$. Moreover, $\mathrm{ad}_A\, a = 0$, so $P(0) = 0$ above. □

§§8.3 Jordan decomposition of Lie elements

Definition 8.10. A **derivation** of an algebra A is any $D \in \mathfrak{gl}(A)$ for which

$$D(ab) = D(a)b + aD(b) \quad \forall a, b \in A.$$

The set of derivations is written $\operatorname{Der} A$. It naturally forms a Lie algebra with the commutator bracket $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$.

Hence given a Lie algebra $\mathfrak{g}$ there is a natural inclusion

$$\mathfrak{g} \hookrightarrow \operatorname{Der} \mathfrak{g} \qquad x \mapsto \operatorname{ad}_x.$$

A derivation of the form ad_x is called an **inner derivation**.

Proposition 8.11

The natural map $\mathfrak{g} \hookrightarrow \operatorname{Der} \mathfrak{g}$ is an isomorphism when $\mathfrak{g}$ is semisimple, i.e. all derivations of a semisimple Lie algebra are inner.

Proof omitted, we'll come back to this.

Theorem 8.12

Let $\mathfrak{g}$ be a semisimple complex Lie algebra. Then any $x \in \mathfrak{g}$ can be uniquely decomposed as

$$x = x_s + x_n$$

where x_s is ad-semisimple, x_n is ad-nilpotent, and x_s and x_n commute.

Moreover, if a $y \in \mathfrak{g}$ commutes with x, it commutes with x_s and x_n as well.

Proof. First we show uniqueness. We obtain

$$(\operatorname{ad}_x)_s = \operatorname{ad}_{x_s} = \operatorname{ad}_{x'_s} \implies \operatorname{ad}_{x_s - x'_s} = 0.$$

So $x_s - x'_s$ is in the center of $\mathfrak{g}$, hence $x_s - x'_s = 0$ since $\mathfrak{g}$ is semisimple.

As for existence, decompose $\mathfrak{g} = \bigoplus \mathfrak{g}_\lambda$ into generalized ad_x-eigenspaces.

Claim 8.13. We have $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}$.

Proof. To see this, note that for $y \in \mathfrak{g}_\lambda$, $z \in \mathfrak{g}_\mu$, we have

$$(\operatorname{ad}_x - \lambda \mathrm{id} - \mu \mathrm{id})[y, z] = [(\operatorname{ad}_x - \lambda \mathrm{id})y, z] + [y, (\operatorname{ad}_x - \mu \mathrm{id})z]$$

by the Jacobi identity. By induction we eventually get

$$(\operatorname{ad}_x - \lambda \mathrm{id} - \mu \mathrm{id})^n [y, z] = \sum_k \binom{n}{k} [(\operatorname{ad}_x - \lambda)^k y, (\operatorname{ad}_x - \mu)^{n-k} z]$$

which implies the result by taking $n > \dim \mathfrak{g}_\lambda + \dim \mathfrak{g}_\mu$. ■

Now by the $(\operatorname{ad}_x)_s$ is a derivation. Hence $(\operatorname{ad}_x)_s = \operatorname{ad}_{x_s}$ for some $x_s \in \mathfrak{g}$. Then, set $(\operatorname{ad}_x)_n = \operatorname{ad}_{x - x_s}$. □

§§8.4 Toral subalegbras

Definition 8.14. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Lie subalgebra. It is called **toral** if it is commutative and consists only of semisimple elements in $\mathfrak{g}$.

Definition 8.15. A bilinear form $\langle - \rangle$ on $\mathfrak{g}$ is invariant if

$$\langle \operatorname{ad}_x y, z \rangle + \langle y, \operatorname{ad}_x z \rangle = 0$$

holds identically for x, y, z.

Theorem 8.16

Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra and let $\mathfrak{h} \subseteq \mathfrak{g}$ be toral. Let $\langle , \rangle$ be a non-degenerate symmetric invariant bilinear form on $\mathfrak{g}$.

1. View $\mathfrak{g}$ as a $\mathfrak{h}$-algebra, and write

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^\vee} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha$ are eigenspaces for all $h \in \mathfrak{h}$ with eigenvalues α, meaning

$$\operatorname{ad}_h(x) = \alpha(h)x \qquad h \in \mathfrak{h}, x \in \mathfrak{g}_\alpha.$$

2. $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.
3. If $\alpha + \beta \neq 0$ then $\mathfrak{g}_\alpha$ is orthogonal to $\mathfrak{g}_\beta$ with respect to $\langle , \rangle$.
4. For all α, we have a nondegenerate pairing

$$\mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha} \to \mathbb{C}.$$

Proof.

1. Since $H \in \mathfrak{h}$, $\{\operatorname{ad}_h\}_{h \in \mathfrak{h}}$ is simultaneously diagonalizable. (We assumed $\mathfrak{g}$ was finite dimensional, so $\mathfrak{g}_\alpha$ vanishes for almost all α).
2. Given $y \in \mathfrak{g}_\alpha$, $z \in \mathfrak{g}_\beta$ we have

$$\begin{aligned} \operatorname{ad}_h[y, z] &= [\operatorname{ad}_h y, z] + [y, \operatorname{ad}_h z] \\ &= \alpha(h)[y, z] + \beta(h)[y, z] \\ &= (\alpha + \beta)(h)[y, z]. \end{aligned}$$

Now $[\operatorname{ad}_h y, z] + [y, \operatorname{ad}_h z] = 0$ because of $\mathfrak{g}$-invariance of the bilinear form.

3. Compute

$$\begin{aligned} 0 &= \langle [h, x], y \rangle + \langle x, [h, y] \rangle \\ &= \langle \alpha(h)x, y \rangle + \langle x, \beta(h)y \rangle \\ &= (\alpha + \beta)(h) \langle x, y \rangle . \end{aligned}$$

4. Similar above. $\square$

From here, we would like to make $\mathfrak{h}$ as big as possible. It turns out there is "only one" way to do this.

We also need to prove that $\operatorname{Der} \mathfrak{g} = \mathfrak{g}$ for $\mathfrak{g}$ semismiple (and in fact the converse is true).

Finally, we need to show $\langle , \rangle$ actually exists.

All of these are related to the existence of a nondegenerate invariant form, whose existence can be used to detect semisimplicity.

§9 March 3, 2016

Let H be a closed subgroup of G, where G is a linear algebraic group. Our goal is to show that G/H has the structure of a variety.

§§9.1 Review of varieties

Recall $\mathbb{A}^n = \operatorname{MSpec} k[x_1, \dots, x_n]$ where k is algebraically closed, with the usual Zariski topology. Since $k[x_1, \dots, x_n]$ is Noetherian, ascending chains of ideals stabilize; a "closed set" then corresponds to the vanishing set of some ideal or polynomials.

Let X be an affine variety closed in $\mathbb{A}^n = \mathbb{C}^n$, thus with the Zariski topology. Then we can consider X_{an} which is X with the inherited analytic topology of $\mathbb{A}^n$.

Recall that X is **separated** if the diagonal morphism $\Delta : X \to X \times X$ has closed image $\operatorname{im} \Delta \subseteq X$.

Proposition 9.1 (Properties of X_{an})

(i) The inclusion $X_{\text{an}} \to X$ is continuous.

(ii) If $X \to Y$ is a morphism of varieties, the corresponding map $X_{\text{an}} \to Y_{\text{an}}$ is continuous.

(iii) If X is separated then X_{an} is Hausdorff.

Lemma 9.2 ($\mathbb{A}^n$ is Noetherian)

Let $X \subseteq \mathbb{A}^n$ be a closed variety. Then

(i) Any family of closed subsets of X contains a minimal one.

(ii) Any descending chain $X_1 \supset X_2 \supset \dots$ of closed subsets must eventually stabilize.

Recall that a space is **irreducible** if it is not the union of two proper closed subspaces.

Lemma 9.3

Let X be a topological space.

(i) $A \subset X$ is irreducible if and only if $\overline{A}$ is.

(ii) If $f : X \to Y$ is continuous and X is irreducible, then $\operatorname{im} f \subseteq Y$ is irreducible as well.

Let U be open on X. A k-valued function f on U is called **regular** at $x \in U$ if in some neighborhood $x \in V \subseteq U \cap D(h)$ we have $f(y) = g(y)h(y)^{-1}$ for $y \in V$.

This gives us a **sheaf** of regular functions on U. Definition of a sheaf deferred to Napkin.

Then, for us we define

Definition 9.4. An abstract **variety** X consists of a pair $(X, \mathcal{O}_X)$ where

- X is quasi-compact topological space. (any open cover has a finite subcover).
- X is locally affine algebraic and separated.
- $\mathcal{O}_X$ is a sheaf of rings of k-valued functions on X.

Definition 9.5.
- A **projective** variety is a closed subvariety of $\mathbb{P}^n$.
- A **quasi-projective** variety is a open subvariety of $\mathbb{P}^n$.
- A **locally closed** subset is the intersection of an open or closed set.

§§9.2 Algebraic groups

Definition 9.6. An **algebraic group** is a group that is an algebraic variety, such that the multiplication and inversion are regular functions.

For example, $\mathrm{GL}(n, \mathbb{C})$ is a distinguished open of $\mathbb{A}^{n^2}$; hence its an affine variety and then an algebraic group.

Proposition 9.7 (Connected components algebraic groups)
The irreducible components of an algebraic group G are the connected components of G as a topological space.

Hence, we generally think about connected algebraic groups.

Definition 9.8. Any (Zariski) closed subgroup of $\mathrm{GL}(n, \mathbb{C})$ is a **linear algebraic group**.

§§9.3 Group actions

Now suppose G is an algebraic group acting on a space X. Given a point $x \in X$. Then

$$\mathrm{Stab}_G\, x = \{g \mid g \cdot x = x\} \subseteq G$$

is the **isotropy group**, a closed subgroup of G.

Lemma 9.9
Let a group G act on a space X.

(i) G orbits are open in their closures.

(ii) There exists closed orbits.

Proof. We skip (1). Now (2) follows from Noetherian property. □

Definition 9.10. We say T is **unipotent** if $T - \mathrm{id}$ is nilpotent.

Proposition 9.11 (Jordan Decomposition)
Let $a \in \mathrm{GL}(V)$, then there exists unique $a_s, a_u \in \mathrm{GL}(V)$ such that a_s is semisimple, a_u is unipotent, and

$$a = a_s a_u = a_u a_s.$$

Proof. Let $a = a_s + a_n$ be the Jordan decomposition. Since a is invertible, it has no zero eigenvalues, thus a_s is invertible and we can take $a_u = 1 + a_s^{-1} a_n$. □

Now, assume X is an affine variety and an algebraic group G act on it. That is, we have an action

$$a : G \times X \to X$$

which we require to be a map of varieties. Thus this induces a map

$$a^* : k[X] \to k[G \times X] = k[G] \otimes_k k[X].$$

Here $k[X]$ is the coordinate ring (set of regular functions on X). Then given $g \in G$, $x \in X$, and $f \in k[X]$, we have

$$s(g) : k[X] \to k[X] \quad \text{by} \quad s(g)f(x) = f(g^{-1}x).$$

This gives us a representation $G \to \mathrm{GL}(k[X])$, but since $k[X]$ is usually infinite dimensional (as a k-vector space: e.g. take $k[\mathbb{A}^1] \cong k[x]$).

Lemma 9.12

Let V be a finite dimensional subspace in $k[X]$.

(1) There exists a finite dimensional W such that $V \subseteq W$ and $G \cdot W \subseteq W$.

(2) $G \cdot V \subseteq V$ if and only if $a^* V \subseteq k[G] \otimes V$.

Proof. Enough if V is one-dimensional, say spanned by f. Then we have

$$a^* f = \sum_{i=1}^{n} u_i \otimes f_i$$

for $u_i \in k[G]$ and $f_i \in k[X]$. Then

$$s(g)f(x) = f(g^{-1}x) = \sum_{i=1}^{n} u_i(g^{-1}) f_i(x).$$

Since $u_i(g^{-1}) \in k$ don't depend on x, we see that $s(g)f$ lives in the span of these finitely many f_i's, for every g. Now we can just let W be the span of $s(g)f$ as $g \in G$, which is finite dimensional.

Now suppose V is G-stable. Choose a basis of V, say $\{f_i\}$, and extend it to a basis of $k[X]$, say $\{f_i\} \cup \{b_i\}$. If $F \in V$ then

$$a^* f = \sum u_i \otimes f_i + \sum v_j \otimes b_j$$

for some $u_i, v_j \in k[G]$. Then, applying g gives

$$s(g)f = \sum u_i(g^{-1}) f_i + \sum v_j(g^{-1}) b_j.$$

If $s(g)V \subseteq V$, then by assumption we need the second sum to be zero, meaning $v_j(g^{-1}) = 0$ for all $g \in G$, i.e. $v_j = 0$. In particular, $a^* f = \sum u_i \otimes f_i + 0$, hence $a^* f = k[G] \otimes V$. □

Lemma 9.13 (Closed subgroups)

Let H be a closed subgroup in G. Then there exists a finite dimensional rational representation of G containing a line $\ell = kv \subseteq H$ such that $h \cdot \ell \subseteq \ell$ for every $h \in H$.

Proof. Let G act on itself by left multiplication. Since H is closed, there is an ideal $I \subseteq k[G]$ such that every function $f \in I$ has $f|_H = 0$.

Let V be a finite dimensional G-stable subspace in $k[G]$ containing a finite set of generators of I. Let $W = V \cap I \subseteq V$ (hence $H \cdot W \subset W$) and set $n = \dim W$, then

$$\ell = \Lambda^n W \subseteq \Lambda^n V$$

We claim this W and ℓ is the desired.

Then ℓ is clearly fixed by H, so we need to show that only the elements of H fix ℓ:

Claim 9.14. If $x \in G$, $xW \subseteq W \iff x\ell \subseteq \ell$.

Proof. Let $(v_1, \dots, v_n)$ be a basis of W and extend it to a basis $(v_1, \dots, v_p)$ of V. We may also assume $v_{k+1}, \dots, v_{k+n}$ is a basis of xW for our given $x \in G$ (for example $k = 0$ if $xW = W$, else $k > 0$).

Then $v_{i_1} \wedge \dots \wedge v_{i_n}$ are a $\Lambda^n V$ basis ($i_1 < \dots < i_n$ as usual). Let

$$e = v_1 \wedge \dots \wedge v_n \in \ell$$

be the basis element of $\ell = \Lambda^n W$, and

$$f = v_{k+1} \wedge \dots \wedge v_{k+n}$$

be a basis element of $x\ell = \Lambda^n xW$. Then $x \cdot e$ is a multiple of f.

Of course, if $k > 0$ then the e and f are linearly independent. ■

Hence the proof of the lemma. □

Corollary 9.15

There exists a G-homogeneous (transitive G-action) quasi-projective X for G together with a point $x \in X$ such that

(1) $\mathrm{Stab}_G\, x = H$ is closed.

(2) The map $\Psi : g \mapsto g \cdot x$ defines a separable morphism $G^0 \to \Psi g^0$.

(3) The fibers Ψ are cosets gH for $g \in H$.

§10 March 8, 2016

§§10.1 Remarks

Chevalley's structure theorem states that a connected algebraic group G over a perfect field has a unique normal affine algebraic subgroup such that the quotient is an abelian variety.

Philosophical musing: projective varieties are more often *compact* or better "more often occurring in nature": when we zoom out, projective varieties get smaller (whereas something like $\mathbb{A}^n$ gets smaller). We've been taught all our lives to look at things locally ("zooming in", or looking at neighborhoods) when for AG purposes we often want to "zoom out".

Other philosophical musing: in algebraic geometry, one should usually think about closed sets (which are more natural and well-behaved) than open sets. For example, the statement of compact (Hausdorff) can be phrased as follows:

Definition 10.1. A **closed void** is an infinite intersection

$$\bigcap C_\alpha = \varnothing.$$

where each C_α is closed. Then a topological space is compact if any any closed void has a finite sub-cover.

Exercise 10.2. On Thursday we checked that $\mathrm{GL}(n)$ is algebraic. Show that in fact $\mathrm{GL}(n)$ is *affine*, meaning we can take a closed embedding

$$\mathrm{GL}(n) \hookrightarrow \mathbb{A}^n.$$

Exercise 10.3. Fill in some details of the proof that G/H is a variety (below).

§§10.2 Review

Last time we showed that

Proposition 10.4

If H is closed in G and we consider the action of G on $k[G]$, then there exists a finite dimensional G-stable vector space $V \subseteq k[G]$ and a one-dimensional subspace $\ell \subseteq V$ such that

$$\{g \in G \mid g \cdot \ell = \ell\} = H.$$

For example, when $H = G$ we can take $\ell = V = \sum_{g \in G} g$, but when $H \subsetneq G$ we need some more work.

We then proved that

Proposition 10.5

There exists a quasi-projective X which is G-homogeneous and $x \in X$ with $\mathrm{Stab}_G\, x = H$.

Proof. Let $\pi : V \setminus \{0\} \to \mathbb{P}(V)$ with V as above, and let G act on $\operatorname{im} \pi$. If we let $x = \pi(\ell) \in \mathbb{P}(V)$, then define X to the be the G-orbit of x. Then we have inclusions

$$X \xhookrightarrow{\text{open}} \overline{X} \xhookrightarrow{\text{closed}} \mathbb{P}^n$$

□

§§10.3 Quotients

Let $\ell(v)$ denote the line $\{\lambda \cdot v \mid \lambda \in k\}$.

Definition 10.6. A **quotient** of G by a closed subset H over k is a pair (Q, a), $a \in Q$, such that

(i) Q is a G-homogeneous space, and $\ell(a)$ is stable under exactly the elements of H,

(ii) $a \in Q$ is a point such that we have the universal property: for every (Y, b) with $\ell(b)$ stable under H there's a unique G-equivariant $\phi : Q \to Y$ such that $\phi(a) = b$.

Theorem 10.7
A quotient (Q, a) exists and is unique up to G-isomorphism.

For existence, (X, x) as in the earlier proposition works.

Now let

$$G/H = \{gH \mid g \in G\}$$

as a set and consider the projection

$$\pi : G \to G/H.$$

If we require π to be an open map, then

Exercise 10.8. Check this defines a topology on G/H.

Now we define $\mathcal{O}$ a sheaf of k-valued functions on G/H as follows. Given $U \subseteq G/H$ open, we set

$$\mathcal{O}(U) = \left\{ f : U \to k \mid f \circ \pi \text{ is regular on } \pi^{-1}(U) \right\}.$$

In other words, $\mathcal{O}$ is the pushforward of the sheaf on G.

Exercise 10.9. Check it's a sheaf.

Now G acts on G/H transitively by left translations (as a set). Consider $x \in G$ and the map

$$gH \xmapsto{\cdot xg} xgH$$

is an automorphism of the ringed space $(G/H, \mathcal{O})$. (Check this too.)

Now assume (Y, b) as in the universal property. Then we wish to show there exists a G-morphism of ringed spaces

$$\phi : G/H \to Y \quad gH \mapsto g \cdot b.$$

This is

1. Well-defined since $\mathrm{Stab}_G \, \ell(b) \supseteq H$.

2. Continuous by looking at the diagram

$$\begin{tikzcd} G \arrow[d, "\pi"] \arrow[dr] & & g \arrow[dr, mapsto] & \\ G/H \arrow[r, "\phi"] & Y & & g \cdot b \end{tikzcd}$$

3. Morphism of ringed spaces. This follows just because we're assuming our sheaves our functions: that is we define
$$\mathcal{O}_Y \to f_* \mathcal{O}_X$$
by composition; if $q : U \to k$ ($U \subseteq Y$) then we compose
$$\left(f^{-1}(U) \xrightarrow{f} Y \xrightarrow{q} k \right) \in \mathcal{O}_X(f^{-1}(U)) = f_*\mathcal{O}_X(U).$$

The uniqueness follows by $\phi a = b$.

In particular, if (X, x) as in the earlier proposition, there is a unique G-morphism of ringed spaces
$$G/H \xrightarrow{\phi} X$$
such that $\phi H = x$. Now ϕ is an open map and has a continuous inverse, so ϕ is a homeomorphism.

Still need to check that the sheaves are isomorphisms. (Use Lie algebra.)

§§10.4 Complete varieties

Definition 10.10. An algebraic variety X is **complete** if for every Y, the map
$$X \times Y \xrightarrow{\pi_Y} Y$$
is closed.

(Note that $X \times Y$ is not the product of Zariski topologies; in general $X \times Y$ has *more* closed sets. Take the diagonal $\mathbb{A}^2$ for example.)

In particular, $\mathbb{A}^n$ is not complete for any $n \geq 1$. For this example, look at the hyperbola $xy - 1 = 0$ in $\mathbb{A}^2$. Then the image of the projection onto the y-coordinate i $\mathbb{A}^1 \setminus \{0\}$, which isn't closed.

Proposition 10.11 (Properties of complete spaces)
Let X be complete.

(1) Any closed $S \hookrightarrow X$ is complete.

(2) If Y is complete, so is $X \times Y$.

(3) If $\phi : X \to Y$ is a morphism then $\operatorname{im} \phi$ closed and complete.

(4) If $X \subseteq Y$ is a subvariety then X closed in Y.

(5) If X is irreducible then $k[X] = k$; the regular functions on X are just scalars.

(6) If X is affine, then X is finite.

Proof. (1) and (2) are easy.

(3) requires the fact that if $\phi : X \to Y$ is a morphism, then $\operatorname{im} \phi$ is constructible, meaning it's the finite union of locally closed sets. (In turn follows from Zariski's main theorem.) We also need the fact that if X is separable then the graph of a morphism from X is closed. ... □

Theorem 10.12 (Projective $\implies$ complete)

If X is projective, then X is complete.

§§10.5 Parabolic subgroups

Definition 10.13. Let G be a linear algebraic group. A closed subgroup P of G is **parabolic** if G/P is complete.

Proposition 10.14

Assume G is connected. Then G contains a *proper* parabolic subgroup if and only if G is *not* solvable.

We will prove this on Thursday.

Theorem 10.15 (Borel's fixed point theorem)

If G is a connected, solvable, algebraic group acting regularly on a non-empty, complete algebraic variety X over an algebraically closed field k, then there is a G fixed-point of V.

Proof. Also there exists a closed orbit, say $G.x$. Then $\operatorname{Stab}_G x$ is parabolic in G, since $G/\operatorname{Stab}_G x \cong G.x$ is closed and complete (complete follows from X complete, $G.x$ closed in X).

Thus by definition $\operatorname{Stab}_G x$ is parabolic. Since G is solvable, there are no proper parabolic subgroups, hence $G = \operatorname{Stab}_G x$ and x is a fixed point. $\square$

Proposition 10.16

X over $\mathbb{C}$ is complete if and only if X_{an} is compact.

§11 March 10, 2016

Taking a break from algebraic geometry, and talking about Lie algebras attached to compact Lie groups.

§§11.1 Preliminaries

Recall that a bilinear form B on a vector space V (thus $B : V \to V^\vee$) is **nondegenerate** if $\ker B = \{0\}$.

Remark 11.1. Check that B is G-invariant if $V \to V^\vee$ is also a G-intertwiner.

Fact 11.2. For any $V \in \operatorname{Rep} G$, and $\mathfrak{g}$ the associated Lie algebra, we also recall that

$$V^G = V^{\mathfrak{g}}$$

whenever G is connected.

Definition 11.3. Let V be a $\mathfrak{g}$-module. Then we say a bilinear form B on V is **$\mathfrak{g}$-invariant** if

$$B(\rho(x)v, w) + B(v, \rho(x)w) = 0$$

for $x \in \mathfrak{g}$.

Definition 11.4. We say B is

- **positive definite** if $B(v, v) \geq 0$ with equality only if $v = 0$.
- **negative definite** if $B(v, v) \leq 0$ with equality only if $v = 0$.
- **(negative) semidefinite** if $B(v, v) \leq 0$ for all v, but equality might hold in places other than $v = 0$.

§§11.2 Reductive Lie algebras

Definition 11.5. A Lie algebra $\mathfrak{g}$ is **reductive** if its radical equals its center, i.e.

$$\operatorname{rad} \mathfrak{g} = \mathfrak{z}(\mathfrak{g})$$

where the **radical** $\operatorname{rad} \mathfrak{g}$ is the unique maximal solvable ideal in $\mathfrak{g}$.

Claim 11.6. If $\mathfrak{g}$ is finite-dimensional, then $\operatorname{rad} \mathfrak{g}$ exists, is unique, and contains all the solvable ideals in $\mathfrak{g}$.

Proof. Check that sums of solvable ideals are solvable. It's easy to see it's an ideal; to see solvable, look at the exact sequence

$$I_1 \hookrightarrow I_1 + I_2 \twoheadrightarrow (I_1 + I_2)/I_1 \simeq I_2/(I_1 \cap I_2)$$

and note the left and right terms are solvable (right is quotient of solvable ideals), hence the center term.

Then let $\operatorname{rad} \mathfrak{g} = \sum_{I \text{ solvable}} I$. □

In what follows, assume always $\mathfrak{g}$ is finite-dimensional.

Let (V, ρ) be a representation of $\mathfrak{g}$. Now consider the bilinear form on $\mathfrak{g}$ by

$$B_V(x, y) = \operatorname{Tr}_V(\rho(x)\rho(y)).$$

We claim that if B_V is invariant under the connected group G, it is also invariant under the action of $\mathfrak{g}$. Compute

$$B_V(gxg^{-1}, gyg^{-1}) = \operatorname{Tr}_V(g\rho(x)\rho(y)g^{-1}) = \operatorname{Tr}_V(\rho(x)\rho(y)). = B_V(x, y)$$

Fact 11.7. If V is an irreducible $\mathbb{C}$-representation, then any $h \in \operatorname{rad}\mathfrak{g}$ acts on V by a scalar. Moreover, it acts by zero if $h \in [\mathfrak{g}, \operatorname{rad}\mathfrak{g}]$.

Proof. Since $\operatorname{rad}\mathfrak{g}$ is solvable, there is a common eigenvector v for all $h \in \operatorname{rad}\mathfrak{g}$ meaning $\rho(h) \cdot v = \lambda(h)v$. Hence we get $\lambda : \operatorname{rad}\mathfrak{g} \to \mathbb{C}$. Then we see that

$$V_\lambda = \{w \in V \mid \rho(h)w = \lambda(h)w \qquad \forall h \in \operatorname{rad}\mathfrak{g}\} \subseteq V$$

is a subrepresentation. Since V is irreducible, $V_\lambda = V$. □

Proposition 11.8

Suppose B_V is non-degenerate for some representation (V, ρ) of $\mathfrak{g}$. Then $\mathfrak{g}$ is reductive.

Proof. We need to show $[\mathfrak{g}, \operatorname{rad}\mathfrak{g}] = 0$. Let $x \in [\mathfrak{g}, \operatorname{rad}\mathfrak{g}]$, so $\operatorname{ad}_x = 0$ for any representation V_i, meaning $x \in \ker B_{V_i}$.

Exercise 11.9 (Kirillov 5.1)**.** Show that if $0 \to V_1 \to W \to V_2 \to 0$ is a short exact sequence of $\mathfrak{g}$-representations then $B_W = B_{V_1} + B_{V_2}$.

Then by induction, we get $x \in \ker B_V$, but B_V was nondegenerate hence $x = 0$ □

Theorem 11.10

$\mathfrak{gl}(n, \mathbb{C})$ is reductive.

Proof. Let $\mathfrak{gl}_n \mathbb{C}$ act on $\mathbb{C}^{\oplus n} = V$ in the usual way. Then

$$B_V(x, y) = \sum x_{ij} y_{ji}$$

is nondegenerate as desired. □

We have

$$\mathfrak{gl}_n = \underbrace{\mathfrak{z}(\mathfrak{gl}_n)}_{=\mathbb{C}} \oplus \mathfrak{sl}_n(\mathbb{C}).$$

Now note that $[\mathfrak{gl}_n, \mathfrak{gl}_n] = \mathfrak{sl}_n$, and $\mathfrak{sl}_n$ is reductive for the same reason as above: with respect to B_V, $\mathfrak{z}(\mathfrak{gl}_n)$ and $\mathfrak{sl}_n$ are orthogonal.

§§11.3 Killing form

View $\mathfrak{g}$ as a representation over itself by ad, i.e.

$$\operatorname{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

Definition 11.11. The **Killing form** is the bilinear form $K_\mathfrak{g} \overset{\text{def}}{=} B_{\operatorname{ad}}(x, y) = \operatorname{Tr}(\operatorname{ad}_x \operatorname{ad}_y)$.

Remark 11.12. Note that:

$$\mathfrak{z}(\mathfrak{g}) = \ker \rho \hookrightarrow \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}).$$

Theorem 11.13 (Cartan's criterion)

The Lie algebra $\mathfrak{g}$ is semisimple if and only if $K_\mathfrak{g}$ is nondegenerate.

§§11.4 Compact Lie groups

Here is an example which will be an important ingredient in the following theorem (but is not an example of the theorem itself)

Example 11.14

Let $\mathfrak{g} = U(n)$ be the skew-Hermitian matrices. Then the trace form $\langle x, y \rangle = \operatorname{Tr}(x, y)$ is negative definite. To see this note, that

$$\operatorname{Tr}(xy) = -\operatorname{Tr}(xy^\dagger) \implies \operatorname{Tr}(x^2) = -\operatorname{Tr}(xx^\dagger) = -\sum_{i,j} |x_{ij}|^2.$$

Thus $\mathfrak{g}$ is reductive.

We'll also need

Theorem 11.15

Let G be a compact real Lie group. Then $\mathfrak{g}$ is reductive, and the Killing form is negative semidefinite. Moreover, $\ker K = \mathfrak{z}(\mathfrak{g})$ and K is negative definite on the semisimple part $\mathfrak{g}/\mathfrak{z}(\mathfrak{g}) = \mathfrak{g}_{ss}$.

Conversely, if $\mathfrak{g}$ is a semisimple real Lie algebra with negative definite Killing form, then $\mathfrak{g}$ is attached to some compact Lie group.

Proof. Since G compact, then every complex representation is unitary (first week of class). Thus $\rho(G) \subseteq U(V)$, and $\rho(\mathfrak{g}) \subseteq \mathfrak{u}(V)$. Thus the trace form $B_V(x, y)$ is negative semidefinite with $\ker B_V = \ker \rho$.

Now apply this to $V = \mathfrak{g}_\mathbb{C}$ with G acting on $\mathfrak{g}_\mathbb{C}$ by adjoint action. Then B_V is the Killing from K, thus negative semidefinite.

Now it's a theorem that if $\mathfrak{g}$ is reductive then $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}_{ss}$. But $[\mathfrak{g}_{ss}, \mathfrak{g}_{ss}] = \mathfrak{g}_{ss}$. Thus $\operatorname{Ker} K = \mathfrak{z}(\mathfrak{g})$.

For the converse, assume $\mathfrak{g}$ is a real semisimple Lie algebra with negative definite killing form K. By Lie's third theorem, let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and define positive definite $B(x, y) = -K(x, y)$. Now B is $\operatorname{Ad} G$ invariant, which means $\operatorname{Ad} G \subseteq \operatorname{SO}(G)$; since $\operatorname{Ad} G$ is connected it follows $\operatorname{Ad} G$ is the connected component of the identity of $\operatorname{Aut} \mathfrak{g}$ (the group of Lie algebra automorphisms).

Regard $\operatorname{Aut} \mathfrak{g} \hookrightarrow \operatorname{GL} \mathfrak{g}$ as a closed Lie subgroup. Then $\operatorname{im} \operatorname{Ad} G$ is closed in the compact $\operatorname{SO}(\mathfrak{g})$ and hence $\operatorname{Ad} G$ is compact. So

$$\operatorname{Ad} G = G/Z(G)$$

whence the Lie algebra attached to $\operatorname{Ad} G$ is $\mathfrak{g}/\mathfrak{z}(\mathfrak{g}) \simeq \mathfrak{g}$ (the center is trivial since K nondegenerate). $\square$

§§11.5 One-dimensional representations

Proposition 11.16

If $\mathfrak{g}$ is semisimple, then the only 1-dimensional representation is the zero one.

Proof. If $\rho : \mathfrak{g} \to \mathbb{C}$ has $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ then notice that $\rho[\mathfrak{g}, \mathfrak{g}] = 0$. $\square$

Proposition 11.17

If $\mathfrak{g}$ is reductive, then one-dimensional representations of $\mathfrak{g}$ correspond to one-dimensional representations of $\mathfrak{z}(\mathfrak{g})$.

Contrast this with the solvable case when all irreducible representations were one-dimensional.

We'd like to find some middle ground here. Suppose $\mathfrak{b} \hookrightarrow \mathfrak{g}$, where $\mathfrak{b}$ is solvable and $\mathfrak{g}$ is a reductive Lie algebra. Then we get a map $\mathfrak{z}(\mathfrak{g}) \hookrightarrow \mathfrak{b}$.

Not all irreps of $\mathfrak{b}$ come from $\mathfrak{z}(\mathfrak{g})$ in this way (for example, if $\mathfrak{g}$ is semisimple then $\mathfrak{g}$ has no center at all). So we want to understand how to uniformly construct all irreducible representations of $\mathfrak{b}$.

Given an irrep V of $\mathfrak{b}$, let $\widetilde{V}$ be $\operatorname{ind}_{\mathfrak{b}}^{\mathfrak{g}} V$. It's not clear $\widetilde{V}$ should be irreducible. This is not always true, but

- If $\mathfrak{g}$ is semisimple over $\mathbb{C}$, then it's true.
- In the modular case, it's not true.

I'm hungry.

§§11.6 Borel and parabolic subgroups

Let V be a representation of $\mathfrak{b}$. Let G be semisimple and reductive. Assume $\mathfrak{b} = \operatorname{Lie} B$, where B is a Borel subgroup in G (defined later). We saw that if P is a parabolic subgroup of G, then G/P was complete and quasi-projective. Then a minimal parabolic P (containing no other proper parabolic) is the **Borel subgroup**; it keeps as much G-information as possible. In fact,

Theorem 11.18 (Borel $\iff$ Maximal solvable)

Let B be a subgroup of a semisimple, reductive G. Then the following are equivalent:

- B is Borel.
- B is the maximal solvable subgroup of G
- B is the smallest subgroup for which G/B is projective.
- G/B is complete and B is solvable.

Theorem 11.19 (Levi's theorem)

Given a general Lie algebra $\mathfrak{g}$, we have

$$\mathfrak{g} = \operatorname{rad} \mathfrak{g} \oplus \mathfrak{g}_{ss}.$$

the Lie algebra $\mathfrak{g}_{ss}$ is not semisimple.

Remark 11.20. A complex variety is complete if and only if it is compact as a complex-analytic variety.

New office hours: W 11:30AM - 1PM.

§12 March 15, 2016

Homework 4 due Thursday. Next week: holiday.

Today: diagonalizable groups.

§§12.1 Proof of proposition from last time

Last time:

Proposition 12.1

A connected linear algebraic group G contains a proper parabolic subgroup if and only if G is not solvable.

Proof. Assume G is closed in $\mathrm{GL}(V)$ and let G act on $\mathbb{P}(V)$. Then there exists a closed orbit X, thus X is projective (hence complete).

Let $x \in X$, and $P = \mathrm{Stab}_G\, x$; this gives a morphism

$$G/P \to X \qquad gP \mapsto g \cdot x.$$

This is bijective. By some lemma, this implies P is parabolic. (Namely, if X and Y are G-homogeneous and $\phi : X \to Y$ is a bijective G-map of varieties then X complete $\iff$ Y complete.)

If $P \neq G$ then done. If $P = G$, set $V_1 = V/x$, and let G act on $\mathbb{P}V_1$. Continue in this fashion. Eventually we'll get either a proper parabolic, or G will be isomorphic to upper triangular matrices in $\mathrm{GL}(n)$ whence G is solvable.

Conversely, assume G is connected and solvable. Assume $P \subsetneq G$ is parabolic and of maximal dimension. (Lemma: P parabolic in $G \implies P^0$ parabolic in G^0.) Thus we can assume P is connected.

Consider (G, G) the group generated by commutators; this is closed and connected. Let Q be the orbit of P on (G, G); this is connected, parabolic, and contains P. Then

- If $Q = G$, then $(G, G)/(G, G) \cap P \to G/P$ is a bijection, hence $(G, G) \cap P$ is parabolic in (G, G). If $(G, G) \neq G$ then induct down; now we may as well assume $(G, G) \cap P = (G, G)$, or $(G, G) \subseteq P$, contradiction.
- Else if $Q = P$, then $(G, G) \subseteq P$, so G/P is affine. □

Here's a lemma that might be useful:

Lemma 12.2

Let H be a closed subgroup of the linear algebraic group G. Assume $\pi : G \to G/H$ has "local sections", and H acts on X. Then the fiber product

$$G \times_H X = \left\{ (g, x) \in G \times X \mid (gh, h^{-1}x) \sim (g, x) \right\}$$

exists. This is a fiber bundle for G/H.

Proof: exercise.

§§12.2 Rational representations

We want to study $\operatorname{Rep} G$ or $\operatorname{Rep} \mathfrak{g}$, where $\operatorname{Rep} G$ means "finite-dimensional rational representations" in a context where G is an algebraic group.

Definition 12.3. Let G be an algebraic group. Then a representation is a map $\phi : G \to \operatorname{GL}(V)$ which is a morphism of varieties (where we also view $\operatorname{GL}(V)$ as a linear algebraic group).

If we have subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ built out of semisimple commuting things (i.e. $\mathfrak{h}$ is toral) then restricting to $\mathfrak{h}$ gives a decomposition

$$\mathfrak{g} = \bigoplus_\alpha \mathfrak{g}_\alpha.$$

OK now consider the following.

- The group G acts on itself by conjugation.
- This induces an action Ad of G on $\mathfrak{g}$.
- This induces an action ad of $\mathfrak{g}$ on $\mathfrak{g}$.

The way to go from the last of these back to the original (for simply connected Lie groups G) is through the exp map, which is not algebraic.

Sometimes we want to view G acting on $k[G]$ as a "rational" representation, but we don't have finite-dimensional.

Definition 12.4. let V be any vector space (possibly infinite dimensional). Then $a \in \operatorname{End} V$ is **locally finite** if we can decompose $V = \bigcup_\alpha V_\alpha$ such that each V_α is a-stable and $\dim V_\alpha < \infty$.

Assuming a is locally finite, we say it is

- **semisimple** if restriction to any finite dimensional a-stable subspace is semisimple.
- **locally nilpotent** if restriction to any finite dimensional a-stable subspace is nilpotent.
- **locally unipotent** if restriction to any finite dimensional a-stable subspace is unipotent.

Then as before, any locally finite a can be written as

$$a = a_s + a_n$$

where a_s is semisimple and a_n is locally nilpotent.

If a is invertible and locally finite, we can also put

$$a = a_s a_u$$

where a_s is semisimple, and a_u is locally unipotent.

Example 12.5

Let $\mathbb{G}_m$ be the **multiplicative group of the field**

$$\mathbb{G}_m = \operatorname{Spec} k[x, x^{-1}]$$

which as a set of points is k^*; this is also $\mathrm{GL}(1, k)$.

We have $\operatorname{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$ since the only algebraic maps from $\mathbb{G}_m$ to itself are the maps $x \mapsto x^n$.

This gives the irreducible one-dimensional representations, since we have computed all maps $\mathbb{G}_m \to \mathrm{GL}(1, k)$.

On the other hand, since $\mathbb{G}_m$ is one-dimensional as a group, so all the irreducibles are one-dimensional.

Consider G acting on $k[G]$, call this ρ. One can show that $\rho(g)$ is locally finite for all G, so we get a decomposition

$$\rho(g) = \rho(g)_s \rho(g)_u.$$

Theorem 12.6

For any $g \in G$, there exist unique commuting $g_s, g_u \in G$ such that $\rho(g)_s = \rho(g_s)$ and $\rho(g)_u = \rho(g_u)$. Thus homomorphisms of algebraic groups preserve the properties "unipotent" and "semisimple".

Definition 12.7. Let $\sigma \in \operatorname{Aut} G$. We say σ is **semisimple** if σ looks like conjugation by a semisimple element in $\mathrm{GL}\, V$.

That is, if we take our embedding $\phi : G \to H \subseteq \mathrm{GL}(n)$ with H a closed subgroup of $\mathrm{GL}(n)$, then there should be $s \in \mathrm{GL}(n)$, in the normalizer of H with respect to $\mathrm{GL}(n)$, so that

$$\phi(\sigma x) = s\phi(x)s^{-1} \qquad \forall x \in G.$$

Exercise 12.8. Let G_u be the set of unipotent elements in G. Show that G_u is always a closed subgroup of G.

§§12.3 Diagonalizable groups

Definition 12.9. We say G is **diagonalizable** if H is isomorphic to a closed subgroup D_n of diagonal matrices in $\mathrm{GL}(n)$. We say H is a **torus** if $H \simeq D_n$.

Theorem 12.10

If G is a commutative linear algebraic group then G_s and G_u are closed subgroups and

$$\pi : G_s \times G_u \to G$$

is an isomorphism of algebraic groups.

Reading: Dedekind's theorem. (Prime ideals in a Dedekind domain.)

Now let G be a linear algebraic group, and let $\chi : G \to \mathbb{G}_m$ a homomorphism of algebraic groups. Then we call χ the rational characters, the set of which is

$$X^*(G) = \operatorname{Hom}(G, \mathbb{G}_m) \subseteq k[G]$$

which is an abelian group; we'll often write the operation additively. (Tangential remark: the dual concept is $X_*(G) = \{\mathbb{G}_m \to G\}$; for G commutative this is also an abelian group. If G is arbitrary, this is still a $\mathbb{Z}$-module.)

Dedekind's theorem says these characters are linearly independent in $k[G]$. Here is the condition for it to be a basis:

Theorem 12.11

Let G be a linear algebraic group. The following are equivalent.

(1) G is diagonalizable.

(2) $X^*(G)$ is a finitely generated abelian group and a k-basis for $k[G]$.

(3) $\operatorname{Rep} G$ is a semisimple abelian category whose irreducibles are one-dimensional.

§13 March 17, 2016

Last time we saw $X^*(G)$ forms a k-basis of $k[G]$.

§§13.1 More on diagonalizable groups

Definition 13.1. Let M be an abelian group. Then $k[M]$ is the algebra with basis $e(m)$ for $m \in M$, with $e(m)e(n) = e(m+n)$.

We can make it into a Hopf algebra via

$$\begin{aligned} \Delta(e(m)) &= e(m) \otimes e(m) \\ i(e(m)) &= e(-m) \\ \varepsilon(e(m)) &= 1. \end{aligned}$$

In particular

Corollary 13.2

if G is diagonalizable, then $X^*(G)$ is a finitely generated abelian group. It has no p-torsion if $\operatorname{char} k = p > 0$.

Moreover, $k[G] \cong k[X^*(G)]$ always.

Proof. If $\operatorname{char} k = p > 0$ then the only pth root of 1 is 1; hence if $X^*(G)$ contains p-torsion this contradicts the assumption that $X^*(G)$ forms a k-basis.

To show the isomorphism, we first note:

Claim 13.3. If M_1 and M_2 are both finitely generated then

$$k[M_1 \oplus M_2] \simeq k[M_1] \otimes_k k[M_2].$$

Claim 13.4. If $k[M]$ is affine, then there exists a diagonal algebraic group $\mathscr{G}(M)$ with $k[\mathscr{G}(M)] = k[M]$ such that Δ, ε, i are comultiplication, inversion, and the unit.

Moreover, there is a canonical isomorphism

$$M \simeq X^*(\mathscr{G}(U)).$$

If G is diagonalzible, we also have a canonical isomorphism

$$\mathscr{G}(X^*(G)) \simeq G.$$

Proof. We can reduce to the case that M is cyclic, due to the result on $k[M_1 \oplus M_2] = k[M_1] \otimes k[M_2]$. If $M = \mathbb{Z}$, then $k[M] \cong k[T, T^{-1}]$ which is an integral domain. If $M = \mathbb{Z}/d$ with $p \nmid d$, then $k[M] \simeq k[T]/(T^d - 1)$. The polynomial $T^d - 1$ doesn't have multiple roots since $d \nmid p$, so we get a reduced ring, hence affine variety. Now we just check the coalgebra properties manually.

Now for $m \in M$, the map is

$$m \mapsto [x \mapsto e(m)(x)]$$

where we regard $e(m) \in k[\mathscr{G}(M)]$ b the isomorphism. This is an isomorphism because Dedekind's theorem implies these characters form a basis.

The third part is a similar construction. ■

This now completes the proof of the proposition. □

We think of $\mathscr{G}$ as an "inverse" to X^* for G diagonalizable. This gives you:

Exercise 13.5. Fix k algebraically closed. The category of diagonalizable linear algebraic groups is equivalent to the opposite category of finitely generated abelian groups, via the $\mathscr{G}$ and X^* maps.

Corollary 13.6

Let G be diagonalizable. Then

(i) G is a direct product of a torus and finite abelian group with order prime to $p = \operatorname{char} k$.

(ii) G is a torus if and only if G is connected.

(iii) G is a torus if and only if $X^*(G)$ is free abelian.

Proof. First we check that $\mathscr{G}(\mathbb{Z}^n)$ is isomorphic to the diagonal matrices in k. Now we know $X^*(G) = \mathbb{Z}^n \oplus M$ for some finite abelian group M, hence $G \simeq D_n \times \mathscr{G}(M)$. ☐

§§13.2 Rigidity

Theorem 13.7

Assume G and H are diagonalizable. Let V be a connected affine algebraic variety. Assume $\phi : V \times G \to H$ is a morphism of varieties such that for all $v \in V$, the map $\phi(v, -) : G \to H$ is a morphism of algebraic groups.

Then for $x \in G$, the function $\phi(-, x)$ is constant.

Proof. Fix $v \in V$ and let $\psi \in X^*(H)$. Then

$$\psi(\phi(v, -)) \in X^*(G)$$

by composing

$$G \xrightarrow{\phi(v,-)} H \xrightarrow{\psi} \mathbb{G}_m.$$

By taking our basis, we have

$$\psi(\phi(v, -)) = \sum_{x \in X^*(G)} \underbrace{f_{x,\psi}(v)}_{\in k[V]} \chi(x).$$

By Dedekind's theorem, for every v, we see $f_{x,\chi}(v)$ is 1 for some x and 0 elsewhere. Thus $f_{x,\chi}^2 = f_{x,\chi}$. Since V is connected we get $f_{x,\chi}$ is 1 for some x and 0 elsewhere. ☐

Application: if G is *any* linear algebraic group (not necessarily diagonalizable) and H is closed in G, then $Z_G(H)$ is closed in $N_G(H)$, and $N_G(H)^0 = Z_(H)^0$; in particular the quotient $N_(H)/Z_G(H)$ is finite.

Fact 13.8. If G is a linear algebraic group then G^0 is normal in G and $|G/G^0| < \infty$.

§§13.3 Tori

§§13.4 Parabolic subgroups

Lemma 13.9 (Parabolic is transitive)

Let P be parabolic in G, and Q be parabolic in P. Then Q is parabolic in G.

Sketch of proof. We'd like G/Q to be complete. We need for all X that $\pi_X : G/Q \times X \to X$ to be closed. The idea is to rewrite π_X as the composition of closed maps (using completeness of G/P, P/Q). □

Assume this for now. Then:

Lemma 13.10

(i) Let P be parabolic in G. If Q is closed in G and contains P, then Q is itself parabolic.

(ii) P is parabolic in G if and only if P^0 is parabolic in G^0.

Proof. (i) We have a surjective morphism $G/P \twoheadrightarrow G/Q$. The image from a complete space is complete.

(ii) P is parabolic in G while P^0 is parabolic in P. Also G^0 parabolic in G, ...

□

§§13.5 Borel subgroups

Definition 13.11. A **Borel subgroup** of G is

- closed, connected, solvable, and
- maximal with respect to the above properties.

Theorem 13.12

(1) A closed subgroup of G is parabolic if and only if it contains a Borel subgroup.

(2) A Borel subgroup is parabolic.

(3) Two Borel groups in G are conjugate.

Proposition 13.13

Let $\phi : G \twoheadrightarrow G$' be a surjective morphism of linear algebraic groups.

(i) If P is parabolic in G, then $\phi(P)$ is parabolic in G'.

(ii) If B is Borel in G, then $\phi(B)$ is Borel in G'.

§14 March 29, 2016

Guest lecture by David Vogan.

§§14.1 Exercises for Tuesday April 5

Exercise 14.1. Let G be a linear algebraic group, and H a closed subgroup. Prove the following are equivalent.

(a) G/H is an irreducible algebraic variety.

(b) H meets each connected component of G.

(c) $G = G^0$.

Exercise 14.2. Give an example of a finite solvable

$$H \subseteq \mathrm{SL}(2, \mathbb{C})$$

so that H is *not* conjugate to a group of upper triangular matrices.

Exercise 14.3. Let G be a connected linear algebraic group, all elements are semisimple. Prove that G is a torus. (Consider a Borel subgroup of G.)

§§14.2 Matrices

Let

$$g = \begin{pmatrix} a_1 & x_{12} & \dots & x_{1n} \\ & a_2 & \dots & x_{2n} \\ & & \ddots & \vdots \\ & & & a_n \end{pmatrix}.$$

Clearly, g is unipotent iff all a_i are equal to 1.

Now consider semisimple. We know that:

- If all $x_{ij} = 0$ then g is semisimple.
- If all a_i are distinct, then g is semisimple.

On the other hand, for example let $n = 2$. Then g is semisimple if and only if one of the above hold.

In fact, in general

$$g \text{ semisimple} \iff x_{ij} = 0 \forall a_i = a_j.$$

For example, we can explicitly write $n = 2$ as

$$\begin{pmatrix} a_1 & x_{12} \\ 0 & a_2 \end{pmatrix} \text{ is semisimple} \iff a_1 \neq a_2 \text{ or } x_{12} = 0.$$

The interesting thing to note is that for the latter case, this condition is actually *neither open nor closed*: instead it is the union of an open set and a closed set.

Conclusion: if G is linear algebraic then we have $G_u \subseteq G$ the set of unipotent elements, which is a *closed* subset of G (not necessarily subgroup). On the other hand the set of semisimple elements G_s usually neither open nor closed.

§§14.3 Lie-Kolchin

Theorem 14.4 (Lie-Kolchin)
If $G \subseteq \mathrm{GL}(n,k)$ is a (closed) **connected** solvable subgroup then there exists $x \in \mathrm{GL}(n,k)$ so that
$$xGx^{-1} \subseteq T_n = \{\text{upper triangular}\}.$$

(Hence the second exercise earlier says that we need the "connected" assumption.)

Proof. Induction on n. Let $\mathrm{GL}(n,k)$ act on projective space $\mathbb{P}^{n-1}$. The Borel fixed point theorem implies G has a fixed point $k \cdot v$: thus v is an eigenvector across all of G. Mod out by v and induct down. □

Remark 14.5. Wikipedia: "the Borel fixed-point theorem is a fixed-point theorem in algebraic geometry generalizing the Lie-Kolchin theorem."

Thus, the right way to picture a (connected) solvable subgroup is just as a subgroup of upper triangular matrix (i.e. these are the *only* examples, up to conjugation).

Lemma 14.6
Let V be an n-dimensional vector space over an algebraically closed field k, and let S be a **commuting** set of diagonalizable endomorphisms. Then S is **simultaneously diagonalizable**.

Proof. Induction on n. If all elements of S are scalars, result is immediate.

Thus assume $s_1 \in S$ is not a scalar, so it has an eigenvalue λ_1 but isn't $\lambda_1 \cdot \mathrm{id}$. Let V_1 be the λ_1 eigenspace, and V_2 the sum of all other eigenspaces of S_1. Thus we can write
$$V = V_1 \oplus V_2$$
with $\dim V_1, \dim V_2 < n$. Thus if $s_2 \in S$ commutes with s_1 then it preserves the decomposition. So induct downwards. □

Theorem 14.7
Let G be a connected **abelian** linear algebraic group. Define G_s as the set of semisimiple elements and G_u as the set of unipotent elements. Then G_s and G_u are closed subgroups, and the group multiplication gives an isomorphism
$$G_s \times G_u \to G$$
of algebraic groups.

Proof. Assume $G \subseteq \mathrm{GL}(n,k)$.

Now, G_s and G_u closed under multiplication since G is commutative, so they are abstract groups.

Recall G_u is a closed algebraic subgroup. Since G_s is a commuting family of diagonalizable matrices, we can simultaneously diagonalize them (previous lemma).

Let D_n be the diagonal matrices of the ambient space in this basis. It consists of semisimple matrices and containts G_s, so $G_s = G \cap D_n$. This completes the proof that G_s and G_u are Zariski closed.

Now $G_s \cap G_u = \{1\}$, and so the morphism $G_s \times G_u \to G$ is injective. It is also surjective by Jordan decomposition. So it's a bijective morphism of algebraic groups. Omit proof that it's actually an isomorphism (obvious in characteristic 0, more work in p). $\square$

Corollary 14.8

Retain the setting of the previous theorem. Then G_s and G_u are connected.

Proposition 14.9 (Connected linear algebraic group of dimension 1)

If G is connected linear algebraic group of dimension 1, then it is commutative and either $G = G_s \cong k^\times = \mathbb{G}_m$ or $G = G_u \cong k = \mathbb{G}_a$.

Proof. Fix $g \in G$. Define $\varphi : G \to G$ by

$$\varphi(x) = xgx^{-1}$$

hence $\operatorname{im} \varphi$ is a conjugacy class of $g \in G$.

Since G is connected it's irreducible as a variety. Thus closure of $\operatorname{im} \varphi$ is irreducible and connected in G. But since G is one-dimensional we either have

(i) $\operatorname{im} \varphi$ is 0-dimensional, which means that g is in the center $Z(G)$.

(ii) The closure $\overline{\operatorname{im}(\varphi)}$ equals G. Thus almost everything in G is conjugate to g.

We expect *a postieri* the second case never occurs, so let's dispense of it. By hypothesis the characteristic polynomial is constant on conjugacy classes, hence on G. In particular, everything in G has the same characteristic polynomial as 1, hence in particular is unipotent. By Engel's theorem this implies that we can choose a basis in which it is upper triangular with all entries 1. Then the commutator $[G, G]$ is a proper subgroup of G, hence of lower dimension, so $[G, G] = 0$, which contradicts the assumption we're in the second case.

Thus $g \in Z(G)$ for every $g \in G$, and we know G is commutative. As before we know have an isomorphism

$$G_s \times G_u \to G$$

and again because G has dimension 1, we either have $G = G_s$ or $G = G_u$.

So now we have to show that

- A commutative connected semisimple algebraic group is copies of $\mathbb{G}_m$. (Easy, maybe do on Thursday.)
- A commutative connected unipotent algebraic group is copies of $\mathbb{G}_a$. $\square$

Remark 14.10. Look at $\mathrm{SL}(2, k)$ and $\mathfrak{sl}(2, k)$ where k has characteristic 2. Then $\mathrm{SL}(2, k)$ is not solvable, but $\mathfrak{sl}(2, k)$ is.

Remark 14.11. Look at

$$T^2 = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right\}$$

again where k has characteristic 2. This is non-abelian and solvable. But t_2 has basis $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ which is abelian.

Consider the following (will discuss Thursday):

Exercise 14.12. Let $g = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and consider $\langle g \rangle \subseteq D_n$. What is its Zariski closure?

On Thursday: will prove existence and conjugacy of maximal tori.

§15 March 31, 2016

§§15.1 Maximal Torus

Recall:

Definition 15.1. A **torus** is a connected abelian linear algebraic group consisting of semisimple elements.

Hence it's isomorphic to $(k^\times)^m$.
Given G a linear algebraic group, a **maximal torus** is a subtorus of maximal dimension.
The main theorem is

Theorem 15.2

Let G be a connected solvable linear algebraic group. Then

- Any semisimple element s of G belongs to some maximal torus.
- The centralizer $Z_G(s)$ is connected.
- Any two maximal tori in G are conjugate by the lower central series $D_\infty G$, the intersection of $D_i(G) = [G, D_{i-1}(G)]$.

Finally if $T \subseteq G$ is a maximal torus, then $G \cong T \rtimes G_u$; any $g \in G$ can be uniquely written as $t \cdot u$.

Remarks:

Corollary 15.3

Maximal tori in *any* connected linear algebraic group G are all conjugate to one another.

Proof. If T and T' are maximal in tori, extend them to maximal solvable subgroups B, B' (Borel subgroups). By conjugacy of Borel subgroups, $gBg^{-1} = B'$. Thus gTg^{-1}, T' are two maximal tori in B', thus conjugate in B'. $\square$

§§15.2 Main idea

The argument for our main theorem will differ slightly from Springer (following Humphreys' "Linear Algebraic Groups").

Let G be solvable. Thus we can think of it as a subset of the upper triangular matrices T_n for some n:

$$G \subseteq T_n = \left\{ \begin{pmatrix} * & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & * \end{pmatrix} \right\}$$

We know in that case $G_u = G \cap U_n$, a closed connected normal subgroup of G; here U_n is the unipotent elements.

Let D_n be the diagonal subgroups now. Of course we have an exact sequence

$$1 \to U_n \hookrightarrow T_n \twoheadrightarrow D_n \to 1.$$

Thus there is an exact sequence

$$1 \to G_u \hookrightarrow G \twoheadrightarrow S \to 1$$

where S is the image of G in D_n (set of diagonal matrices). The projection above is a morphism of algebraic groups, hence it has a connected images. Thus S is a connected subgroup of D_n, meaning it is a torus of some dimension $\le n$.

Consequently, if G is solvable then G/G_u is a torus S. Moreover, if T is any subtorus of G, then T maps injective into a subtorus of G/G_u, hence $\dim T \le \dim S$.

The main content is actually finding a subtrous $T \subseteq G$ such that $\dim T = \dim S$. To do this, we use induction on $\dim G$.

Let's first consider the case where G is nilpotent, meaning that for some $N > 0$ we have $D_N(G) = 1$.

Proposition 15.4

A linear algebraic group G is nilpotent if and only if G_s is a subgroup of $Z(G)$

Proof. Assume G nilpotent, $s \in G_s$. Consider the map

$$\chi : G \to G \quad \text{by} \quad x \mapsto sxs^{-1}x^{-1}$$

Since G is nilpotent, $\chi^N(x) = 1$ for N large; But χ is a morphism of algebraic varieties, and the differential $d\chi$ is $\operatorname{Ad}_s - \operatorname{id}$.

Since s is semisimple, Ad_s is semisimple. So $d\chi$ is semisimple as well. But it's also nilpotent as $(d\chi)^N = 0$, thus $d\chi = 0$. Hence $\operatorname{Ad}_s = \operatorname{id}$, which means $s \in Z(G)$. ☐

Proposition 15.5

If G is nilpotent and connected, then

(i) G_s, G_u are closed subgroups.

(ii) G_s is a central torus.

(iii) $G \cong G_s \times G_u$.

We proved this last time for commutative groups G. We can repeat this proof from before using just using the fact that G is central.

Corollary 15.6

If G is connected and solvable, then

(i) $[G, G]$ is closed, connected, unipotent, and normal.

(ii) G_u is closed, connected, unipotent, normal.

(iii) G/G_u is a torus.

(We more or less did this earlier.)

§§15.3 Proof of main theorem

If G_s is central in G, then G is nilpotent, and we're done by earlier work. (Here G_s is the unique maximal torus.)

Otherwise, pick $s \in G_s$ which is not in $Z(G)$. So Ad_s is diagonalizable, but not zero. Let H denote the centralizer of s in G, which is a (possibly disconnected) linear solvable algebraic. The corresponding Lie algebra $\mathfrak{h}$ is the zero eigenspace of Ad_s which is a *proper* subalgebra of the Lie algebra $\mathfrak{g}$ of G. Consequently,

$$\dim H < \dim G.$$

Consider $G \subseteq T_n$ and S_G the image described earlier. By induction, H has a maximal torus $T_H \cong S_H$. The aim is to show that $S_G = S_H$. We won't be able to prove this properly in this class.

§§15.4 An example!

Let

$$G = T_2 = \left\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \right\}.$$

Then

$$G_u = U_2 == \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \right\}.$$

The maximal tori are elements of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & (b-a)x \\ 0 & b \end{pmatrix}.$$

We call the set of these T_x (fix x, varying a, b).

The semisimple elements are

$$\begin{pmatrix} a & y \\ 0 & b \end{pmatrix}$$

such that $a \neq b$ or $y = 0$. If $y = 0$ then it belongs to every maximal torus, otherwise if $y \neq 0$ (hence $a \neq b$) it belongs to the maximal torus corresponding to

$$x = \frac{y}{b-a}.$$

§§15.5 Cartan subgroups

Assuming Theorem 10, it follows that maximal tori in any connected linear algebraic group are all conjugate to each other.

Definition 15.7. A **Cartan subgroup** of a connected linear algebraic group is the identity component of the centralizer of some maximal torus (which are all conjugate).

These are "not important": used by Cartan in structure theory of complex Lie algebras before there were algebraic groups. These days it is mostly a "historical curiosity".

§§15.6 Radicals

If G is connected, then define the **nilpotent radical** of G, $R_u(G)$, to be the largest unipotent normal subgroup. This is the intersection of unipotent radicals of all the Borel subgroups of G.

The **radical** of G is the largest solvable normal subgroup, which is the intersection of all Borel subgroups. Of course $R(G) \supseteq R_u(G)$.

We say G is **semisimple** if $R(G) = 1$ and **reductive** if $R_u(G) = 1$.

§16 April 5, 2016

To do: conjugacy and existence of maximal tori in connected, solvable groups.

§§16.1 Review

Quick review of connected solvable groups B. We can assume B is a closed subgroup of $\mathbf{T}_n$ by Lie-Kolchin.

Given connected commutative algebraic group C, we have $C = C_s \times C_u$ (isomorphism), where C_s and C_u are closed connected subgroups. Given connected nilpotent algebraic group N, we have $N = N_s \times N_u$ where N_s is central torus and N_u is normal in N.

But given connected solvable algebraic group B, we would like $B_s \times B_u \to B$ to be an isomorphism. But B_s is too big to be a diagonal subgroup.

Note that if T is a torus then $Z_G(T)$ contains lots of semisimple elements. However, we want to prove that

Theorem 16.1

The maximal torus T gives an isomorphism of varieties

$$T \times B_u \to B$$

where T is no longer central, but B_u is normal in B.

Note that $Z(B)_s \subseteq T$, because $z \in Z(B)_s$ lies in a maximal torus, and all maximal tori are conjugate to one another. But the central elements are stable under conjugation.

§§16.2 More on maximal tori

Corollaries of this theorem:

Corollary 16.2

Let G be a connected solvable linear algebraic group. Suppose $H \subseteq G$ is a closed subgroup consisting only of semisimple elements. Then

(1) H lives in a maximal torus.

(2) $Z_G(H)$ is connected and equal to $N_G(H)$.

Proof. First, let's see H is commutative We have a diagram

$$\begin{tikzcd} G \arrow[r] & G/G_u \\ H \arrow[u, hook] \arrow[r, equal] & H/H_u \arrow[u, hook] \end{tikzcd}$$

where G/G_u is a torus. So H is isomorphic to a subgroup of a torus, hence commutative.

If $H \subseteq Z(G)$, we're okay. Otherwise, if $s \in H$ with s not central, we know $Z_G(s)$ is connected.

So $H \subseteq Z_G(s)$, the latter being solvable (since it's a subgroup of G).

Use induction on $\dim G$; we simply need $\dim Z_G(s) < \dim G$. As $s \notin Z(G)$, we get $Z_G(s) \subsetneq G$. Thus $Z_G(s)$ is a proper, closed, connected irreducible subgroup of G.

The fact that $Z_G(H)$ is connected follows from the same induction as above. For the second part, if $x \in N_G(H)$ and $h \in H$, then $xhx^{-1}h^{-1} \in H \cap (G, G) \subseteq H \cap G_u = \{1\}$ since H was assumed to consist of semisimple elements. Therefore $x \in Z_G H)$ too. □

§§16.3 Maximal tori are conjugate

From now on G is connected, linear algebraic group, but we don't have hypotheses like "solvable". We prove anyways that

Theorem 16.3

Maximal tori are conjugate.

Proof. Fix a maximal torus T in G. Fix a Borel B in G and note B has a maximal torus T'. Since T is connected, solvable, it's contained in some Borel B'. But B' is conjugate to B, say

$$B' = gBg^{-1}.$$

Thus T is conjugate to a maximal torus in gBg^{-1} which is conjugate to T'. □

Proposition 16.4

Let T be a maximal torus in G, and let $C = Z_G(T)^\circ$.

(1) C is nilpotent, and T is a maximal torus in C.

(2) $\exists t \in T$ lying in only finitely many conjugacy classes of C.

(Here C is an example of a Cartan subgroup. We'll show in fact that $Z_G(T)$ was connected anyways.)

Proof. Observe $T \subseteq Z(C)$, since the torus T commutes with all of $Z_G(T)$, hence with C.

Now let B be a Borel subgroup of C with $T \subseteq B \subseteq C$. Then also, $T \subseteq Z(B)$, so T is normal in B. Thus B nilpotent, and $B/T \cong B_u$. We now show that

Lemma 16.5

If G is a linear algebraic group and its Borel B is nilpotent, then $G^0 = B$. Moreover, $Z(G)^0 \subseteq Z(B) \subseteq Z(G)$.

Proof of Lemma. For the inclusion, $Z(G)^\circ$ is closed, connected, commutative, so $Z(G)^\circ$ lives in a Borel. Since they're all conjugate, it lives in all of them, in particular it lives in B.

Also, let $g \in Z(B)$. We have a map $G \to G$ by

$$x \mapsto gxg^{-1}x^{-1} \qquad \text{for } g \in Z(B)$$

which vanishes on B, so it gives a map $G/B \to G$. Since G/B is projective, G is affine, it follows it is the constant map. Thus it is trivial, so $gx = xg$ and finally $g \in Z(B)$; thus $Z(B) \subseteq Z(G)$.

B is nilpotent solvable and a connected nilpotent group, so it contains a nontrivial, closed connected group H in its center. We can take the subgroup generated by maximal length commutators. ... ■ fill in

Exercise 16.6. If $\phi : G \to G'$ is a homomorphism of algebraic groups, then $\varphi(G^\circ) = \varphi(G)^\circ$.

Thus $C = B$, and C is nilpotent. □

Lemma 16.7

If S is a subtorus of G, then there exists $s \in S$ such that $Z_G(s) = Z_G(S)$.

Proof. We can assume $G = \mathrm{GL}_n$ and $S \cong \mathbf{D}_n$. Let $\chi_1, \dots, \chi_m$ be distinct characters of χ. Thus there exists an s such that $\chi_i(s) \neq \chi_j(s)$. We claim that for such an s, we have $Z_G(s) = Z_G(S)$. IT suffices to check $Z_G(s) \subseteq Z_G(S)$, but if $q \in Z_G(s)$ then q must fix all of $\chi_1, \dots, \chi_m$ which is a basis for the character group of S. □

Remark 16.8. The set of such s are dense, open in S.

We can now prove the part of the proposition: let $t \in T$ with $Z_G(t) = Z_G(T)$. If $t \in gCg^{-1}$, then $g^{-1}tg$ si semisimple, and lives in $C = Z_G(T)^\circ$. Thus $g^{-1}tg \in T$.

Check that $Z_G(g^{-1}tg) = g^{-1}Z_G(t)g \supseteq T$. So $g \in N_G(t)$.

Lemma 16.9

H is a closed subgroup of connected G. Consider the subvariety $X = \bigcup_{x\in G} xHx^{-1}$ of G. Then

(1) X contains nonempty open subsets of $\overline{X}$. Moreover, if H is parabolic, then X is closed.

(2) Assume $[N_G(H) : H] < \infty$. Moreover, there exists elements of H in only finitely many conjugates of H. Then $\overline{X} = G$.

The proof of this is mostly algebraic varieties. Proof later, possibly.

Now we prove the big theorem.

Theorem 16.10 (1) Every element of G lives in a Borel subgroup.

(2) Every semisimple element in G lives in a maximal torus.

(3) The union of the Cartan subgroups of G contains a dense open subset.

The last thing is what we'll use to prove the Cartan subgroup is all of the centralizer.

Proof. Let's assume T is a maximal torus, and let C denote its Cartan, meaning $T \subseteq C = Z_G(T)^\circ$. Let B be a Borel containing C (this exists because C is nilpotent and connected).

Apply the lemma with $H = C$. Let

$$X = \bigcup gCg^{-1}$$

so $\overline{X} = G$. Then C nilpotent implies $C \cong T \times C_u$.

We claim this implies that $N_G(T) = N_G(C)$. Since $N_G(C) \subseteq N_G(T)$ is tautological, it suffices to show that $N_G(T) \subseteq N_G(C)$. Assume $g \in N_G(T)$, so $GTg^{-1} = T$. Thus $gTC_ugi^{-1} = gTg^{-1}gC_ug^{-1} = TgC_ug^{-1}$. This is nilpotent, but T is the unique central torus, so T commutes with gC_ug^{-1}. Thus $gC_ug^{-1} \subseteq Z_G(T)^\circ = C$.

Proof to be finished next time. $\square$

§17 April 14, 2016

(Note: was out sick the last week.)

Let G be a linear algebraic group. Then we can identify the Borel subgroups of G, say $\mathscr{B}$, with the "flag variety" G/B by the map

$$G/B \ni gB \mapsto gBg^{-1} \in \mathscr{B}$$

This is a G-equivariant map, if G acts on G/B by left multiplication and on $\mathscr{B}$ by conjugation.

We can also do this for parabolics.

§§17.1 Vector bundles

Let V be a representation of a Borel B. Then we have a vector bundle

$$\pi : G \times_B V \to G/B$$

Note that $H^0(G/B, V)$ gives a G-representation.

As a special case, let $T \subseteq B$ be a maximal torus. Given $\lambda \in X^*(T)$, we have k_λ an irreducible representation of B.

Now let V be a representation of G. We can restrict it to a representation of T, giving

$$V = \bigoplus_{\lambda \in X^*(T)} V_\lambda.$$

For instance, if $\lambda \in x^*(T)$, then we want to find which $\mu \in X^*(T)$ are weights in $H^0(\lambda)$.

§§17.2 Interlude

For G connected reductive, we will actually see

$$G = BWB = \bigsqcup_{w \in W} B\widetilde{w}B$$

for fixed B and T. Here $w \in W = W(G,T) = N_G(T)/C_G(T)$, and $\widetilde{w}$ is the lift of W and $G/B = \bigsqcup B\widetilde{w}B/B$,

There are finitely many B-orbits in bijection with W. Let $U = B_u$ be the unipotent radical of B. In fact, as a variety $U \cong \mathbb{A}^n$ (not as a group).

So G/B has a cellular decomposition in $\mathbb{C}$, or an "étale topology" for $\overline{\mathbb{F}_q}$. So computing $H^\bullet(G/B)$ or at least its dimensions is easy, determined by W and the action of W on X and ...

§§17.3 Weyl group

Let $W = N_G(T)/Z_G(T)$ be a Weyl group now, acting on $X = X^*(T)$. Convince yourself this makes sense, and that the action is faithful.

We can identify W with some group of automorphisms of X. Let P denote the weights already in $\mathfrak{g}$, and let P' denote the weights

$$P' = \{\alpha \in P \mid G_\alpha = C_G((\ker \alpha)^\circ) \text{ not solvable}\}.$$

If S is a subtorus of T then

$$W(C_G(S), T) \subseteq W(G,T)$$

is a subgroup, and $S \subseteq Z(G)$. Moreover, the map $G \twoheadrightarrow G/S$ induces

$$W(G,T) \simeq W(G/S, T/S).$$

Additional exercise: W is in bijection with the Borels containing T.

Finally for $B \supset T$, there is a bijection

$$W \leftrightarrow (G/B)^T$$

where the right-hand side are the T-fixed points in G/B, acted on by left multiplication. To see this, note that if $tgB = gB \iff gtg^{-1} \in B \forall t \in T$, then $\exists b \in B$ such that $b^{-1}g^{-1}tgb \in T$. So $gb \in N_G(T) \supset N_G(B) = B$, thus $g \in N_G(T)$.

Now fix $\alpha \in P'$, $S = (\ker \alpha)^\circ \subseteq G_\alpha = Z_G((\ker \alpha)^\circ)$. Then $S \subseteq Z(G_\alpha)$, and

$$W_\alpha = W(G_\alpha, T) \cong W(G_\alpha/S, T/S).$$

We have $T/S \cong \mathbb{G}_m$ (why is this)? Thus $|W_\alpha| \le 2$, based on automorphisms of $X^*(\mathbb{G}_m)$.

Proposition 17.1

Assume G is *not* solvable and $\dim T = 1$. Then $|W| = 2$, and $\dim G/B = 1$.

Simple special case: look at SL_2, and look at the Borel groups

$$B^+ = \left\{ \begin{bmatrix} a & * & 0 & a^{-1} \end{bmatrix} \right\} \qquad B^- = \left\{ \begin{bmatrix} a & 0 & * & a^{-1} \end{bmatrix} \right\}.$$

They both contain the torus T of diagonal matrices.

Proof. Fix an isomorphism $\lambda : \mathbb{G}_m \to T$, and $B \supset T$. Recall we had a closed map $G/B \to \mathbb{P}V$, and identified G/B with its image in this map. (Here V is as in the proof that G/H is a variety.) Let $\phi : G \to \mathrm{GL}(V)$ be the corresponding representation.

Let $e_1, \dots, e_n$ be a basis of V. ... $\square$

Example: let $G = \mathrm{SL}_2$. Then $G/B \cong \mathbb{P}^1$, with south pole $[0,1]$ and north pole $[1,0]$, say (think of $\mathbb{P}^1$ as Riemann sphere). Let SL_2 act on G/B in the usual fashion:

$$\begin{bmatrix} a & \\ & a^{-1} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} a & \\ & a^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ a^{-1} \end{bmatrix}.$$

There are two $\mathbb{G}_m$-fixed points.

§18 April 21, 2016

§§18.1 Setup

Let G be a connected linear algebraic group, and $T \subseteq G$ a maximal torus. Let P be the weights of T acting on $\mathfrak{g}$ via the adjoint action, thus $P \subseteq X \overset{\text{def}}{=} X^*(T)$. We let P' be the subset of $x \in P$ for which $G_x \overset{\text{def}}{=} Z_G\left((\ker x)^\circ\right)$ is not solvable. (We'll actually use G_α later, $\alpha \in P'$.) Eventually, we'll start calling P' the "roots".

Last time, we saw that if G is not solvable and $\dim T = 1$, then they Weyl group W has order $|W| = 2$ and the dimension of the flag variety G/B is 1. The Weyl group W acts on $X^*(T)$ in the same way as $\mathrm{GL}_n(\mathbb{Z})$. In the situation as above, $W \cong \mathbb{Z}/2$ acts on $X^*(T) = X^*(\mathbb{G}_m) = \mathbb{Z}$ by negation; on the level of tori this means $ntn^{-1} = t^{-1}$.

For $\alpha \in P'$, let $W_\alpha = W(G_\alpha, T) \subseteq W(G,T)$ have order two and let $n_\alpha \in N_{G_\alpha}(T) - Z_{G_\alpha}(T)$. Let s_α be the image of $n_\alpha \in W = W(G,T)$, and let $X^\vee = \mathrm{Hom}(X, \mathbb{Z}) = X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$. We get a pairing

$$\langle , \rangle \; X \times X^\vee \to \mathbb{Z}.$$

We can view $X \subseteq X \otimes_{\mathbb{Z}} \mathbb{R} = V$, and $X^\vee \subseteq X^\vee \otimes_{\mathbb{Z}} \mathbb{R} = V^\vee$. Thus we get a form on V, $V^\vee$.

Take some symmetric bilinear positive definite form, say $f : V \times V \to \mathbb{R}$, and average it with respect to W to get a W-invariant form

$$(x, y) \overset{\text{def}}{=} \sum_{w \in W} f(w \cdot x, w \cdot y)$$

which induces a metric on V and is invariant with respect to W.

§§18.2 Euclidean reflections

This implies s_α is a Euclidean reflection, which implies

$$s_\alpha(x) = x - 2(\alpha, \alpha)^{-1}(x, \alpha)\alpha.$$

Example 18.1

Let $G = \mathrm{SL}_2(\mathbb{C})$. Then W has order two, so we can put $W = (\mathrm{id}, s)$. The vector space spanned by s has dimension 1. Get $s_\alpha(\alpha) = -\alpha$.

Here T is the diagonal matrices $T = \mathbb{G}_m = \left[a \quad 0 | \; 0 \quad a^{-1}\right]$, and $X = X^*(T) = \mathbb{Z}$. So $V = X \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}$.

Example 18.2 (A_2)

Let $G = \mathrm{SL}_3(\mathbb{C})$, so that T is the torus of diagonal matrices (of dimension 2), and $X = X^*(T) = \mathbb{Z}^{\oplus 2}$. So $V = X \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}^2$.

Now consider two-space, and let α be on the real axis, and let β be a cube root of unity. (Thus $\alpha + \beta$ is a sixth root of unity.) So we get six roots α, $-\alpha$, β, $-\beta$, $\alpha + \beta$, $-(\alpha + \beta)$.

Allow reflections orthogonal to these elements. This gives S_3.

Lemma 18.3

There exists a unique $\alpha^\vee \in V^\vee$ with $\langle \alpha, \alpha^\vee \rangle = 2$ and for any $x \in X$, we have

$$s_\alpha(x) = x - \left\langle x, \alpha^\vee \right\rangle \alpha.$$

Also, if $\beta \in P'$ and $G_\alpha = G_\beta$ then $s_\alpha = s_\beta$.

Proof. First part: $\alpha^\vee$ has ta satisfy $\langle x, \alpha^\vee \rangle = 2(\alpha, \alpha)^{-1}(x, \alpha)$ for $x \in V$, which is uniquely determined. Second part: if $G_\alpha = G_\beta$ we can choose $n_\alpha = n_\beta$, thus $s_\alpha = s_\beta$. □

Theorem 18.4

W is generated by s_α for $\alpha \in P'$.

Proof. Induction on $\dim G$. Blargh. □

Definition 18.5. The **rank** of G is defined as $\dim T$, and its **semisimiple rank** is the rank of $G/R(G)$.

Lemma 18.6

We have

- G is the disjoint of B and UnB,
- $R(G) = (U \cap nUn^{-1})^\circ$,
- $\dim U/U \cap nUn^{-1} = 1$.

Lemma 18.7

Assume G has semisimple rank one. Then

- $\dim U = 1$, $Z_G(T) = T$, and $U \cap nUn^{-1} = \{\mathrm{id}\}$.
- There is a unique weight α of T in $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_\alpha \oplus t \oplus \mathfrak{g}_{-\alpha}$, where $\mathfrak{g}_\alpha = \mathrm{Lie}(U)$ and $\mathfrak{g}_\alpha = \mathrm{Lie}(nUn^{-1})$.

§19 May 3, 2016

Homework 6 due officially May 13. See webpage for details.

§§19.1 Some functors

Let G be a linear algebraic group, and $H \subset G$ a closed subgroup. In what follows the category

- Mod_G is G-modules (allows infinite dimensional modules). (One correct way to do this is to use a group scheme.)
- $\mathrm{Rep}\, G$ is rational representations, all finite dimensional.

One can show directly that any G-module is locally finite: for all $m \in M$, then $kG \cdot m$ is finite dimensional over k. (Follows by multiplicative Jordan decomposition.)

Then we have the functor

$$(-)^H : \mathrm{Mod}\, G \to \mathrm{Vect}\, k$$

which is left-exact.

Lemma 19.1

Let H, H' be closed subgroups in G with H' normalizing H. Then for $M \in \mathrm{Mod}\, G$, we can regard $M^H \in \mathrm{Mod}\, H'$.

Proof. If $m \in M^H$, $\widetilde{h} \in H'$, we wish to check $\widetilde{h} \cdot m \in M^H$. If $h \in H$, we have $h(\widetilde{h} \cdot m) = \widetilde{h}(h'm) = \widetilde{h}m$. □

Another functor is the restriction functor

$$\mathrm{Res}_H^G : \mathrm{Mod}\, G \to \mathrm{Mod}\, H$$

This is exact, and is the identity if we take the forget functor to $\mathrm{Vect}\, k$.

More interesting is the induction functor

$$\mathrm{Ind}_H^G : \mathrm{Mod}\, G \to \mathrm{Mod}\, H$$

It's defined as follows. Let $M \in \mathrm{Mod}\, H$; then $M \otimes_K k[G]$ is a $G \times H$ module under:

- G acts trivially on M.
- G acts via the left regular representation on $k[G]$
- H acts as usual on M
- H acts via right regular representation.

In symbols,

$$(g, h) \cdot (m \otimes f) = (h \cdot m) \otimes \rho_r(h)\rho_\ell(g)f.$$

Here, $\rho_r(h)\rho_\ell(g)f(q) = f(g^{-1}qh)$.

Now: $1 \times H$ is a subgroup of $G \times H$ normalized by $G \times 1$, so we can apply the previous lemma. In short:

$$\mathrm{Ind}_H^G\, M \overset{\text{def}}{=} (M \otimes kk[G])^H \in \mathrm{Mod}\, G.$$

Example 19.2
We have $\operatorname{Ind}_1^G k = k[G]$, the latter being infinite dimensional.

This functor is only left exact in general, but if H is diagonalizable then Ind_H^G is exact.

All these functors are additive and commute with direct sum, intersection, direct limits, etc.

Now, we seek the relation between Ind and Res. Recall we have the injection $1 \hookrightarrow G$, which gives the counit

$$\varepsilon_G : k[G] \to k$$

which is "evaluation at 1".

Proposition 19.3 (Frobenius reciprocity)
Let $M \in \operatorname{Mod} H$, define $\varepsilon_M : M \otimes k[G] \to M$ by $\varepsilon_M = \mathrm{id}_M \varepsilon_G$. Then

- ε_M induces a map $\operatorname{Ind}_H^G M \to M$ which is an H-module homomorphism.
- For all $N \in \operatorname{Mod} G$, then $\varphi \mapsto \varepsilon_M \circ \varphi$ is an isomorphism

$$\operatorname{Hom}_G(N, \operatorname{Ind}_H^G M) \to \operatorname{Hom}_H(\operatorname{Res}_H^G N, M).$$

Thus Ind_H^G and Res_H^G are adjoint.

Thus, we have an isomorphism of functors

$$\operatorname{Ind}_{H_2}^G \operatorname{Ind}_{H_1}^{H_2} \simeq \operatorname{Ind}_{H_1}^G$$

from the corresponding (obvious) property for Res.

§§19.2 Root systems

From now on: G connected, reductive and $T \subset G$.

Choose a system of positive roots R^+ such that $B^+ \to R^+(B^+)$; thus B is the opposite Borel.

Then W takes a system of positive roots to a system of positive roots.

There exists a unique longest $w_0 \in W$ so that $w_0(R^+) = -R^+$. Thus B is the Borel corresponding to $-r^+$, and $B \cap B^+ = t$. Let $B^u = U$, $B_u^+ = U^+$.

Definition 19.4. Let $\lambda, \mu \in X^*(T)$. Say $\lambda \le \mu$ if $\mu - \lambda$ is a sum of positive roots.

Let B be the Borel Since U, U^+ are unipotent for any nonzero $M \in \operatorname{Mod} G$, we have $M^u \neq 0 \neq M^{U^+}$. Now: T normalizes U and U^+ and T acts on

$$M^{U^+} = \bigoplus_{\lambda \in X^*(T)} M_\lambda^{U^+}.$$

So there exists $\lambda, \lambda' \in X^*(T)$ with

$$\operatorname{Hom}_B(k_\lambda, M) \neq 0 \neq \operatorname{Hom}_{B^+}(M, k_{\lambda'}).$$

We have $B \twoheadrightarrow T \xrightarrow{\lambda} \mathbb{G}_m$.

Now if $\dim M < \infty$, then we can apply the above to M^* to get there exists λ, λ' such that

$$\operatorname{Hom}_B(M, k_\lambda) \neq 0 \neq \operatorname{Hom}_{B+}(M, k_{\lambda'}).$$

Applying Frobenius reciprocity, we get

$$\operatorname{Hom}_G(M, \operatorname{Ind}_G^B k_\lambda) \neq 0 \neq \operatorname{Hom}_G(M, \operatorname{Ind}_{B+}^G k_{\lambda'}).$$

Notation: if N is a B-module, we set

$$\begin{aligned} H^i(N) &= R^i \operatorname{Ind}_B^G(N) \\ H^0(N) &= R^0 \operatorname{Ind}_B^G(N) = \operatorname{Ind}_B^G(N). \end{aligned}$$

This notation comes from sections of line bundles on G/B. Note G/T and G/B are homotopy equivalent.

§20 May 12, 2016

Reference: chapters 1-8 of Humphrey's book on category of $\mathcal{O}$-modules.

Let $\mathfrak{g}$ be a semisimple over $\mathbb{C}$, decomposing as

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}.$$

Recall the PBW theorem tells us that ordered monomials in a basis of $\mathfrak{g}$ correspond to a basis of

$$U(\mathfrak{g}) \cong U(\mathfrak{n}^-) \oplus U\mathfrak{h} \oplus U\mathfrak{n}.$$

Define for $\lambda \in \mathfrak{h}^*$, where C_λ acts on $U\mathfrak{b}$, the induced representation

$$\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda = U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}_\lambda = M(\lambda).$$

We call $M(\lambda)$ a **Verma module**, which is infinite dimensional. Thus $M(\lambda)$ is a $U(\mathfrak{n}^-)$ module.

Can also deduce: there's a mximal vector $v^=$ with $U(\mathfrak{n}) \cdot v^+ = 0$, and $M(\lambda) = U(\mathfrak{g}) \cdot v^+ = U(\mathfrak{n}^-) \cdot v^+$. The upshot of this is that characters of $M(\lambda)$ are not polynomial in e^μ, $\mu \in \mathfrak{h}^*$; rather we use Laurent series, which are easy to describe.

Each irreducible $\mathfrak{g}$-rep occurs as a quotient of Verma modules, and is the unique irreducible quotient; $L(\lambda)$ is irreducible for each $\lambda \in \mathfrak{h}^*$.