Math 55b Lecture Notes

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This is Harvard College's famous *Math 55b*, instructed by Dennis Gaitsgory. The formal name for this class is "Honors Real and Complex Analysis" but it generally goes by simply "Math 55b". It is an accelerated one-semester class covering the basics of analysis, primarily real but also some complex analysis.

The permanent URL is `http://web.evanchen.cc/~evanchen/coursework.html`, along with all my other course notes.

Special thanks to W Mackey for providing notes on the several day that I slept through class.

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§1 January 29, 2015

Grading system for 55B with $\simeq$ 55A.

We'll be more or less following Baby Rudin.

§1.1 Definition and Examples

This subsection is copied from my Napkin project.

Definition 1.1. A **metric space** is a pair (M, d) consisting of a set of points M and a **metric** $d : M \times M \to \mathbb{R}_{\geq 0}$. The distance function must obey the following axioms.

- For any $x, y \in M$, we have $d(x, y) = d(y, x)$; i.e. d is symmetric.
- The function d must be **positive definite** which means that $d(x, y) \geq 0$ with equality if and only if $x = y$.
- The function d should satisfy the **triangle inequality**: for all $x, y, z \in M$,

$$d(x, z) + d(z, y) \geq d(x, y).$$

Abuse of Notation 1.2. Just like with groups, we will abbreviate (M, d) as just M.

Example 1.3 (Metric Spaces of $\mathbb{R}$)

(a) The real line $\mathbb{R}$ is a metric space under the metric $d(x, y) = |x - y|$.

(b) The interval $[0, 1]$ is also a metric space with the same distance function.

(c) In fact, any subset S of $\mathbb{R}$ can be made into a metric space in this way.

Example 1.4 (Metric Spaces of $\mathbb{R}^2$)

(a) We can make $\mathbb{R}^2$ into a metric space by imposing the Euclidean distance function

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

(b) Just like with the first example, any subset of $\mathbb{R}^2$ also becomes a metric space after we inherit it. The unit disk, unit circle, and the unit square $[0, 1]^2$ are special cases.

Example 1.5 (Taxicab on $\mathbb{R}^2$)

It is also possible to place the following **taxicab distance** on $\mathbb{R}^2$:

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|.$$

For now, we will use the more natural Euclidean metric. (One can also use max instead of a sum.)

Example 1.6 (Metric Spaces of $\mathbb{R}^n$)

We can generalize the above examples easily. Let n be a positive integer. We define the following metric spaces.

(a) We let $\mathbb{R}^n$ be the metric space whose points are points in n-dimensional Euclidean space, and whose metric is the Euclidean metric

$$d\left((a_1, \dots, a_n), (b_1, \dots, b_n)\right) = \sqrt{(a_1 - b_1)^2 + \dots + (a_n - b_n)^2}.$$

This is the n-dimensional **Euclidean space**.

(b) The open **unit ball** B^n is the subset of $\mathbb{R}^n$ consisting of those points $(x_1, \dots, x_n)$ such that $x_1^2 + \dots + x_n^2 < 1$.

(c) The **unit sphere** S^{n-1} is the subset of $\mathbb{R}^n$ consisting of those points $(x_1, \dots, x_n)$ such that $x_1^2 + \dots + x_n^2 = 1$, with the inherited metric. (The superscript $n-1$ indicates that S^{n-1} is an $n-1$ dimensional space, even though it lives in n-dimensional space.) For example, $S^1 \subseteq \mathbb{R}^2$ is the unit circle, whose distance between two points is the length of the chord joining them. You can also think of it as the "boundary" of the unit ball B^n.

Example 1.7 (Discrete Space)

Let S be any set of points (either finite or infinite). We can make S into a **discrete space** by declaring the following distance function.

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

If $|S| = 4$ you might think of this space as the vertices of an regular tetrahedron, living in $\mathbb{R}^3$. But for larger S it's not so easy to visualize...

Example 1.8 (Graphs are Metric Spaces)

Any connected simple graph G can be made into a metric space by defining the distance between two vertices to be the graph-theoretic distance between them. (The discrete metric is the special case when G is the complete graph on S.)

Example 1.9 (Function Space)

We can let M be the space of integrable functions $[0,1] \to \mathbb{R}$ and define the metric by $d(f,g) = \int_0^1 |f-g| \ dx$.

§1.2 Continuous Maps

This is again largely excerpted from my Napkin project.

Abuse of Notation 1.10. For a function f and its argument x, we will begin abbreviating $f(x)$ to just fx when there is no risk of confusion.

In calculus you were also told (or have at least heard) of what it means for a function to be continuous. Probably something like

> A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at a point $p \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x - p| < \delta \implies |fx - fp| < \varepsilon$.

Question 1.11. Can you guess what the corresponding definition for metric spaces is?

All we have do is replace the absolute values with the more general distance functions: this gives us a definition of continuity for any function $M \to N$.

Definition 1.12. Let $M = (M, d_M)$ and $N = (N, d_N)$ be metric spaces. A function $f : M \to N$ is **continuous** at a point $p \in M$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d_M(x, p) < \delta \implies d_N(fx, fp) < \varepsilon.$$

Moreover, the entire function f is continuous if it is continuous at every point $p \in M$.

Notice that, just like in our definition of an isomorphism of a group, we use both the metric of M for one condition and the metric of N on the other condition.

Example 1.13

Let M be any metric space and D a discrete space. When is a map $f : D \to M$ continuous? Any map $D \to M$ is continuous.

Proof. Take an open ball of radius $\frac{1}{2}$. □

Example 1.14

The map $\mathbb{R} \to \mathbb{R}$ by $x \mapsto x^2$ is continuous. So is $x \mapsto x^3$.

Proof. Homework. □

Example 1.15

Let $X = \mathbb{R}^n$ with one product metric and let $Y = \mathbb{R}^n$ with another product metric. Then $\mathrm{id} : X \to Y$ is continuous. Thus we will generally not be pedantic about the choice of metric.

§1.3 Forgetting the Metric – Topological Spaces

Again excerpted from the Napkin project.

Definition 1.16. A **topological space** is a pair $(X, \mathcal{T})$, where X is a set of points, and $\mathcal{T}$ is the **topology**, which consists of several subsets of X, called the **open sets** of X. The topology must obey the following four axioms.

- $\varnothing$ and X are both in $\mathcal{T}$.
- Finite intersections of open sets are also in $\mathcal{T}$.
- Arbitrary unions (possibly infinite) of open sets are also in $\mathcal{T}$.

Abuse of Notation 1.17. We refer to the space $(X, \mathcal{T})$ by just X. (Do you see a pattern here?)

Example 1.18
We can declare all sets are open: a **discrete space** is a topological space in which every set is open.

Example 1.19
Declare U open if and only if $\forall x \in U$, the ball of radius r centered at x is contained in U.

Proof. Check all the axioms. Blah. □

Proposition 1.20
Let M be a metric space. Then for all $x \in M$ and $r > 0$, the ball

$$B(x,r) = \{y \mid d_M(x,y) < r\}$$

is open.

Proof. Pick y in the ball. Let $t = d(x,y)$. Then $t < r$, so pick ε with $t + \varepsilon < r$. You can check using the triangle inequality that $B(y,\varepsilon) \subseteq B(x,r)$. □

Example 1.21
$[0,1]$ is not open since no ball at 0 is contained inside it.

Definition 1.22. A subset Y is closed iff $X - Y$ is open.

Definition 1.23. A function $f : X \to Y$ of topological spaces is **continuous** at $p \in X$ if the pre-image of any neighborhood of fp is also a neighborhood of p.

With some effort, we can show this is the same definition of continuity as with metric spaces.

§1.4 Intermediate Value Theorem

Theorem 1.24 (IVT)
Let $f : [0,1] \to \mathbb{R}$ be continuous such that $f(0) < 0$ and $f(1) > 0$. Then $\exists a \in [0,1]$ such that $f(a) = 0$.

This theorem is not cheap, and requires the following theorem.

Theorem 1.25
Let $A \subseteq \mathbb{R}$ be bounded above. Then there exists a **least** upper bound $y \in \mathbb{R}$.

Proof of IVT. Long and boring. Just draw a picture. The main point is to take y to be the least upper bound of the a for which $f(a) \le 0$. □

"This is just a game of quantifiers. To do this, all you have to do is be sober, which is not a problem since you are all under 21.

. . . I cannot do this past 6PM, not because I'm not sober, but because I am old."

– Dennis Gaitsgory

§2 February 3, 2015

Didn't attend class.

§2.1 Sequential Continuity

Continuous functions send convergent sequences to convergent sequences, limits sent to limits.

The converse is true in metric spaces.

§2.2 Cauchy Completeness

So far we can only talk about sequences converging if they have a limit. But consider the sequence $x_1 = 1$, $x_2 = 1.4$, $x_3 = 1.41$, $x_4 = 1.414$, $\dots$. It converges to $\sqrt{2}$ in $\mathbb{R}$, of course. But it fails to converge in $\mathbb{Q}$. And so somehow, if we didn't know about the existence of $\mathbb{R}$, we would have *no idea* that the sequence (x_n) is "approaching" something.

That seems to be a shame. Let's set up a new definition to describe these sequences whose terms eventually get close to each other, but don't necessarily converge to a point.

Definition 2.1. Let $x_1, x_2, \dots$ be a sequence which lives in a metric space $M = (M, d_M)$. We say it is **Cauchy** if for any $\varepsilon > 0$, we have

$$d_M(x_m, x_n) < \varepsilon$$

for all sufficiently large m and n.

Note that, unlike the rest of this chapter, this is a notion which applies only to metric spaces. In a general topological space there is not a good enough notion of "distance" to make this definition work. make sure

Question 2.2. Show that a sequence which converges is automatically Cauchy. (Draw a picture.)

Now we can define the following.

Definition 2.3. A metric space M is **complete** if every Cauchy sequence converges.

Most metric spaces aren't complete, like $\mathbb{Q}$. But it turns out that every metric space can be *completed* by "filling in the gaps" somehow, resulting in a space called the **completion** of the metric space. The construction is left as an (in my opinion) fun problem.

It's a theorem that $\mathbb{R}$ is complete. To prove this I'd have to define $\mathbb{R}$ rigorously, which I won't do here (yet). In fact, there are some competing definitions of $\mathbb{R}$. It is sometimes defined as the completion of the space $\mathbb{Q}$. Other times it is defined using something called *Dedekind cuts*. For now, let's just accept that $\mathbb{R}$ behaves as we expect and is complete.

Example 2.4 (Examples of Complete Sets)

(a) $\mathbb{R}$ is complete.

(b) The discrete space is complete, as the only Cauchy sequences are eventually constant.

(c) The closed interval $[0, 1]$ is complete.

(d) $\mathbb{R}^n$ is in fact complete as well. (You are welcome to prove this by induction on n.)

Example 2.5 (Non-Examples of Complete Sets)

(a) The rationals $\mathbb{Q}$ are not complete.

(b) The open interval $(0, 1)$ is not complete, as the sequence $x_n = \frac{1}{n}$ is Cauchy but does not converge.

§2.3 $\mathbb{R}$ is Complete

Theorem 2.6 (Bolzano-Weirestraß)

Any sequence in $[0, 1]$ has a convergent subsequence.

§2.4 1-countable and Hausdorff

Most of the "nice" properties of metric spaces carry over to 1-countable Hausdorff general topological subspaces.

§3 February 5, 2015

Last time we defined Cauchy sequences. Cool.

Today we will complete a metric space!

Theorem 3.1

Let X be a metric space.

(a) There exists an $\overline{X}$ and an isometric embedding $\iota : X \to \overline{X}$ so that $\overline{X}$ is complete and $\operatorname{im} \iota$ is dense in X.

(b) Any isometric embedding $f : X \to Y$ with Y complete factors uniquely through $\overline{X}$ via an isometric embedding. (In other words, $\overline{X}$ is universal with this property.)

(c) In (b), if $f``(X)$ is dense than $\overline{f}$ is an isometry.

Here a **dense** set $X' \subseteq X$ means that every neighborhood of X contains a point of X'. Also, $f``(X)$ just means $\{f(x) \mid x \in X\}$.

Corollary 3.2

The completion of $\mathbb{Q}$ is $\mathbb{R}$.

Proof. We have a dense isometry $\mathbb{Q} \hookrightarrow \mathbb{R}$. □

Okay, it's not actually that straightforward, you need some properties of $\mathbb{R}$...

We will assume in what follows that $\mathbb{R}$ is complete.

§3.1 Completion

Consider the set of all Cauchy sequncees of X. We define a metric on them by

$$\rho\left(\{x_n\}, \{y_n\}\right) = \lim_n \left(\rho(x_n, y_n)\right).$$

Actually, we first need to show that this limit exists.

Lemma 3.3

If $\{x_n\}$, $\{y_n\}$ are Cauchy, then the sequence $\rho(x_n, y_n) \in \mathbb{R}$ as $n = 1, 2, \dots$ is Cauchy.

Proof. By "Quadrilateral Inequality",

$$|\rho(x_n, y_n) - \rho(x_N, y_N)| \le \rho(x_n, x_N) + \rho(y_n, y_N).$$

Using the $\varepsilon/2$ trick completes the proof. □

Assuming that $\mathbb{R}$ is complete (COUGH COUGH), this now implies that $\lim_n \left(\rho(x_n, y_n)\right)$ exists. You can check it also satisfies the Triangle Inequality.

Unfortunately, it's easy to find examples with $\rho(\{x_n\}, \{y_n\}) = 0$, with $(x_n) \ne (y_n)$. So we "mod out" by this.

Hence, we define $C(X)$ as the set of all Cauchy sequences modded out by the relation $x_n \sim y_n$ if $\rho(\{x_n\}, \{y_n\}) = 0$. The rest is left as homework.

§3.2 Why no compactness

Lemma 3.4

A function $f : [0,1] \to \mathbb{R}$ is bounded above.

Proof. Otherwise we get a sequence of points x_1, x_2, ... such that $f(x_m) > m$ for all m. Then we can find a convergent subsequence using Bolzano-Weierestraß. This breaks sequential continuity. □

Theorem 3.5

Any function $f : [0,1] \to \mathbb{R}$ has a global maximum.

Proof. By the lemma, $f``[0,1]$ is bounded above and we can take the least upper bound y. We claim this is actually in the limit. If not, then we can construct a sequence $x_1, x_2, \dots$ such that $f(x_m) \ge y - \frac{1}{m}$. Take a convergent subsequence by Bolzano-Weierestraß. Then (x_m) converges to some $x \in [0,1]$, but then we must have $f(x) = y$. □

This sucks. Compactness is better. Rawr.

Second Proof, from Jane. As before, there is a least upper bound y. Assume for contradiction that the bound is never obtained. Then the function $[0,1] \to \mathbb{R}$ by

$$x \mapsto \frac{1}{y - f(x)}$$

is an unbounded continuous function on $[0,1]$, impossible. □

§3.3 Sequential Compactness

Definition 3.6. A topological space is **sequentially compact** if every sequence has a convergent subsequence.

Proposition 3.7

Let $f : X \to Y$. If X is sequentially compact then $f(X)$ is sequentially compact.

Proof. Trivial. If (fx_n) is a sequence, take a converge subsequence of $(x_n) \in X$. □

Theorem 3.8 (Heine-Borel)

$A \subseteq \mathbb{R}$ is sequentially compact if and only if it closed and bounded.

We can now kill the original maximum theorem. Given $f : [0,1] \to \mathbb{R}$, its image in $\mathbb{R}$ is bounded and closed and we can easily use this to show that f has a maximum.

To prove Heine-Borel, we first prove one direction in greater generality.

Lemma 3.9

Let $Y' \subseteq Y$, where Y' is sequentially compact and Y is 1-countable and Hausdorff. Then Y' is closed in Y.

Proof. Because Y is 1-countable, we can use the sequence definition of closed in the tricky direction: it suffices to prove that if $y_1, y_2, \dots \to y$ with $y_i \in Y'$ then $y \in Y'$. But y_i has a convergent subsequence to some y'. It had also better converge to y. By Hausdorff, $y = y'$. □

Lemma 3.10

Suppose $Y' \subseteq Y$ for some topological spaces, with Y a Hausdorff space. If Y' is closed and Y is sequentially compact then Y' is sequentially compact.

Proof. Triviality: given y_n in Y' use sequential compactness of Y to force some subsequence to converge to $y \in Y$. Since Y' is closed, $y \in Y'$. □

Proof of Heine-Borel. First, suppose A is bounded and closed; by bounded-ness it lives in some $A \subseteq [a, b]$. By Bolzano-Weirestraß, the set $[a, b]$ is sequentially compact. By our lemma, $A \subseteq [a, b]$ is sequentially compact.

For the converse, suppose A is sequentially compact. We did a lemma that show A is closed. Hence A is bounded. □

§3.4 Compactness

Apparently Gaitsgory does not like sequential compactness QQ.

Definition 3.11. A topological space is compact if every open cover has a finite subcover.

Proposition 3.12

If $f : X \to Y$ is continuous and Y is compact then $f(X)$ is compact.

Proof. Tautological using the definition of continuity. □

We can mirror this for sequential compactness now.

Lemma 3.13

Let Y be a Hausdorff space and Y' a compact subspace. Then Y' is closed.

Proof. We will show that for any $y \in Y \setminus Y'$, there is a neighborhood $U \subseteq Y \setminus Y'$ containing y. For every $y' \in Y'$, we can use the Hausdorff condition to find neighborhoods $U_{y'} \ni y'$ and $V_{y'} \ni y$. By definition, Y' is covered by $U_{y'}$, and we can find a finite subcover

$$Y' \subseteq \bigcup_{i=1}^{n} U_{y_i}$$

Now take

$$\bigcap_{i=1}^{n} V_{y_i} \ni y.$$

This is a neighborhood of $y \in Y$ disjoint from Y'. $\square$

Proposition 3.14

If X is first-countable and X is compact, then it is sequentially compact.

Proof. Suppose not. Let $\{x_n\}$ be a sequence with no convergent subsequence. Using 1-countable, we find that for every $x \in X$, there exists a U_x contains only finitely many elements of the sequence. But then we can take a finite subcover $\bigcup U_x$. $\square$

Proposition 3.15

If X is *second-countable*, then sequentially compact implies compact.

Here **second countable** means there is a countable base.

§4 February 12, 2015

"Today we will be doing applied math" – Gaitsgory
"This can't be happening"
"Applied by my standards".

§4.1 Warm-Up

Lemma 4.1

Let a_i, b_i be sequences converging to a and b. Then $a_i + b_i \to a + b$, $a_i b_i \to ab$.

Proof. The maps $+, \times : \mathbb{R}^2 \to \mathbb{R}$ are continuous. □

§4.2 "Applied" Math

Also known as: cooking up stupid bounds.

Also known as: suppose $a_n \le x_n \le b_n$ for all n, where a_n and b_n are the locations of two cops and x_n is the location of a drunkard.

Now a surprisingly nontrivial statement.

Example 4.2

Let $|a| < 1$. Then the sequence $(a^n)_n$ converges to 0.

Proof. WLOG $a > 0$ (the other case is easy). We bound the sequence a^n between two guys. Put the estimate

$$\left(\frac{1}{a}\right)^n \ge n\left(\frac{1}{a} - 1\right).$$

From this we deduce

$$0 \le a^n \le \frac{1}{\frac{1}{a} - 1} \cdot \frac{1}{n}.$$

□

Example 4.3

For any real number $a > 0$, the sequence $(\sqrt[n]{a})_n$ converges to 1.

The existence of $\sqrt[n]{a}$ follows from the Intermediate Value Theorem.

Proof. It suffices to consider $a > 1$, since otherwise we can consider $1/a$. Let x_n be such that $x_n^n = a$ for each n. Then

$$1 \le a < \left(1 + \frac{a}{n}\right)^n$$

so $1 \le x \le 1 + \frac{a}{n}$. □

Example 4.4

The sequence $(\sqrt[n]{n})_n$ converges to 1.

Proof. $1 \le n \le (1 + \frac{1}{n})^n$ by Bernoulli's inequality. □

§4.3 Series

We'll put curly brackets around series for emphasis.

Definition 4.5. A series $\{a_n\}$ **converges** if the *sequence* $b_n = a_1 + \cdots + a_n$ converges.

Here is a better notion.

Definition 4.6. The series $\{a_n\}$ **converges absolutely** if $b_n = |a_1| + \cdots + |a_n|$ converges.

Example 4.7

For $|a| < 1$ the series $\{a^n\}_n$ converges absolutely.

Proof. $1 + \cdots + a^n = \frac{1-a^{n+1}}{1-a} \to \frac{1}{1-a}$. □

Lemma 4.8

If the series $\{a_n\}$ converges then the sequence (a_n) converges to 0.

Remark 4.9. The converse is false; for example, take the harmonic series.

Proof. The partial sums S_n converge as sequences to some b. Then $(S_{n+1} - S_n)_n$ is a convergent sequence to $b - b = 0$. □

Lemma 4.10

The series $\{a_n\}$ converges if and only if $\forall \varepsilon > 0$ there is a natural N such that

$$\left| \sum_{i=n_1}^{n_2} a_i \right| < \varepsilon$$

for all $n_1, n_2 > N$.

Proof. The condition just says that the partial sums are a Cauchy sequence. Since $\mathbb{R}$ is complete, that's equivalent to partial sums converging. □

Lemma 4.11

A series converging absolutely also converges.

Proof. Use the triangle inequality in the previous lemma in the form

$$\varepsilon > \left| \sum |a_i| \right| \geq \left| \sum a_i \right|.$$

□

§4.4 Convergent Things

Theorem 4.12

Consider a convergent series $\{a_n\}$.

(a) If $\{a_n\}$ converges absolutely, then for any permutation $\sigma : \mathbb{N} \to \mathbb{N}$, the series $\{a_{\sigma(n)}\}_n$ converges absolutely to the same value.

(b) If $\{a_n\}$ converges but not absoluetly, then for any $L \in \mathbb{R}$ we can permute the sequence $\sigma : \mathbb{N} \to \mathbb{N}$ so that $\{a_{\sigma(n)}\}$ converges to L.

This is why absolutely convergent series are better than non-convergent series.

Lemma 4.13

A monotonically increasing sequence b_n converges if and only if it's bounded above.

Proof. One direction's clear. Now assume it's increasing and bounded by L. By compactness on $[b_1, L]$ we get a convergent subsequence and we're happy. □

§4.5 Convergence Tests

Proposition 4.14

Let $\{a_n\}$ have positive terms. Then $\{a_n\}$ converges if and only if there exists A such that

$$\sum_{i=1}^{n} a_n \le A$$

for all n.

Proof. Let S_n be the partial sums and apply the previous lemma. □

Corollary 4.15

Let $\{a_n\}$ and $\{b_n\}$ have positive terms. If all partial sums of a_n are at most the partial sums of b_n, then convergence of b_n implies convergence of a_n.

Proposition 4.16 (Cauchy Root Test)

Let $\{a_n\}$ be a series.

(a) Assume $\limsup \sqrt[n]{|a_n|} < 1$. Then $\{a_n\}$ converges absolutely.

(b) Assume $\limsup \sqrt[n]{|a_n|} > 1$. Then $\{a_n\}$ diverges.

Proof. For part (a), let $a = \limsup \sqrt[n]{|a_n|}$. Let ε such that $a + \varepsilon < 1$. For all but finitely many n's,

$$\sqrt[n]{a_n} \le a + \varepsilon.$$

So for sufficiently large n we have $a_n \le (a + \varepsilon)^n$, done.

For part (b), note that infinitely many terms are actually greater than 1 which is impossible. ☐

No conclusion when the lim sup equals 1.

Proposition 4.17 (Ratio Test)
Let $\{a_n\}$ be a series and assume that

$$\limsup \left| \frac{a_{n+1}}{a_n} \right| < 1.$$

Then $\{a_n\}$ converges absolutely.

Proof. Let r be the lim sup and let $\varepsilon > 0$ such that $r + \varepsilon < 1$. For sufficiently large n we have $|a_{n+1}| < (r+\varepsilon)|a_n|$ and bound the sequence above. ☐

Example 4.18
The series $\{a_n\}$ given by $a_n = \frac{x^n}{n!}$ converges absolutely.

Proof. Direct application of ratio test. ☐

§4.6 The Series n^{-a}

In homework, we will define what x^c means for real numbers x and c. In fact I can tell you:

$$x^c \stackrel{\text{def}}{=} \lim_{\substack{q \to c \\ q \in \mathbb{Q}}} x^q.$$

Now let's consider the sequence $a_n = n^{-c}$, where $c > 0$ is a real number. The Ratio and Root tests both fail. Here is the answer.

Theorem 4.19 (Zeta Series)
The sequence $\{a_n\}$ given by $a_n = n^{-c}$ converges for $c > 1$ and diverges for $c \le 1$. In particular the **harmonic** series

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots$$

is divergent.

The idea is the "powers of two" estimate, which we prove in full generality here.

Lemma 4.20
Let $\{a_n\}$ be a series of positive real numbers which is monotically decreasing, and let

$$b_n = 2^n \cdot a_{2^n}.$$

Then $\{b_n\}$ converges if and only if $\{a_n\}$ converges.

The lemma immediately implies the theorem on zeta series.

§4.7 Exponential Function

We define

$$\exp(x) \overset{\text{def}}{=} \sum_{n=0}^{\infty} \frac{x_n}{n!}.$$

We showed earlier that this always converges.

Lemma 4.21 (Physicist's Lemma)
Let $\{a_n\}$ and $\{b_n\}$ be absolutely convergent series. Define

$$c_n = \sum_{i=0}^{n} a_i b_{n-i}.$$

Then $\{c_n\}$ converges to the product of $\sum a_n \cdot \sum b_n$.

Proof. We want to show that there for an $\varepsilon > 0$ there exists an N such that when $n \geq N$

$$\sum_{i=0}^{2n} c_i - AB < \varepsilon$$

where A and B are sums of $\{a_n\}$ and $\{b_n\}$.

For sufficiently large N,

$$\left| \left(\sum_{i=0}^{n} a_n \right) \left(\sum_{i=0}^{n} b_n \right) - AB \right| \leq \frac{1}{3}\varepsilon.$$

So we want to estimate the value of

$$\left| \sum_{i=0}^{2n} c_i - \left(\sum_{i=0}^{n} a_n \right) \left(\sum_{i=0}^{n} b_n \right) \right|.$$

Expanding, we find that the quantity is actually

$$\left| \sum_{\substack{n<i\leq 2n \\ j\leq n}} a_i b_j + \sum_{\substack{n<j\leq 2n \\ i\leq n}} a_i b_j \right| = \left| \sum_{1\leq j\leq n} b_j \sum_{n<i\leq 2n} a_i \right| + \left| \sum_{1\leq i\leq n} b_i \sum_{n<j\leq 2n} a_j \right|$$

So it suffices to prove that for large enough n we have

$$\left| \sum_{1\leq j\leq n} b_j \sum_{n<i\leq 2n} a_i \right| < \frac{1}{3}\varepsilon.$$

Using absolute convergency,

$$\sum_{1\leq j\leq n} |b_j| \sum_{n<i\leq 2n} |a_i|$$

The left term is at most B, while the right term can be made to be at most $\frac{\varepsilon/1000}{B}$, as desired. $\square$

You can actually weaken the condition to just one series being absolutely convergent.

Corollary 4.22

$\exp(x)\exp(y) = \exp(x + y)$.

§5 February 17, 2015

Office hours are now 4PM on Friday.

Today we'll define an integral. It might seem a bit counter-intuitive, but it is compatible with reality: you can integrate almost anything, but it's much harder to differentiate things.

§5.1 L^1 norm

We can consider the set $\text{Func}_{\text{cont}}(X, \mathbb{R})$ (also called the "max norm" or "L^∞" or $C(X)$) of continuous functions from a sequentially compact space X to the real numbers. As functions of this type attain a maximum value, we can define

$$\rho(f, g) = \max_{x \in X} |f(x) - g(x)|$$

and it's easy to check that this is a norm. There will be plenty of homework problems on this.

We will mostly be concerned with the space $X = [a, b]$ for this lecture.

§5.2 Defining the integral

We want to define the integral as a continuous map

$$\text{Func}_{\text{cont}}([a, b], \mathbb{R}) \xrightarrow{\int} \mathbb{R}.$$

Of course we already know how to do this for *piecewise linear functions*, those that are linear over finitely many intervals of $[a, b]$. Indeed, we define the integral as the sum of the linear pieces, where for a linear function $f(x) = cx + d$ we define

$$\int_{[a,b]} f = (b - a)d + \frac{b^2 - a^2}{2} \cdot c.$$

Remark 5.1. This takes several minutes to actually compute correctly.

So we want to do an extension

$$\begin{tikzcd}
\text{Func}_{\text{cont}}([a, b], \mathbb{R}) \arrow[r, dotted, "\int"] & \mathbb{R} \\
\text{Func}_{\text{piecewise}}([a, b], \mathbb{R}) \arrow[u, hook] \arrow[ur, "\int"'] &
\end{tikzcd}$$

To do so we first need to define uniform continuity.

§5.3 Uniform Continuity

Definition 5.2. Let $f : X \to Y$ be a map between metric spaces. We say that f is **uniformly continuous** if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\rho(x, y) < \delta \implies \rho(fx, fy) < \varepsilon.$$

This difference is that given an $\varepsilon > 0$ we must specify a δ which works for *every* input X. Also, this definition can't be ported to a general metric space.

Example 5.3

The functions $(0,1) \to \mathbb{R}$ by $x \mapsto x^{-1}$, $x \mapsto x^2$ are not uniformly continuous.

Example 5.4

If $f : X \to Y$ satisfies a Lipschitz condition

$$\rho(fx, fy) \le C\rho(x,y)$$

then f is uniformly continuous.

Remark 5.5. For a differentiable function this is equivalent to saying that the derivatives are bounded.

Proposition 5.6

Suppose we have a diagram

$$\begin{tikzcd} X \arrow[r, dotted, "f"] & Y \\ X' \arrow[u, "\iota"] \arrow[ur, phantom, "f'"] & \end{tikzcd}$$

so that ι is a dense embedding, f' is uniformly continuous and Y is a complete metric space. Then the map factors uniquely through f as above, and moreover f is also uniformly continuous.

Proof. We want to define $f(x)$ for $x \in X$. Choose a sequence $(x_n) \in X'$ converging to X (possible since the embedding is dense), and consider the sequence $f'(x_1), \dots$. We claim that it converges. Indeed, $\{x_n\}$ is Cauchy, so $\{f'(x_n)\}$ is Cauchy by directly applying uniform continuity, and hence continuous by completeness of Y.

Hence for any $x \in X$ we define $f(x)$ to be a limit of such a sequence; it's not hard to check that this is well-defined.

Uniform continuity is easy but annoying to check. Uniqueness was done on the problem set. $\square$

§5.4 Defining the Integral

We want to apply the proposition to our commutative diagram

$$\begin{tikzcd} \mathrm{Func}_{\mathrm{cont}}([a,b],\mathbb{R}) \arrow[r, dotted, "\int"] & \mathbb{R} \\ \mathrm{Func}_{\mathrm{piecewise}}([a,b],\mathbb{R}) \arrow[u, hook] \arrow[ur, "\int"'] & \end{tikzcd}$$

To do this we need to check three conditions. First, $\mathbb{R}$ is complete, which we already know. Then, we need to check that the integral is uniformly continuous for piecewise linear continuous functions. In fact we claim it is *Lipschitz*: for piecewise linear functions, we have

$$\left| \int_{[a,b]} f - \int_{[a,b]} g \right| < (b-a) \cdot \max_x |f(x) - g(x)| .$$

This is pretty obvious. I won't bother you.

Now for the actually hard part.

Theorem 5.7

$\text{Func}_{\text{piecewise}}([a,b], \mathbb{R})$ is dense in $\text{Func}_{\text{cont}}([a,b], \mathbb{R})$.

This looks really annoying. Indeed, it would be without the following result.

Theorem 5.8

Let X be a compact metric space and let $f : X \to Y$ be continuous where Y is another metric space. Then f is uniformly continuous. In other words, for compact sources continuity and uniform continuity are equivalent.

In practice we will take $X = [a,b]$.

Proof. We're given $\varepsilon > 0$ we want a universal δ. For every $x \in X$ let δ_x be such that $\rho_X(x', x) < \delta_x$ implies $\rho_Y(fx, fx') < \frac{1}{2}\varepsilon$. Thus we have an open cover

$$X = \bigcup_x B(x, \frac{1}{2}\delta_x).$$

By compactness we can take a finite subcover. Now let $\delta = \frac{1}{2015} \min \delta_x$ for those x in the finite subcover. Consequently, given $x_1, x_2 \in X$ a distance less than δ from each other, we see they are both inside some ball centered at some x with radius δ_x. (Specifically, take any x which contains x_1 in its $\frac{1}{2}\delta_x$ ball.) Then

$$\rho_Y(fx_1, fx_2) \le \rho(fx_1, fx) + \rho(fx, fx_2) < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon. \qquad \square$$

Proof of density. Recall that $[a,b]$ is compact. Hence given $f : [a,b] \to \mathbb{R}$ continuous we may assume it is *uniformly* continuous.

Given $\varepsilon > 0$, suffices to approximate our $f : [a,b] \to \mathbb{R}$ within ε by some $\tilde{f} \in \text{Func}_{\text{piecewise}}([a,b], \mathbb{R})$. Choose δ required by uniform continuity of f to get within $\frac{1}{2015}\varepsilon$ in f, and divide $[a,b]$ into a **mesh**

$$a = x_0 < x_1 < \cdots < x_{n-1} = x_n = b$$

with $|x_{k+1} - x_k| < \delta$ for each k. Now we do what biologists (and high school hooligans) do: connect our plot by straight lines. This gives us our $\tilde{f}$, and we can check that it works: any $x \in [a,b]$ is within δ of an x_k which is exactly correct. $\square$

§5.5 Properties of Integrals

Lemma 5.9

In what follows abbreviate $\int_{[a,b]}$ to $\int$.

(a) $\int f_1 + f_2 = \int f_1 + \int f_2$

(b) $\int cf = c \int f$.

(c) $\int_{[a,b]} f = \int_{[a,c]} f + \int_{[c,b]} f$.

(d) $f \ge 0 \implies \int f \ge 0$.

(e) Equality holds in the previous point if and only if $f = 0$.

Proof. The point is $\int$ is a unique extension. The identities hold for piecewise linear functions; hence they hold for arbitrary continuous functions.

More precisely, we have

$$\begin{tikzcd}
\mathrm{Func}_{\mathrm{piecewise}}([a,b],\mathbb{R}) \arrow[r, "c\int f - \int cf"] \arrow[d, hook] & \mathbb{R} \\
\mathrm{Func}_{\mathrm{cont}}([a,b],\mathbb{R}) \arrow[ur, dotted] &
\end{tikzcd}$$

This resolves (b) and (c). For (a), we instead deal with two maps

$$\mathrm{Func}_{\mathrm{cont}}([a,b],\mathbb{R}) \times \mathrm{Func}_{\mathrm{cont}}([a,b],\mathbb{R}) \underset{\int f_1 + \int f_2}{\overset{\int (f_1+f_2)}{\longrightarrow}} \mathbb{R}.$$

For (d) and (e), we can work in

$$\begin{tikzcd}
\mathrm{Func}_{\mathrm{piecewise}}([a,b],\mathbb{R}^{\ge 0}) \arrow[r, "\int"] \arrow[d, hook] & \mathbb{R} \\
\mathrm{Func}_{\mathrm{cont}}([a,b],\mathbb{R}^{\ge 0}) \arrow[ur, dotted] &
\end{tikzcd}$$

showing again the embedding is dense.

To prove (e), suppose $x \in [a,b]$ such that $f(x) = r > 0$. Pick an $\delta > 0$ so that $[x-\delta, x+\delta]$ so that $f(x) > 1 - \frac{1}{2015}r > 0$ on this interval. From this you have an interval with area greater than zero. $\square$

§5.6 Riemann Sums

Define a mesh p by

$$a = x_0 < x_1 < \cdots < x_{n-1} = x_n = b$$

with distinguished points $x_i' \in [x_{i-1}, x_i]$ for $i = 1, \dots, n$. We define

$$S_p(f) = \sum (x_i - x_{i-1}) f(x_i').$$

Its **width** is $\max_{i=1,\dots,n}(x_i - x_{i-1})$.

Theorem 5.10 (Riemann)

Let p_n be a sequence of partitions with width converging to 0. Then $S_{p_n}(f)$ converges to $\int f$.

Proof. Given $\varepsilon > 0$, and let $\delta > 0$ be the magic number for

$$\frac{\varepsilon}{b-a}$$

in f. The claim is that if a mesh p has width less than δ then $S_p(f)$ is within ε of $\int f$. Check this. $\square$

§6 February 19, 2015

Didn't attend lecture for HMMT reasons; however the lecture was just a review of homework questions. The notes below are due to W Mackey.

§6.1 Bonus Problem Sets

The first bonus PSet on normed vector spaces has been posted. The plan for the bonus PSets hereon will be:

1. Banach Spaces
2. Topological vector spaces
3. Operator Theory
4. Spectral Theory in Hilbert Spaces
5. Lebesgue measure
6. Lebesgue Integration

Today we'll mostly be doing a 'problem solving exercise' rather than any new material.

§6.2 PSet 2, Problem 7b

Let X be a metric space; let $\text{Cauch}(X)$ and let $\overline{X}$ be the quotient of $\text{Cauch}(X)$ by the equivalence relation that $\{x_n\} \sim \{y_n\}$ if and only if $\rho(\{x_n\}, \{y_n\})$. Show that $\overline{X}$ is complete in this metric.

Let n be fixed and $y_n = \{x_n^m\}$ be a Cauchy sequence of elements indexed by m. We want to show that $y_n \to y$. Define a sequence by fixing a sequence of real numbers $r_n \to 0$. Let $y = \{x_n\}$, and we want to construct the sequence x_n. Pick N so that for $n_1, n_2 \geq N$, $\rho(y_{n_1}, y_{n_2}) < \frac{1}{3}r_n$. Pick any $n' \geq N$, and consider $y_{n'} = \{x_{n'}^m\}$. Let M be so that $m_1, m_2 \geq m$ implies $\rho(x_{n'}^{m_1}, x_{n'}^{m_2}) < \frac{1}{3}r_n$. Pick $m' \geq M$. Set $x_n := x_{n'}^{m'}$. Then we claim this is Cauchy. For any ϵ, if $r_N < \epsilon$ then $n_1, n_2 \geq N$, $\rho(x_{n_1}, x_{n_2}) < \epsilon$.

Alternate method: Suppose we prove X is dense. Then $X \hookrightarrow \overline{X}$.

Lemma 6.1

X is dense in $\overline{X}$.

Assume for the moment the lemma. Then take $\{\overline{x_n}\}$ a Cauchy sequence. Then pick $x_n \in X$ such that $\rho(x_n, \overline{x_n}) < \frac{1}{n}$, then this sequence will converge as desired.

The lemma is almost definitional.

§6.3 PSet 3, Problem 8

Let X and Y be topological spaces with Y compact. Show that the projection $X \times Y \to X$ sends closed subsets to closed subsets.

We want $P(Z)$ to be closed for every Z. We'll prove this on the opens instead. Let $x \in X$, $x \notin P(Z)$. We want $x \in U \subseteq X$ such that $U \cap P(Z) = \emptyset$. For all $y \in Y$, we have $(x, y) \notin Z$. Then there exist U_x, U_y such that $(x, y) \in U_x \times U_Y$ and $U_x \times U_y \subseteq X \times Y - Z$. Now take an open cover $\cup U_y = Y$ for every point, then take a finite subcover $\cup U_{y_i} = Y$. Then fix $U = \cap_i U_{Y_i}$, and we claim $U \cap P(Z) = \emptyset$, proving the complement is empty, hence the projection of Z is closed.

§6.4 PSet 3, Problem 10

Consider the set $X = (0, 1) \cup \infty$. Define a topology on it, so that in addition to X and $\emptyset$, the open subsets are: (i) any U which is open in $(0, 1)$; (ii) the subsets $\infty \cup (0, a) \cup (b, 1)$ for $0 < a < b < 1$; (iii) unions of the above. Problem 10b was showing that X is compact.

Let X be a topological space such that for every $x \in X$ and every open $x \in U_x$ open, there exists a $x \in U'_x \subseteq Y_x \subseteq U_x$ where U'_x is open and Y_x is compact. We call this space locally compact. Let $\overline{X} = X \cup \{\infty\}$, with X Hausdorff. A set in $\overline{X}$ is open if either it's empty or the entire thing, it's contained X and open there, or its complement is a compact subset of X. Note that in question 10, this is exactly the topology we described.

Lemma 6.2

$\overline{X}$ is compact.

Proof. Say $\overline{X} = \cup_\alpha U_\alpha$. There exists an α_0 such that $\infty \in U_{\alpha_0}$ since it's a cover. Consider $Z = \overline{X} - U_{\alpha_0}$, is compact by definition. Then $Z \subseteq U_\alpha (U_\alpha - \infty)$, then compactness of Z means we can take finitely many there, so the adjoining U_{α_0} gives us our finite cover. □

Now we just want to verify that $\overline{X}$ is a topology. Take U open and Z compact, then $((X - Z) \cup \{\infty\}) \cup U$ is open, $(X - U) \cap Z$ is closed in Z and therefore still compact. Now for finite intersections, take $U \cap ((X - Z) \cup \{\infty\}) = (X - Z) \cap U$ is open.

Lemma 6.3

$\overline{X}$ is Hausdorff if and only if X is locally compact.

Proof. Fix $x \in X$, $\infty \in \overline{X}$. Hausdorff means there is a $U \subseteq X$ open and an open set that contains ∞ and not x such that the two are disjoint, so there is compact $Z \subseteq X$ such that $U \cap (X - Z) = \emptyset$, implying $U \subseteq Z$. (TL;DR Hausdorff gives us an open set that hides in the complement, which is compact definitionally.) □

§6.5 PSet 3, Problem 4

(a) Let r be a real number Show that for $x \in \mathbb{R}^{>0}$ for any sequence of rationals $r_i \to r$, the sequence x^{r_i} is Cauchy. We set x^r to be its limit.

Write $x^{r_m} - x^{r_n} = x^{r_n}(x^{r_m - r_n} - 1)$. Let's assume $x \geq 1$, we'll get $x < 1$ by inversion. Then we want to say $x^{1/n} - 1 \leq \frac{1}{n}(x - 1)$. Our first lemma will be x^r for r in $\mathbb{Q}$ is monotonic. Our second will be that $x^{1/n} \to 1$. This will work out.

(b) Prove that the function $x \mapsto x^r : \mathbb{R}^{>0} \to \mathbb{R}^{>0}$ is continuous and satisfies

$$(x_1 \cdot x_2)^r = (x_1)^r \cdot (x_2)^r.$$

We know that $x_i^r - x^r = x^r \left(\left(\frac{x_i}{x}\right)^r - 1 \right)$, and $\left(\frac{x_i}{x}\right)^r$ is monotonic in r. Then

$$\left| \left(\frac{x_i}{x}\right)^r - 1 \right| \le \left| \left(\frac{x_i}{x}\right)^{r'} - 1 \right|.$$

by dominating with a rational number. This sandwiches

$$\left(\frac{x_i}{x}\right)^R \ge \left(\frac{x_i}{x}\right)^r \ge 1$$

for $x_i \ge x$, and the other way around for $x_i \le x$, then we squeeze.

(c) Prove that for a fixed $a \in \mathbb{R}^{>0}$, the function $x \mapsto a^x : \mathbb{R} \to \mathbb{R}^{>0}$ is continuous and satisfies

$$a^{x_1 + x_2} = a^{x_1} \cdot a^{x_2}.$$

Another squeeze argument. We have $a^x(a^{x - x_i} - 1)$, and $r_i' \ge x - x_i \ge r_i$, we know $a^{r_i} \to 1$ for r_i rationals that tend to 0, so we get what we want.

(d) Assume that $a \neq 1$. Show that for $y \in \mathbb{R}^{>0}$ there exists an element $\log_a(y) \in \mathbb{R}$ so that $a^{\log_a(y)} = y$, and that the resulting function $y \mapsto \log_a(y) : \mathbb{R}^{>0} \to \mathbb{R}$ is continuous.

This is just since the inverse function is also continuous here, which we've proved before.

Lemma 6.4

$f : [a, b] \to \mathbb{R}$ is continuous and strictly monotonic on the rationals. Then it is monotonic.

Proof. Take $x < y$, let $x \leftarrow x_i$ and $y_i \to y$, then $f(x) = \lim f(x_i) < \lim f(y_i) = f(y)$. $\square$

Then we get that a^x is surjective by the intermediate value theorem.

§6.6 Filters

(Something Gaitsgory specifically didn't want to do) You can test limits in arbitrary topological spaces with a filter. Sequences aren't always enough. To test convergence in a topological space that isn't first countable, you have to map other types of ordered sets. Gaitsgory says that he doesn't find it very useful. It's much better to check compactness and such by the general topological, rather than sequential definitions. For more, see: `http://en.wikipedia.org/wiki/Filter_%28mathematics%29`.

§6.7 PSet 4

PSet 4 is mostly functional analysis stuff. For problem 1, $\text{Funct}_{\text{cont}}(X, Y)$, for X compact and Y metric (complete). We'll say $\rho(f, g) = \max_{x \in X} \rho(f(x), g(x))$. The theorem is that the space of functions is complete.

For problem 2, we have $Y = \mathbb{R}$, and $\rho(f,g) = \rho(f-h, g-h)$, further $\rho(cf, cg) = |c|\rho(f,g)$. Problem 2 is pretty boring.

Problem 3 is interesting. We say two norms are equivalent if there exist scalars A, B such that the first norm is bounded by the second up to A, and the second is bounded by the first up to B. Problem 3 is proving that any two norms are equivalent in finite dimensional vector spaces.

Problem 4 shows this is emphatically not the case for infinite dimensional vector spaces. In particular, we compare the max norm and the L_1 norm, defined by $||f||_{L_1} = \int_{[a,b]} |f|$. We'll show that these two norms are not equivalent.

Problem 5 introduces the L_2 norm and is fairly easy.

Problem 6 is showing a way to prove second countability.

Problem 7 is also kind of cool. We consider the completion of $\text{Funct}_{\text{cont}}(X, Y)$ with the L_1 norm, $L_1([a,b])$. We then do it for $L_2([a,b])$.

§7 February 24, 2014

Let $f : X \to Y$ be a map of topological spaces.

§7.1 Limits

Definition 7.1. We say f is **continuous at x** if for all neighborhoods $U_y \ni fx$ there exists a neighborhood $U_x \ni x$ then

$$f``(U_x) \subseteq U_y.$$

Definition 7.2. Let $\mathring{f} : X - \{x\} \to Y$. We say that the **limit** of $\mathring{f}$ at x is y if $\mathring{f}$ can be extended to $f : X \to Y$ so that $f(x) = y$ and f is continuous at x.

Equivalently, the limit of f at x is y if for every neighborhood $U_y \ni y$ there exists a neighborhood $U_x \ni x$ such that $\forall x' \in U_x - \{x'\}$ we have $f(x') \in U_y$.

We as usual write

$$\lim_{x \to c} f(x) = L$$

to mean that L is the limit of f at c.

Remark 7.3. Given $Y = \mathbb{R}$, limits are additive. Moreover, for any bounded function $g : X \to \mathbb{R}$, if $\lim_{x \to c} f(x) = 0$ then $\lim_{x \to c} f(x)g(x) = 0$.

Assume now that X is a metric space. We introduce little o notation:

Definition 7.4. A function $f : X - \{x_0\} \to \mathbb{R}$ is $o(\rho(x, x_0)^n)$ if

$$\lim_{x \to x_0} \frac{f(x)}{\rho(x, x_0)^n} = 0.$$

Equivalently, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$\rho(x, x_0) < \delta \implies |f(x)| \le \varepsilon \left|\rho(x, x_0)\right|^n.$$

We restrict our attention to nonnegative integers n.

In particular, f is $o(1)$ if $\lim_{x \to x_0} f(x) = 0$. (This is $n = 0$ above.)

Example 7.5
Let $X = \mathbb{R}$, $f(x) = x^3 + x \sin x$. Then $f(x)$ is $o(x^n)$ for $n = 0, 1, 2, 3$ only.

§7.2 Differentiation

Consider a function $f : (a, b) \to \mathbb{R}$.

Definition 7.6. We say f is **differentiable** at $x_0 \in (a, b)$ if there exists a constant C such that

$$f(x) - f(x_0) - (x - x_0)C = o(x - x_0)$$

id est,

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - C(x - x_0)}{x - x_0} = 0 \iff \lim_{x \to x_0} \frac{f(x) - f(x - 0)}{x - x_0} = C.$$

In that case we write $f'(x_0)$ for the constant C.

Lemma 7.7

If $f'(x_0)$ exists then it is unique, and f is continuous at x_0.

Proof. For the first part, limits are unique. The second part is some boring calculation (the point is that differentiability is stronger than continuity). □

> "If you find this exceedingly boring, quietly go to sleep for the next ten minutes. Actually, maybe all of today will be boring. Go check your Facebook or something."
>
> – Gaitsgory

Example 7.8

If $f(x) = x$ then $f'(x_0) = 1$ for any x_0.

Lemma 7.9 (Product Rule)

If f and g are differentiable at x then so is $f \cdot g$ and the value is given by

$$(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x_0).$$

Proof. Bash. □

As usual, we can by induction get the derivative for x^n. It doesn't work for $x^{1/n}$ because you don't know a priori that $x^{1/n}$ is differentiable.

§7.3 Chain Rule

Lemma 7.10 (Chain Rule)

Let $(a, b) \xrightarrow{f} (c, d) \xrightarrow{g} \mathbb{R}$. Suppose f is differentiable at x_0 and g is differentiable at $y_0 = f(x_0)$. Then $(g \circ f)'(x_0)$ exists and has value

$$g'(y_0) \cdot f'(x_0).$$

Proof. Do a few tricks:

$$\begin{aligned}
&\lim_{x \to x_0} \frac{gfx - gfx_0 - g'(y_0)f'(x_0)(x - x_0)}{x - x_0} \\
&= \frac{gfx - gy_0 - g'y_0(fx - y_0)}{x - x_0} + g'y_0 \cdot \frac{fx - y_0 - (x - x_0)f'x_0}{x - x_0} \\
&= \frac{gfx - gy_0 - g'y_0(fx - y_0)}{x - x_0}
\end{aligned}$$

Thus this amounts to showing that

$$\frac{gfx - gy_0 - g'y_0(fx - y_0)}{x - x_0} \to 0.$$

For $\varepsilon > 0$ there's a $\mu > 0$ such that for $|y - y_0| < \mu$ we have

$$\left| \frac{gy - gy_0 - g'y_0(y - y_0)}{y - y_0} \right| < \varepsilon$$

and take $\delta > 0$ such that

$$|x - x_0| < \delta \implies |fx - y_0| < \mu$$

so

$$\left| \frac{gy - gy_0 - g'y_0(y - y_0)}{y - y_0} \right| \le \varepsilon \cdot \frac{f(x) - y_0}{x - x_0}.$$

□

§7.4 More "applied math"

Theorem 7.11 (Critical Points)

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then if y attains a maximum at $x_0 \in (a, b)$ then $f'(x_0) = 0$.

Proof. Assume not, and WLOG $f'(x_0) > 0$. Take a small perturbation. □

Theorem 7.12 (Rolle)

Let $f : [a, b] \to \mathbb{R}$ be continuous and differentiable on (a, b). Then for some $x \in (a, b)$ we have $f'(x) = 0$.

Proof. Compactness to get maximum.

> "Imagine you're going on I-90, going east until at some point you stopped and turned back because Massachusetts is the best."

By previous theorem, $f'(x) = 0$.

But what if $x = a$?

> "So imagine you didn't travel east, you travelled *west* instead."

Take a minimum instead.

What if the minimum and maximum are both endpoints?

> "If the maximum is at the endpoint and the minimum is at the endpoint, then you *really* have a problem; the function is constant so you didn't go anywhere. No place like home." □

Theorem 7.13 (Mean Value Theorem)

With the same assumptions, there exists $x \in (a, b)$ such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

Proof. By Gaitsgory quote.

> "Suppose you're going at I-90. In an hour you cover 95 miles. The cop claims at some point you were travelling at least 95 miles per hour." □

Theorem 7.14

Let f be differentiable on (a, b).

(a) If $f'(x) \geq 0$ for all x then f is nondecreasing.

(b) If $f'(x) \leq 0$ for all x then f is nonincreasing.

(c) If $f'(x) = 0$ for all x then f is constant.

Proof. Use MVT for (a) and (b); then (c) follows. □

§7.5 Higher Derivatives

Definition 7.15. Let $f : (a, b) \to \mathbb{R}$. We say f is differentiable n times at x_0 if it's differentiable $n-1$ times on (a', b') and then $f^{(n-1)}$ is differentiable at x_0.

Theorem 7.16 (Stop Sign Theorem)

Let f be differentiable n times at x_0 such that

$$f^{(n)}(x_0) = 0 \quad k = 0, \dots, n.$$

Then

$$f = o\left((x - x_0)^n\right).$$

Proof. Induction on n. □

Theorem 7.17 (Taylor)

Let f be differentiable n times at x_0. Then

$$f(x) - \sum_{i=0}^{n} \frac{f^{(i)}(x_0)(x - x_0)^i}{i!}$$

is $o\left((x - x_0)^n\right)$.

Proof. Apply previous theorem. □

Unfortunately, even "infinitely differentiable with all zero derivatives" is not enough to guarantee that f is identically zero in any neighborhood.

Proposition 7.18

The function $f : \mathbb{R} \to \mathbb{R}$ by

$$x \mapsto \begin{cases} 0 & x \leq 0 \\ e^{-1/x} & x > 0. \end{cases}$$

is infinitely differentiable at $x = 0$, with all derivatives zero.

Proof. The point is to check that $\frac{e^{-1/x}}{x^n} \to 0$ as $x \to 0$ for every n. □

§8 February 26, 2015

Today we prove the Fundamental Theorem of Calculus.

Lemma 8.1

A function $[a, b] \to \mathbb{R}$ whose derivative on (a, b) is everywhere zero is constant.

Proof. Mean Value Theorem. $\square$

§8.1 Fundamental Theorem of Calculus

Let $f : [a, b] \to \mathbb{R}$ be continuous. We define $F : (a, b) \to \mathbb{R}$ by

$$F(x) = \int_{[a,x]} f.$$

Theorem 8.2

F is differentiable and

$$F'(x) = f(x).$$

Proof. It suffices to show that

$$\lim_{x' \to x} \frac{\int_{[x,x']} f}{x' - x} \to f(x)$$

which is not difficult (just ε and δ using continuity of f, and bound the integrand). $\square$

Corollary 8.3 (Fundamental Theorem of Calculus)

Let F be continuous on $[a, b]$ and differentiable on (a, b) such that F' extends to a continuous function on $[a, b]$. Then

$$F(b) - F(a) = \int_{[a,b]} F'.$$

Proof. Let $G(x) = \int_a^x F'$. By the previous theorem, G is differentiable on (a, b) and $G' = F'$. You can check it's also continuous at $x = a$ and $x = b$. Since $G - F$ has zero derivative, it is constant. Thus

$$G(b) - F(b) = G(a) - F(a) = -F(a) \implies F(b) - F(a) = G(b) = \int_{[a,b]} F'. \qquad \square$$

You can extend this all to work for certain noncontinuous functions, but

> "Trying to Riemann integrate discontinuous functions is kind of outdated."
> – Gaitsgory

We want Lebesgue integrals.

Remark 8.4. As we've said many times, and will continue to see: integrals are much nicer than derivatives. It's easy to control the integral of continuous functions compared to controlling derivatives (or even proving derivatives exist).

§8.2 Pointwise converge sucks

Recall that

$$\exp(x) = \sum_{n \ge 0} \frac{x^n}{n!}.$$

We wish to show $\exp' = \exp$. In an ideal world, we would have the following result:

> Let f_n be a sequence of functions $[a,b] \to \mathbb{R}$ differentiable on (a,b) such that for every x, the sequence $f_1(x), f_2(x), \dots$ converges to fx. Then f is differentiable and
> $$f'(x) = \lim_{n \to infty} f_n'(x).$$

Unfortunately this notion of "point-wise convergence" is not good enough, which is to say that the theorem is false. Here's a counterexample. We define a sequence g_n by

$$g_n(x) = \begin{cases} 0 & 0 \le x \le \frac{1}{2} - \frac{1}{n} \\ \frac{1}{2}n\left(x - \left(\frac{1}{2} - \frac{1}{n}\right)\right) & \frac{1}{2} - \frac{1}{n} \le x \le \frac{1}{2} + \frac{1}{n} \\ 1 & \frac{1}{2} + \frac{1}{n} \le x \le 1 \end{cases}$$

and let $f_n(x) = \int_0^x g_n(t)\, dt$. It is not difficult to see that

$$\lim_{n \to \infty} f_n = x \mapsto \begin{cases} 0 & x \le \frac{1}{2} \\ x - \frac{1}{2} & x \ge \frac{1}{2}. \end{cases}$$

§8.3 Uniform Convergence of Functions

Definition 8.5. Let $f_1, \dots$ be a sequence of functions $X \to \mathbb{R}$. We say that (f_n) **converges uniformly** to $f : X \to \mathbb{R}$ if they converge in the *sup metric*, *id est*

$$\lim_{n \to \infty} \sup_{x \in X} |f(x) - f_n(x)| = 0.$$

In other words, for every $\varepsilon > 0$ the quantity

$$|f(x) - f_n(x)|$$

is bounded by ε for all sufficiently large n.

The less happy notion of just

$$\lim_{n \to \infty} |f(x) - f_n(x)| = 0$$

is the one we tried above.

Here is another false theorem.

Theorem 8.6

Let f be a sequence of continuous functions on $[a,b]$ differentiable on (a,b). Assume that each f_n' extends to a continuous function on $[a,b]$, and moreover the sequence $f_1', f_2', \dots$ converges *uniformly* to g. Assume also that for some $c \in [a,b]$ the sequence $f_1(c), f_2(c), \dots$ converges.

Then f_n converges uniformly to some differentiable function f whose derivative equals g.

Proof. Let

$$f(x) = \int_{[a,x]} g - C$$

where $C = \left(\lim_{n\to\infty} f_n(c) - \int_{[a,c]} g\right)$ (the constant C is contrived so that $f(c) = \lim_{n\to\infty} f_n(c)$). By the Fundamental Theorem of Calculus, $f' = g$. So now we simply want to show $f_n \to f$.

We write f and f_n so they look similar:

$$f_n(x) = \int_{[a,x]} f_n' + \left(f_n(c) - \int_{[a,c]} f_n'\right)$$

$$f(x) = \int_{[a,x]} g + \left(\lim_{n\to\infty} f_n(c) - \int_{[a,c]} g\right)$$

We claim that each term converges uniformly to the corresponding term. That is, we have $f_n(c) \to \lim_{n\to\infty} f_n(c)$ by definition. Given $\varepsilon \to 0$ have for suitably large n that

$$\int_{[a,x]} |f_n' - g| < \int_{[a,x]} \varepsilon \to 0$$

and similarly for the $\int_{[a,c]} f_n'$ term. □

§8.4 Applied Math – Differentiating e^x

Now we finally prove the following result.

Theorem 8.7

The function $\exp : \mathbb{R} \to \mathbb{R}$ is differentiable and equals its own derivative.

Proof. Restrict to any interval $[a,b]$. Since

$$\exp(x) = \sum_{n\ge 0} \frac{x^n}{n!}$$

we define

$$f_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}.$$

It is pretty amusing to show that $f_n' = f_{n-1}$. Hence, $f_n \to \exp(x)$ and $f_n' \to \exp(x)$ as well. Thus we only need to show that the convergence is *uniform*, and we'll be done.

For any n we have

$$|\exp(x) - f_n(x)| = \left| \sum_{k\ge n+1} \frac{x^k}{k!} \right| \le \sum_{k\ge n+1} \frac{(\max\{|a|,|b|\})^k}{k!} \overset{n\to\infty}{\longrightarrow} 0.$$

Also $1 = f_1(0) = f_2(0) = \dots$. So the theorem applies and we're done. □

Second Proof. We can do this directly using the fact that we proved $\exp(x+y) = \exp(x) + \exp(y)$. Observe that

$$\frac{e^{x+h} - e^x}{h} = e^x \cdot \frac{e^h - 1}{h}.$$

So we merely need to show that $\lim_{h\to 0} \frac{e^h-1}{h} = 1$, which is

$$\lim_{h\to 0} \sum_{k\ge 1} \frac{h^{k-1}}{k!} = 1.$$

The idea is to bound $k!$ by a geometric series now:

$$\left| \sum_{k\ge 1} \frac{h^{k-1}}{k!} - 1 \right| \le \left| \sum_{k\ge 1} \frac{h^{k-1}}{2^{k-1}} - 1 \right| = \left| \frac{1}{2}h + (\frac{1}{2}h)^2 + \dots \right| = \left| \frac{\frac{1}{2}h}{1-\frac{1}{2}h} \right| \to 0. \qquad \square$$

§8.5 Differentiable Paths in $\mathbb{R}^n$

Consider a function $\gamma : (a,b) \to \mathbb{R}^n$.

Definition 8.8. We say γ is **differentiable** if each of the coordinates is also differentiable.

Let $\gamma : [a,b] \to \mathbb{R}^n$ be a continuous function differentiable on (a,b). We wish to define a notion of *length* on γ.

Definition 8.9. Given a norm $\|\bullet\|$ on $\mathbb{R}^n$ we define the **length** by

$$\operatorname{length}(\gamma) = \int_{[a,b]} \|\gamma'\|.$$

A mesh of points $\mathfrak{p}$ is, as before, a sequence

$$a = c_0 \le c_1 \le \cdots \le c_m = b.$$

The width is $\max_k c_{k+1} - c_k$. Then

Theorem 8.10

Given any sequence of partitions $\mathfrak{p}_n$ with width approaching zero, we have that

$$\operatorname{length}(\gamma) = \lim_{n\to\infty} \left(\sum_{[c_k,c_{k+1}] \text{ in } \mathfrak{p}_n} \|\gamma(c_{k+1}) - \gamma(c_k)\| \right).$$

Proof. Homework, but via Riemann sums the problem boils down to showing that

$$\lim_{n\to\infty} \left(\sum_{d_i\in[c_k,c_{k+1}] \text{ in } \mathfrak{p}_n} \left\|(c_{k+1} - c_k)\gamma'(d_i)\right\| \right).$$

converges to the same thing as

$$\lim_{n\to\infty} \left(\sum_{[c_k,c_{k+1}] \text{ in } \mathfrak{p}_n} \|\gamma(c_{k+1}) - \gamma(c_k)\| \right).$$

For this we will need to show that

$$\left\| \int_{[a,b]} f \right\| \le \int_{[a,b]} \|f\|.$$

$\square$

§9 March 3, 2015

Guess who showed up half an hour late to class? Blocking is hard.

§9.1 Differentiable forms

In what follows, V and W are real finite-dimensional vector spaces and U is an open subset of V.

Definition 9.1. Let V and W be normed vector spaces. We say $f : U \to W$ is **differentiable at x** (here $x \in U$) if there exists a function $T \in \operatorname{Hom}_{\mathbb{R}}(V, W)$ such that

$$\lim_{v \to 0} \frac{\|f(x+v) - f(x) - T(v)\|_W}{\|v\|_V} = 0.$$

We write $Df(x)$ for the linear transformation T.

Here $Df(x)$ is a *linear* map of vector spaces. If $W = \mathbb{R}$, we find $\operatorname{Hom}(V, \mathbb{R}) = V^\vee$ is the *dual module*. When $V = \mathbb{R}$, $V^\vee$ consists of scalar functions and this looks like our usual notion of derivative.

The following definitions are my best guesses, since I was very late to lecture.

Definition 9.2. Let $x \in U$ and $v \in V$. For $f : U \to W$ differentiable we write

$$(D_v f)(x) = (Df(x))(v).$$

Hence $D_v f : V \to W$.

Note that $D_{cv}(f) = cD_v f$ for any constant c.

Definition 9.3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable and let $e_1, \dots, e_n$ be a distinguished basis. Then we define $\partial_i f$ as shorthand $D_{e_i}(f)$.

Lemma 9.4

A function $f : U \to W$ which is differentiable at x is also continuous at x.

Proof. Same as one variable case. $\square$

Theorem 9.5

Assume that the partials $\partial_i f$ exist and are continuous on some $x \in U' \subseteq U$. Then f is differentiable.

Proof. Note that

$$\frac{f(x,y) - f(0,0) - x\partial_x f(0,0) - y\partial_y(0,0)}{\sqrt{x^2+y^2}}$$

equals

$$\frac{f(x,y) - f(0,y) - x\partial_x f(0,0)}{\sqrt{x^2+y^2}} + \frac{f(x,y) - f(0,) - y\partial_y f(0,0)}{\sqrt{x^2+y^2}}$$

The first part equals

$$\frac{f(x,y) - f(0,y) - x\partial_x f(0,0)}{\sqrt{x^2+y^2}} \le \partial_x f(x',y) - \partial_x f(0,0) \to 0.$$

$\square$

§9.2 Vector-Valued Functions

Let $U_1 \xrightarrow{f} U_2$, where $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$ are open.

Lemma 9.6

Let $f = (f_1, \dots, f_m)$ be a function $V_1 \to \mathbb{R}^m$. Then f is differentiable if and only if each f_i is differentiable and

$$\pi_i \left(Df(x) \right) = Df_i(x).$$

Theorem 9.7 (Chain Rule)

Consider $U_1 \xrightarrow{f} U_2 \xrightarrow{g} U_3$, where U_i is open in a real vector space V_i. Suppose f is differentiable at $x_1 \in U_1$ and g is differentiable at $x_2 = f(y_1) \in U_2$. Then

$$D(g \circ f)(x_1) = Dg(x_2) \circ Df(x_1).$$

Proof. Same as one-variable case. □

§9.3 Higher-Order Derivatives

Definition 9.8. Let $f : U' \to W$, where V and W are vector spaces (here $U' \subseteq V$ is open). Then f is **differentiable twice** at x if

(1) f is differentiable on some neighborhood U of x, so we may consider Df.

(2) The map $Df : U \to \operatorname{Hom}(V, W)$ is itself differentiable at x.

For three times, we want

$$D^2 f : U \to \operatorname{Hom}(V, \operatorname{Hom}(V, W)) \cong \operatorname{Hom}(V^{\otimes 2}, W)$$

to itself be differentiable at x.

Theorem 9.9

Let $f : V \to W$ be twice differentiable. Let $v, w \in V$ and consider $D_w(D_v(f))$ and $D_v(D_w(f))$. If both are continuous then they coincide.

In particular,

Corollary 9.10

"Let f be diff be differentiable twice" (sic). Assume that $\partial_j(\partial_i f)$ and $\partial_i(\partial_j f)$ are continuous. Then they coincide.

In fact this is the only case we need consider, since if v and w are linearly dependent it's trivial.

Proof. Assume $V = \mathbb{R}^2$, $v = e_1$, $w = e_2$, since the case where v and w are linearly dependent is trivial.

We may assume by shifting and subtracting linear factors that $x = (0,0)$, $f(0,0) = 0$ and $Df(0,0) = 0$.

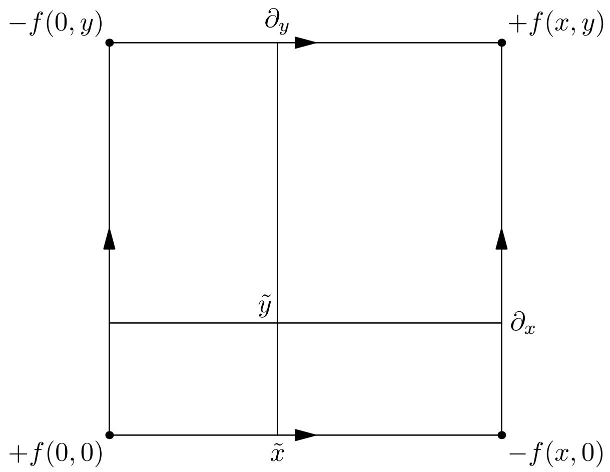
Define

$$H(x,y) = \frac{f(x,y) - f(x,0) - f(0,y) + f(0,0)}{xy}.$$

We claim that

$$\lim_{\substack{x \to 0 \\ y \to 0}} H(x,y) = \partial_y \partial_x f.$$

The claim will then follow by symmetry.



By using Mean Value Theorem twice as in the diagram above, we can do a trick where we force

$$H(x,y) = \partial_y \partial_x (\tilde{x}, \tilde{y})$$

where $0 \le \tilde{x} \le x$, $0 \le \tilde{y} \le y$. Since $x, y \to 0$ we get the desired lemma. $\square$

§10 March 5, 2015

§10.1 Inverse function theorem

Let $f : U_1 \to U_2$ be continuously differentiable, where $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$ are open. We're interested in whether there exists a g such that g is the inverse of f.

Suppose for now g does exist, so that we have

$$V_1 \supseteq U_1 \underset{g}{\overset{f}{\rightleftarrows}} U_2 \subseteq V_2$$

We have $f \circ g = \mathrm{id}_1$ and $g \circ f = \mathrm{id}_2$, so given $x_1 \in U_1$ and $x_2 = f(x_1) \in U_2$ we obtain

$$Df(x_1) : V_1 \to V_2 \quad \text{and} \quad Dg(x_2) : V_2 \to V_1$$

The chain rule implies that

$$Df(x_1) \circ Dg(x_2) = \mathrm{id}_2 \quad \text{and} \quad Dg(x_2) \circ Df(x_1) = \mathrm{id}_1.$$

Thus it's *necessary* that $Df(x_1)$ be invertible.

Remark 10.1. Remember that $Df(x_1)$ is a *linear map*. Hence the condition that $Df(x_1)$ is invertible is not terribly impressive; it's equivalent to some determinant not being zero.

The inverse function theorem states that this weak condition is in fact*locally* sufficient.

Theorem 10.2 (Inverse Function Theorem)

Let $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$ be vector spaces, and let $f : U_1 \to U_2$ be continuously differentiable. Suppose $Df(x_1)$ is invertible for some $x_1 \in U_1$ (in particular $\dim V_1 = \dim V_2$). Then there exists $U_1' \ni x_1$ and $U_2' \ni x_2 = f(x_1)$ neighborhoods such that the restriction $f : U_1' \to U_2'$ admits a differentiable inverse.

Remark 10.3. It turns out that in order to get a global inverse we require U_2 to be simply connected.

Example 10.4

For example, if $f : \mathbb{R} \to \mathbb{R}$ the fact that Dfx is invertible is just the condition that $f'(x_1) \neq 0$. So a function $f : \mathbb{R} \to \mathbb{R}$ is locally invertible if and only if the derivative at that point is not zero.

Example 10.5

Needless to say, global inverses may not exist. For example, consider $\mathbb{C} \mapsto \mathbb{C} \setminus \{0\}$ (where $\mathbb{C} \simeq \mathbb{R}^2$) by

$$z \mapsto \exp(2\pi i z).$$

Every neighborhood is invertible but the target space is globally not.

§10.2 Proof of the Inverse Function Theorem

The proof follows the following three steps.

1. We show that it's possible to restrict to a small enough neighborhood so that f is injective.
2. We show that as long as Df is invertible, the function is **open**, meaning it maps open sets to open sets. Hence we can let $g = f^{-1}$ as a function of *sets*; the fact that f is open then yields continuity of g.
3. Finally, we show that g is differentiable.

The hard part is the second step.

Let $U = U_1$. Throughout the proof, we will continuously shrink U and let U shrink; we'll continue using the same letter U.

§10.3 Step 0

Consider norms $\|-\|_1$ and $\|-\|_2$ on V_1 and V_2. Recall (from the homework) that for a linear map $T : V_1 \to V_2$ we defined the **norm of a linear operator** is given by

$$\|T\|_{\text{op}} \overset{\text{def}}{=} \max_{\|v_1\|_1 \le 1} \|T(v_1)\|_2 .$$

This endows $\text{Hom}_{\mathbb{R}}(V_1, V_2)$ with a metric topology.

Lemma 10.6

Let $\text{Hom}_{\mathbb{R}}(V_1, V_2)^{\text{inv}}$ denote the set of invertible linear maps. Then it is open in $\text{Hom}_{\mathbb{R}}(V_1, V_2)$.

Proof. It's vacuous if $\dim V_1 \neq \dim V_2$ since $\text{Hom}_{\mathbb{R}}(V_1, V_2)^{\text{inv}}$ is empty in that case.

Now recall that amp is invertible if and only if it has zero determinant, and

$$\det : \mathbf{Mat}_{n\times n}(\mathbb{R}) \to \mathbb{R}$$

is continuous because "it consists of multiplying things".

> "This is classic Math 55, we spend a whole semester learning abstract linear algebra and then we're here and no one knows what to do." – James Tao

□

Lemma 10.7

The map from $\text{Hom}_{\mathbb{R}}(V_1, V_2)^{\text{inv}}$ to itself by $T \mapsto T^{-1}$ is continuous.

Proof. Use applied math – there is a formula for T^{-1} in terms of multiplying blah. □

Corollary 10.8

IF $Df(x_1)$ is invertible then there is a neighborhood $U \subseteq U_1$ such that $Df(x_1^{-1})$ is invertible for all $x \in U$ and the map

$$x \mapsto (Df(x))^{-1}$$

is continuous.

§10.4 First Step

Proposition 10.9

Let $U \subseteq U_1$ be convex, and let $\left\|(Df(x_1))^{-1}\right\|_{\text{op}} \le \Lambda$ over $f : U \to U_2$. Moreover, assume Df is uniformly continuous. Then for some $\delta > 0$, if $\|x_1' - x_1''\| < \delta$ then

$$\left\|f(x_1') - f(x_1'')\right\|_2 \ge \frac{1}{2\Lambda} \left\|x_1' - x_1''\right\|_1.$$

This will imply Step 1, because by taking r small enough we can consider a closed r-ball B of x_1 on which $Df(x_1')$ is invertible. This closed ball is compact, so Df continuous implies Df is uniformly continuous; thus $\left\|(Df)^{-1}\right\|_{\text{op}}$ is bounded on B. Applying the proposition, we let U be an open ball of radius less than half the specified δ, contained entirely inside B. Then the proposition implies the conclusion.

To prove Proposition 10.9 we first show the following Proposition 10.10.

Proposition 10.10 (“Uniform” Differentiability)

Let $g : U_1 \to U_2$ be such that Dg is uniformly continuous, where U_1 is convex. Then for all $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $\|x' - x''\|_1 < \delta$ we have

$$\frac{\|g(x'') - g(x') - Dg(x')(x'' - x')\|_2}{\|x'' - x'\|_1} < \varepsilon.$$

Proof. Identify $V_2 = \mathbb{R}^n$ with the max norm. In this way it suffices to consider $V_2 = \mathbb{R}$ (i.e. we consider everything componentwise).

This allows us to appeal to the Mean Value Theorem. Thus we find that there exists $\tilde{x}$ lying on the line segment joining x' to x'' for which

$$Dg(\tilde{x})(x' - x'') = g(x') - g(x'').$$

(Here we use convexity so that Dg is defined on the entire segment from x' to x''.) Thus the fraction is question is bounded by the operator norm

$$\left\|Dg(\tilde{x}) - Dg(x')\right\|_{\text{op}}.$$

Hence the conclusion follows by uniform continuity. □

Proof of Proposition 10.9 from Proposition 10.10. Let $\varepsilon = \frac{1}{2\Lambda}$ and let δ be as in Proposition 10.10.

Define

$$v = f(x_1'') - f(x_1') - Df(x_1')(x_1'' - x_1').$$

We are given that

$$\|v\|_2 < \frac{1}{2\Lambda} \left\| x'' - x' \right\|_1 .$$

Apply $(Df(x_1'))^{-1}$ to v. On one hand, we have

$$(Df(x_1'))^{-1}(v) \le \frac{1}{2} \left\| x_1'' - x_1' \right\|_1 .$$

On the other hand

$$(Df(x_1'))^{-1}(v) = \left\| (Df(x_1')^{-1})(f(x_1'') - f(x_1')) - (x_1'' - x_1') \right\|_1 .$$

By Triangle Inequality, we thus derive

$$\left\| (Df(x_1')^{-1})(f(x_1'') - f(x_1')) \right\|_1 \ge \frac{1}{2} \left\| x_1'' - x_1' \right\| .$$

Hence

$$\left\| f(x_1'') - f(x_1') \right\|_2 \ge \frac{1}{2\Lambda} \left\| x_1'' - x_1' \right\| .$$

□

§10.5 Third Step

In the spirit of procrastination, we now skip to step three.

Let $f : U_1' \to U_2'$ be a injective and surjective. Assume that Df is invertible and open, so that it has a continuous inverse g. We wish to show that g is in fact continuous.

We claim that if $f(x_1) = x_2$, then

$$Dg(x_2) = Df(x_1)^{-1}$$

(indeed, we know *a priori* that this ought to be the case).

We wish to show now that

$$\lim_{x_2' \to x_2} \frac{\left\| g(x_2') - g(x_2) - (Df(x_1)^{-1})(x_2' - x_2) \right\|_1}{\|x_2' - x_2\|_2} = 0.$$

Let $x_2 = f(x_1)$ and $x_2' = f(x_1')$. So the expression rewrites as

$$\frac{\left\| \left(Df(x_1)^{-1}\right) \left(Df(x_1)(x_1' - x_1) - (f(x_1') - f(x_1))\right) \right\|_1}{\|f(x_1') - f(x_1)\|_2}.$$

This is at most

$$\left\| Df(x_1)^{-1} \right\| \cdot \frac{\|f(x_1') - f(x_1) - Df(x_1)(x_1' - x_1)\|}{\|x_1' - x_1\|} \cdot \frac{\|x_1' - x_1\|}{\|f(x_1') - f(x_1)\|}.$$

The first term is bounded, the middle term tends to zero (by definition), and the last term is bounded by some constant due to Proposition 10.9.

§10.6 Second Step

We do so in complete generality.

Proposition 10.11

Let $f : U_1 \to U_2$ be *any* continuous function (not necessarily injective), so that $Df(x_1)$ is invertible at each $x_1 \in U_1$.

Then for all $x_1 \in U_1$, there exists $\varepsilon > 0$ and $r > 0$ such that

$$B_2(f(x_1), \varepsilon) \subseteq f``(B_1(x, r)).$$

(Here as usual, $f``(S) = \{f(s) \mid s \in S\}$ is the pointwise image, while $B_i$ denotes open balls in $V_i$). This is enough to show that $f``(\text{open})$ is open (the balls form a basis), meaning that f is an open mapping.

(Note: the "U_1" here is not the same as for our original function Specifically, to show that $g : U_1 \to U_2$ is an open mapping we are going to apply the proposition $g|_U$ for some $U \subseteq U_1$.)

First, we prove the following theorem on contractions, which we'll continue to use in the next lecture repeatedly!

Theorem 10.12 (Contractions of Complete Spaces Have Unique Fixed Points)

Let X be a complete metric space with metric ρ, and let $\phi : X \to X$ a **contraction**, meaning for some $0 < \lambda < 1$ such that

$$\rho(\phi(x_1), \phi(x_2)) \le \lambda \rho(x_1, x_2).$$

Then ϕ has a unique fixed point.

> "There is a traffic interpretation of this. The gas tank of your car has a leak so eventually you run out of gas." – James Tao

Proof. Uniqueness is straightforward; if $\phi(x_1) = x_1$ and $\phi(x_2) = x_2$ then

$$\rho(x_1, x_2) < \lambda \phi(x_1, x_2).$$

For existence, consider the Cauchy sequence

$$x \; \phi(x) \; \phi(x^2) \; \dots.$$

By completeness, it converges to some point p. You can check that $\phi(p) = p$. $\square$

To be finished next week...

§11 March 10, 2015

§11.1 Completing the Proof of Inverse Function Theorem

Definition 11.1. A function $f : X_1 \to X_2$ is **open** if it sends open sets in X_1 to open sets in X_2.

This is the reverse of our typical continuity in which we require the *pre-image* of an open set in X_2 to be an open set in X_1. In the same way that this gives an ε-δ continuity for metric spaces, we can rewrite this as

$$(\forall x_1 \in X_1)(\forall r > 0)(\exists \varepsilon > 0) f``(B(x,r)) \supseteq B(fx_1, \varepsilon).$$

By fiddling with the radius, we can replace this with

$$(\forall x_1 \in X_1)(\forall r > 0)(\exists \varepsilon > 0) f``\left(\overline{B}(x,r)\right) \supseteq B(fx_1, \varepsilon).$$

where $\overline{B}$ is the closed a ball of radius r.

We need only prove the following result now.

Theorem 11.2

If $Df(x_1)$ is invertible, then f is open on a neighborhood of x_1.

Proof of Theorem. Let $\varepsilon > 0$ be small enough to be determined later, and let $x_2' \in B(fx_1, \varepsilon)$ be arbitrary. We wish to show there exists an x_1' so that $fx_1' = x_2'$.

We define a map $\Phi : \overline{B}(x_1, r) \to V_1$ by

$$\Phi(x_1') = x_1' - (Dfx_1)^{-1}\left(f(x_1') - x_2'\right)$$

Observe that finding x_1' as above amounts to finding a fixed point of Φ.

We wish to shrink r sufficiently so that

$$\left\|\Phi(x_1'') - \Phi(x_1')\right\| \le \frac{1}{2}\left\|x_1'' - x_1'\right\|.$$

Compute

$$D\phi(x_1') = (Df(x_1))^{-1}\left(Df(x_1) - Df(x_1')\right).$$

Let $\Lambda = \left\|Dff(x_1)^{-1}\right\|$, so that

$$\left\|D\phi(x_1'')\right\| \le \Lambda \left\|Df(x_1) - Df(x_1')\right\|.$$

We can choose r small enough that the right-hand side is at most $\frac{1}{2}$, by forcing $\|Df(x_1) - Df(x_1')\| \le \frac{1}{2\Lambda}$ (continuous functions can have arbitrarily small difference).

Recall from a previous problem set that said that We now invoke the following result from Problem Set 6.

1(c). Let $U \subset V$ be convex and let $f : U \to W$ be a continuously differentiable function. Let W be also endowed with a norm, and suppose that $\|Df(x)\| \le \Lambda$ for all $x \in U$. Show that

$$\left\|f(x') - f(x'')\right\| \le \Lambda \cdot \left\|x' - x''\right\|, \quad \forall x', x'' \in U.$$

Applying the result to $g = D\phi$ gives the desired estimate, so that Φ is a contraction by a factor of $\frac{1}{2}$.

It remains to show that Φ maps $\overline{B}(x_1, r)$ into itself, at which point we can apply the fact that contractions have fixed points. We have that

$$\begin{aligned}
\left\|\Phi(x_1') - x_1\right\| &= \left\|\Phi(x_1') - \Phi(x_1)\right\| + \left\|\Phi(x_1) - x\right\| \\
&\le \frac{1}{2}\left\|x_1' - x_1\right\| + \left\|\Phi(x_1) - x_1\right\| \\
&\le \frac{1}{2}r + \left\|\Phi(x_1) - x_1\right\| \\
&\le \frac{1}{2}r + \left\|(Dfx_1)^{-1}(x_2 - x_2')\right\| \\
&\le \frac{1}{2}r + \left\|(Dfx_1)^{-1}\right\| \left\|(x_2 - x_2')\right\| \\
&= \frac{1}{2}r + \Lambda\left\|(x_2 - x_2')\right\|.
\end{aligned}$$

If we select the ε earlier to guarantee that $\|x_2 - x_2'\| < \frac{r}{2\Lambda}$ then we're done. $\square$

§11.2 Implicit Function Theorem

Here is the "high-school version" Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ and $f(x_0, y_0) = z_0$, and moreover $\partial_x f(x_0, y_0) \neq 0$. Then in some neighborhood of (x_0, y_0), for all y there is a unique x so that $f(x, y) = z_0$.

As an example, consider $x^2 + y^2 = 1$ at the point $(\frac{3}{5}, \frac{4}{5})$. Then in some small neighborhood, the function $y = \sqrt{1 - y^2}$ gives a unique x for each y.

We now prove the full Implicit Function Theorem from the Inverse Function Theorem.

Theorem 11.3 (Implicit Function Theorem)

Consider a continuously differentiable function $f : U \to W$, where U is an open set in the vector spaces V and W is another vector space. Let $p \in U$, and let $f(p) = z$. We decompose $V = V_1 \oplus V_2$, and suppose that $Df(p)$ restricted to V_1 is an isomorphism onto W.

Show that we can find neighborhoods $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$ so that $p \in U_1 \times U_2$ and the following property holds: For any $b \in U_2$ we there exists a unique $a \in U_1$ so that $f(a + b) = z$. Moreover the function $b \mapsto a$ is continuously differentiable.

Proof. The trick is to consider the diagram

$$\begin{array}{ccc}
V_1 \oplus V_2 & \overset{Df \oplus \mathrm{id}}{\underset{\simeq}{\longleftrightarrow}} & W \oplus V_2 \\
a + b & \overset{g}{\underset{h}{\rightleftarrows}} & (f(a+b), b)
\end{array}$$

This function g is defined from the left to the right, but its differential (Df, id) at p is invertible, and hence the Inverse Function Theorem applies and gives us an inverse h with suitable restrictions.

Then given a b, we simply call h on (z, b). This gives us $a + b$ from which we can extract a. $\square$

§11.3 Differential Equations

Definition 11.4. Let $U \subseteq V$ be open. A **vector field** is a function $\xi : U \to V$.

> "You should imagine a vector field as a domain, and at every point there is a little vector growing out of it . . . What is it good for? It's good for differentiable equations." – Gaitsgory

(What happens if you water a vector field?)

Definition 11.5. Let $\gamma : (-d, d) \to U$ be a continuous path. We say γ is a solution to the differential equation defined by ξ if for each $t \in (-d, d)$ we have

$$\gamma'(t) = \xi(\gamma(t)).$$

> "As you go further along the real line, the grass grows taller and taller." – Gaitsgory

Example 11.6 (Examples of DE's)
Let $U = V = \mathbb{R}$.

(a) Consider the remarkable vector field $\xi(x) = x$. Then γ is a solution exactly when
$$\gamma'(t) = \xi(\gamma(t)) = \gamma(t).$$
This has solutions $\gamma(t) = c \exp(t)$.

(b) If we replace ξ with $\xi(x) = bx$, then we are trying to solve $\gamma'(t) = b\gamma(t)$, which has solutions $\gamma(t) = c \exp(bt)$.

(c) If $\gamma(x) = x^2$, we are trying to solve $\gamma'(t) = \gamma(t)^2$. BLAH.

> "Now we are back to theoretical math. We will not try to solve them, but we will try to ensure that the solutions exist.
>
> . . . And moreover they're unique!" – Gaitsgory

In general if $\xi : \mathbb{R}^m \to \mathbb{R}^n$ then we can consider $\xi = (\xi_1, \xi_2, \dots, \xi_n)$, and $\gamma_t = (\gamma_1, \dots, \gamma_n)$ by consider an appropriate projection. Then the differential equation reads

$$\gamma_i'(t) = \xi_i(\gamma_1(t), \dots, \gamma_n(t)).$$

Remark 11.7. You might see differential equations such as $f'(t) = \lambda/tf(t)$, where t itself appears in the right-hand side. On the problem set we will see how to convert such *time-dependent* differential equations into time-independent differential equations to make it fit into the right-hand side.

Remark 11.8. We will also be able to convert $\gamma''(t) = -c/\gamma(t)^2$ into the form prescribed above (see problem set).

Theorem 11.9

Let $U \subseteq V$, $\xi : U \to V$ and assume ξ satisfies the **Lipschitz condition**, meaning

$$\left\|\xi(x') - \xi(x'')\right\| \le \Lambda \left\|x' - x''\right\|$$

holds identically for some fixed Λ.

(a) For every $x_0 \in U$ there exists $(-d, d)$ and $\gamma : (-d, d) \to U$ such that $D\gamma(t) = \xi(\gamma(t))$ and $\gamma(0) = x_0$. (In other words, given an initial condition $\gamma(0) = x_0$ we can find a solution.)

(b) If γ_1 and γ_2 are two solutions and $\gamma_1(t) = \gamma_2(t)$ for some t, then $\gamma_1 = \gamma_2$.

Note that continuous differentiability is enough to imply the Lipschitz condition.

Example 11.10 (Counterexample if ξ is not differentiable)

Let $U = V = \mathbb{R}$ and consider $\xi(x) = x^{\frac{2}{3}}$, with $x_0 = 0$. Then $\gamma(t) = 0$ and $\gamma(t) = (t/3)^3$ are both solutions to the differential equation

$$\gamma'(t) = \gamma(t)^{\frac{2}{3}}.$$

Proof. Let X be the metric space of continuous functions from $(-d, d)$ to the complete metric space $\overline{B}(x_0, r)$ (since the target space is bounded all functions are bounded). By Problem 1 on PSet 4, we know X is a complete metric space.

We wish to use the contraction principle on X, so we'll rig a function $\Phi : X \to X$ with the property that its fixed points are solutions to the differential equation.

Define

$$\Phi(\gamma) : t \mapsto \begin{cases} x_0 + \int_{s\in[0,t]} \xi(\gamma(s)) & \text{if } t \ge 0 \\ x_0 - \int_{s\in[-t,0]} \xi(\gamma(s)) & \text{if } t < 0. \end{cases}$$

(The cases are just to handle the fact that we never bothered to define $\int_b^a$ for $a < b$. In fact, for all future calculations in this proof we only consider the case $t > 0$ so one can ignore the second line.) This function is contrived so that $(\Phi\gamma)(0) = x_0$ and $\Phi\gamma$ is both continuous and differentiable. By the Fundamental Theorem of Calculus, the derivative is exhibited by

$$(\Phi\gamma)'(t) = \left(\int_{s\in[0,t]} \xi(\gamma(s))\right) = \xi(\gamma(t)).$$

In particular, fixed points correspond exactly to solutions to our differential equation.

A priori this output has signature $\Phi\gamma : (-d, d) \to V$, so we need to check that $\Phi\gamma(t) \in \overline{B}(x_0, r)$. We can check that

$$\begin{aligned} \|(\Phi\gamma)(t) - x_0\| &= \left\| \int_{s\in[0,t]} \xi(\gamma(s) \right\| \\ &\le \int_{s\in[0,t]} \|\xi(\gamma(s))\| \\ &\le t \max_s \|\xi\gamma(s)\| \\ &< dA \end{aligned}$$

where $A = \max_{x \in \overline{B}(x,r)} \|\xi(x)\|$; we have $A < \infty$ since $\overline{B}(x,r)$ is compact. Hence by selecting $d < r/A$, the above is bounded by r, so $\Phi\gamma$ indeed maps into $\overline{B}(x_0, r)$. (Note that at this point we have not used the fact that ξ is uniformly continuous.)

It remains to show that Φ is contracting. We wish to show that

$$\rho\left(\Psi(\gamma_1), \Psi(\gamma_2)\right) \le \lambda \rho(\gamma_1, \gamma_2)$$

where ρ is the sup norm given by

$$\rho(fx, gx) = \sup_{t \in (-d,d)} \rho_X\left(ft, gt\right).$$

Write

$$\begin{aligned}
\|(\Phi\gamma_1)(t) - (\Phi\gamma_2)(t)\| &= \left\| \int_{s \in [0,t]} \left(\xi(\gamma_1(s)) - \xi(\gamma_2(s))\right) \right\| \\
&= \int_{s \in [0,t]} \|\xi(\gamma_1(s)) - \xi(\gamma_2(s))\| \\
&\le t\Lambda \sup_{s \in [0,t]} \|\gamma_1(s) - \gamma_2(s)\| \\
&< d\Lambda \sup_{s \in [0,t]} \|\gamma_1(s) - \gamma_2(s)\| \\
&= d\Lambda \rho(\gamma_1, \gamma_2).
\end{aligned}$$

Hence once again for d sufficiently small we get $d\Lambda \le \frac{1}{2}$. Since the above holds identically for t, this implies

$$\rho(\Psi\gamma_1, \Psi\gamma_2) \le \frac{1}{2}\rho(\gamma_1, \gamma_2)$$

as needed. $\square$

§11.4 Lawns

Let ξ be continuously differentiable. Next week we will consider a function $U \times \mathbb{R} \to U$ by sending (x_0, t) to $\gamma(t)$, where $\gamma : (-d, d) \to U$ is the solution to the differential equation $\gamma(0) = x_0$ (and $\gamma' = \xi \circ \gamma$). We will show this function is nice.

From this, we will deduce that if $\xi(x) \ne 0$ for all x, then U can be diffeomorph'ed into another U_1 for which ξ is just a constant function. Hence,

> The right way to make your lawn look nice is to apply a diffeomorphism to the soil.

§12 March 12, 2015

'Twas Housing Day. Since the last thing I wanted was to hear more about Houses, I elected to sleep through Math 55. Notes today are again from W Mackey.

Let $U \subseteq V$, and ξ be a vector field. Recall that a solution to a differential equation is a γ such that $D\gamma(t) = \xi(\gamma(t))$.

Example 12.1

If $\xi(x) = v$ for every x, then we have constant speed. So $\gamma(t) = x + tv$.

> "Wouldn't it be nice if every differential equation was like this? The good thing is, it kind of is." –Gaitsgory

By this, we mean that we can always diffeomorph into our constant vector field, so long as the original vector field vanishes nowhere.

Assume there is a Λ such that $||\xi(x_1) - \xi(x_2)|| \le \Lambda||x_1 - x_2||$. Then for every $x \in U$, there is an interval $(-d, d)$ and a solution γ such that γ is a solution to the differential equation and $\gamma(0) = x$. In our proof last class, we actually had an algorithm: set $\gamma_0(t) = x$, and $\gamma_{n+1}(t) = x + \int_{[0,t]} \xi(\gamma_n(s))$. Then there exist small enough d such that each $\gamma_n(-d, d) \subseteq U$, and γ_n converges uniformly to a solution.

Proposition 12.2

For every $x \in U$, there is $x \in U' \subseteq U$ and d such that $\gamma_n^{\text{master}}(x, t)$, defined inductively by

$$\begin{aligned} \gamma_0^{\text{master}}(x,t) &= x \\ \gamma_n^{\text{master}}(x,t) &= x + \int \xi(\gamma_{n-1}(x,s)) \end{aligned}$$

are continuous functions from $U' \times (-d, d) \to U$ that converge uniformly. Let the limit be γ^{master}. Then γ^{master} is a function not just of time, but the original position. Then γ^{master} is continuous, and $D_{\text{time}}\gamma^{\text{master}}(x,t) = \xi(\gamma^{\text{master}}(x,t))$.

Proof. This is the same as last time–d is only dependent on Λ, so by taking smaller U', it just works. $\square$

> "I could say about 60% of math is about differential equations."

Theorem 12.3

Consider $\gamma^{\text{master}}(x,t) : U' \times (-d, d) \to U$. If ξ is continuous differentiable $n+1$ times, then γ^{master} will be continuously differentiable n times (possible on a smaller U', d).

> "The proof of this theorem I find fun... It's kind of like the ultimate use of the chain rule." –Gaitsgory

Proof. To construct this, we'll consider something else that spits it out. Set $\widetilde{V} := V \oplus \text{End}(V)$. Note $\dim(\widetilde{V}) = n + n^2$. Then set $\widetilde{U} = U \oplus \text{Hom}(V, V)$, and $\widetilde{\xi}(x, T) = (\xi(x), D\xi(x) \circ T)$.

> Matt: "Is there any intuition for why we're doing this?"
>
> Gaitsgory: "That's a great question. No." (This is because we get seemingly unnatural choices since we're doing a special case of the theorem. It makes far more sense we we do the general theorem on manifolds.)

Then there is

$$\gamma^{\text{master}} : \widetilde{U}' \times (-d, d) \to \widetilde{V}$$

is a solution to $\widetilde{\xi}$. We're interested in (x, id_V). Then

$$\widetilde{\gamma}^{\text{master}} : U' \times (-d, d) \to V \times \text{End}(V)$$

is given by $({}'\gamma^{\text{master}}(x,t), \delta(x,t))$. We claim that ${}'\gamma^{\text{master}} = \gamma^{\text{master}}$. We know $\widetilde{\gamma}^{\text{master}}$ satisfies

- $D_{\text{time}}({}'\gamma^{\text{master}}(x,t)) = \xi({}'\gamma^{\text{master}}(x,t))$
- $D_{\text{time}}\delta(x,t) = D\xi({}'\gamma^{\text{master}}(x,t)) \circ \delta(x,t)$

Then

$$D_{\text{time}}\widetilde{\gamma}^{\text{master}}(x,t) = \widetilde{\xi}(\widetilde{\gamma}^{\text{master}}(x,t)),$$

and

$$\widetilde{\gamma}^{\text{master}}(x,t) = ({}'\gamma^{\text{master}}(x,t), \delta(x,t)).$$

Then just follow through on the components. Note that ${}'\gamma(x,0) = x$, so we have the same initial condition, hence, ${}'\gamma = \gamma$. From here, we'll deduce the theorem from the next theorem. $\square$

Theorem 12.4

We have $\delta(x,t) = D_{\text{space}}(\gamma(x,t))$.

Corollary 12.5

Theorem 12.4 implies

$$D_{\text{time}}D_{\text{space}}\gamma(x,t) = D\xi(\gamma(x,t)) \circ D_{\text{space}}(\gamma(x,t)).$$

We now claim that Theorem 12.4 implies Theorem 12.3: We'll proceed by induction on n. For the base case, we want to show that the partials are continuously differentiable. Note $D_{\text{time}}\gamma(x,t) = \xi(\gamma(x,t))$ is continuous, immediately. Then $D_{\text{space}}\gamma(x,t) = \delta(x,t)$ is continuous since it's a component of $\widetilde{\gamma}$.

Now suppose we know the theorem for n, and ξ is differentiable $n+2$ times. We want γ to be continuously differentiable $n+1$ times. Then we have $D_{\text{time}}\gamma$ differentiable n times and $D_{\text{space}}\gamma$ differentiable n times. Then just take the derivatives and apply the hypothesis, it works. It just remains to prove Theorem 12.4.

Proof. Set $\gamma_0(x,t) = x$, $\delta_0(x,t) = \text{id}$, and let

$$\gamma_{n+1} = x + \int_0^t \xi(\gamma_n(x,s)) \quad \delta_{n+1} = \text{id} + \int_0^t D\xi(\gamma_n(x,s) \circ \delta_n(x,s).$$

Then we claim that $D_{\text{space}}\gamma_n(x,t) = \delta_n(x,t)$. We know that

$$\gamma_n \xrightarrow{\text{uniformly}} \gamma$$

and

$$\delta_n \xrightarrow{\text{uniformly}} \delta$$

so by our theorem on the derivative of a limit, $D_{\text{space}}\gamma = \delta$, so long as it's true on each component. Now by induction–it's true on the base case. Suppose it's true for n. Then

$$D_{\text{space}}\gamma_{n+1}(x,t) = D_{\text{space}}(x + \int \xi(\gamma_n(x,s)) = \text{id} + D_{\text{space}} \int \xi(\gamma_n(x,s)),$$

which by midterm problem 2, is

$$\text{id} + \int D_{\text{space}}\xi(\gamma_n(x,s)).$$

Then we want

$$D_{\text{space}}(\xi(\gamma_n(x,s))) = D\xi(\gamma_n(x,s)) \circ \delta_n(x,s).$$

This is just the chain rule and the inductive hypothesis, however. $\square$

Theorem 12.6

Let ξ be a vector field such that $\xi(x) \neq 0$ at x. There there is $U \ni x$ with $U \xrightarrow{f} \widetilde{U} \subseteq V' \times \mathbb{R}$ such that f is a diffeomorphism and f transforms ξ to the vector field $(0,1)$.

Definition 12.7. Let

$$\begin{array}{ccc} U & \xrightarrow{f} & \widetilde{U} \\ \cap & & \cap \\ V & & \widetilde{V} \end{array}$$

with f a diffeomorphism (both it and its inverse are differentiable), and ξ be a vector field on V. Then suppose $\widetilde{\xi}(V) = Df(x)(\xi(x))$. Then we say that f **transforms** ξ to the vector field $\widetilde{\xi}$. This is a sensible map, in the sense that it maps solutions to the differential equation to other solutions of the differential equation.

Proof. Assume ξ is continuously differentiable, and $\gamma^{\text{master}} : U' \times (-d,d) \to U$. Let $V' \subsetneq V$ be such that

$$V' \oplus \text{Span}(\xi(x)) = V.$$

Then define $g : V' \times (-d,d) \to U$ by

$$g(v',t) = \gamma^{\text{master}}(x+v',t).$$

so $Dg(0,0)$ is an invertible map from $V' \times \mathbb{R} \to V$. Then $D_{\text{time}}g(0,0) = \xi(x)$, and $D_{\text{space}}g(0,0) = \delta(x) = \text{id}$. Hence the Inverse Function Theorem applies. Then we want that g transforms the constant vector field $(0,1)$ to ξ. That is, we want to show $Df(x',t)(0,1) = \xi(g(v',t))$. We have

$$\begin{aligned} \lim_{s \to 0} \frac{g(v',t+s) - g(v,t)}{s} &= D_{\text{time}}\gamma^{\text{master}}(v',t) \\ &= \xi(\gamma^{\text{master}}(v',t)) \end{aligned}$$

as desired. $\square$

§13 March 24, 2015

§13.1 Midterm Solutions

3. Let $\gamma : [0,1] \to V$ be a continuous function such that its derivative $\gamma' : (0,1) \to V$ extends to a continuous function on $[0,1]$. Fix a norm on V, and assume the following additional property:

$$||v_1 + v_2|| = ||v_1|| + ||v_2|, \Rightarrow v_1 \text{ and } v_2 \text{ are proportional.}$$

Show that the following conditions are equivalent:

- $\ell(\gamma) = ||\gamma(1) - \gamma(0)||$.
- There exists $v \in V$ such that $\gamma'(t) = a(t) \cdot v$ for all $t \in (0,1)$ with $a(t) \in \mathbb{R}^{\ge 0}$.
- There exists $v \in V$ such that $\gamma(t) = \gamma(0) + b(t) \cdot v$ for all $t \in [0,1]$ with $b(t)$ satisfying $t_1 \le t_2 \Rightarrow b(t_1) \le b(t_2)$.

The only hard part is showing that the first implies the third. If not, we have

$$\|\gamma(1) - \gamma(0)\| < \|\gamma(1) - \gamma(t)\| + \|\gamma(t) - \gamma(0)\| \le \ell(\gamma : 0 \to t) + \ell(\gamma : t \to 1) = \ell(\gamma).$$

This is a contradiction.

2. Let f be a continuous function $[a_1, b_1] \times (a_2, b_2) \to \mathbb{R}$, such that for any $t_1 \in [a_1, b_1]$, the function $f(t_1, -) : (a_2, b_2) \to \mathbb{R}$ is differentiable, and the resulting function

$$\partial_2 f(t_1, t_2) : [a_1, b_1] \times (a_2, b_2) \to \mathbb{R}$$

is continuous. Consider the function

$$F : (a_2, b_2) \to \mathbb{R}, \quad F(t_2) := \int\limits_{t_1 \in [a_1, b_1]} f(t_1, t_2).$$

Show that F is differentiable and

$$F'(t_2) = \int_{t_1 \in [a_1, b_1]} \partial_2 f(t_1, t_2).$$

The point is to show that

$$0 = \lim_{h \to 0} \left(\frac{\int_{s \in [a,b]} f(s, t+h) - \int_{s \in [a,b]} f(s,t)}{h} - \int_{s \in [a,b]} \partial_t f(s,t) \right)$$

Fix $\varepsilon > 0$ and let δ be such that

$$\rho\left((t_1', t_2'), (t_1, t_2)\right) < \delta \implies \left\|\partial_2 f(t_1', t_2') - \partial_2 f(t_1, t_2)\right\| < \frac{\varepsilon}{b-a}.$$

We claim that

$$\left\| \frac{f(t_1, t_2') - f(t_1, t_2)}{t_2' - t_2} - \partial_2 f(t_1, t_2) \right\| \le \frac{\varepsilon}{b_1 - a_1}.$$

By the Mean Value Theorem there is a $|tildet_2$ such that the fraction above equals $\partial_2 f(t_1, \tilde{t_2}) - \partial_2 f(t_1, t_2)$.

1. (Arzela-Ascoli) Let X and Y be compact metric spaces, and let $\{f_n\}$ be a sequence of elements in $\text{Funct}_{\text{cont}}(X, Y)$. Assume that the sequence satisfies the following additional hypothesis:

> For every ϵ there exists δ such that for $\rho(x_1, x_2) < \delta$ we have $\rho(f_n(x_1), f_n(x_2)) < \epsilon$ for every n.

Show that under this hypothesis, the sequence $\{f_n\}$ does have a convergent subsequence.

Let X be compact, and let x_1, x_2, … be dense in X (possible since X is compact).

Lemma 13.1

The sequence (f_n) contains a subsequence g_n such that $g_n(x_i)$ converges.

Proof. Uses a diagonal argument. □

Now we claim that the g_n are Cauchy. For all $\varepsilon > 0$ we seek N such that for all X, we have

$$\|g_n(x) - g_m(x)\| < \varepsilon.$$

Let δ be such that $\rho(x', x'') < \delta$ implies $\rho(f_n x', f_n x'') < \frac{1}{3}\varepsilon$ (here we use the equi continuous assumption). In particular,

$$\rho(g_n(x'), f_n(x'')) < \frac{1}{3}\varepsilon \quad \forall n.$$

There exist finitely many balls of radius $\frac{1}{2}\delta$ that cover X, since X is compact. Thus there exists for each j a point from our dense set x_{i_j}. Hence for all x there exists j such that

$$\rho(x, x_{i_j}) < \delta.$$

The rest is clear.

§13.2 PSet Review

7. Let ξ be a vector field on a domain $U \subset V$. Let $U' \subset U$ and $(-d, d) \subset \mathbb{R}$ be such that there exists a function

$$\gamma^{\text{master}} : U' \times (-d, d) \to U$$

that satisfies:

- $D_{\text{time}}\gamma^{\text{master}}(x, t) = \xi(\gamma(x, t))$;
- $\gamma^{\text{master}}(x, 0) = x$.

In this case, we shall say that the *solution to the differential equation defined by ξ is defined on $U' \times (-d, d)$*. For $-d < t < d$ denote

$$\phi_t(x) = \gamma^{\text{master}}(x, t),$$

viewed as a function $U' \to U$.

(a) Let $0 < d' \le d$ and $U'' \subset U'$ be such that $\gamma^{\text{master}}(x,t) \in U'$ for $x \in U''$ and $-d' < t < d'$. Show that the solution

$$\gamma^{\text{master}} : U'' \times (-d-d', d+d') \to U$$

is defined and the corresponding maps ϕ_t satisfy

$$\phi_{t+t'}(x) = \phi_t(\phi_{t'}(x)) \text{ for } -d' < t' < d' \text{ and } -d < t < d.$$

Because of the above property, we call the maps ϕ_t the *flow* associated to ξ.

Fix x. To show that it's defined for $t < d + d'$, note that $\gamma^{\text{master}}(x, t-d) \in U'$, and so we may apply the flow again (shifted by d) to define $\gamma^{\text{master}}(x,t) \in U$. Similarly for $t > -(d+d')$.

(b, (mandatory, but still bonus 1pt)) Assume that ξ is differentiable $(n+1)$ times with $n \ge 1$. Show that γ^{master} is differentiable n times and

$$D_{\text{time}} D_{\text{space}} \gamma^{\text{master}}(x,t) = D\xi(\gamma^{\text{master}}(x,t)) \circ D_{\text{space}} \gamma^{\text{master}}(x,t).$$

NB: in class we proved the above assertion for some U' and d small enough. So, in this problem you're being asked to go from small (U', d) to any (U', d) on which the master solution is defined.

Abbreviate $\gamma = \gamma^{\text{master}}$. We have a function

$$D_{\text{space}} \gamma : U' \times (-d, d) \to \operatorname{Hom}(V, V).$$

We want to check

$$D_{\text{time}} D_{\text{space}} \gamma(x,t) = D\xi(\gamma(x,t)) \circ D_{\text{space}} \gamma(x,t).$$

Observe that the partials commute, so we can rewrite the LHS as

$$D_{\text{space}} D_{\text{time}} \gamma(x,t) = D_{\text{space}} \xi(\gamma(x,t)) = (D_{\text{space}} \xi \gamma(x,t)) \circ D_{\text{space}} \gamma(x,t)$$

the last step following from the chain rule.

(c, (mandatory, but still bonus 2pts)) Show that the maps $\phi_t : U' \to U$ have an invertible differential (see Problem 6) for every $-d < t < d$ at every $x \in U'$.

It's true when $t = 0$, and hence for small neighborhoods (say by looking at matrices). Then use compactness.

8. Let ξ_1 and ξ_2 be vector fields on a domain $U \subset V$, both continuously differentiable twice. Let $U' \subset U$ and $0 < d$ be such that master solutions both $\gamma_{\xi_2}^{\text{master}}$ and $\gamma_{\xi_1}^{\text{master}}$ for both ξ_1 and ξ_2 are defined on $U' \times (-d, d)$.

Let $U'' \subset U'$ let $0 < d' \le d$ be such that for $i = 1, 2$ we have

$$\gamma_{\xi_i}^{\text{master}}(x,t) \in U' \text{ for } x \in U'' \text{ and } -d' < t < d'.$$

In particular, the expressions

$$\gamma_{\xi_2}^{\text{master}}(\gamma_{\xi_1}^{\text{master}}(x,t_1),t_2) \text{ and } \gamma_{\xi_1}^{\text{master}}(\gamma_{\xi_2}^{\text{master}}(x,t_2),t_1)$$

are defined for $x \in U''$ and $-d' < t_1, t_2 < d'$.

(a, (mandatory, but still bonus 2pts)) Assume that $[\xi_1, \xi_2] = 0$. Show that

$$\gamma_{\xi_2}^{\text{master}}(\gamma_{\xi_1}^{\text{master}}(x,t_1),t_2) = \gamma_{\xi_1}^{\text{master}}(\gamma_{\xi_2}^{\text{master}}(x,t_2),t_1)$$

for every $x \in U''$ and $-d'' < t_1, t_2 < d''$ for some $0 < d'' \le d'$.

NB: in what follows we shall say that two vector fields ξ_1 and ξ_2 *commute* if $[\xi_1, \xi_2] = 0$. Thus, the above problem says that if vector fields commute, then the flows that they define commute as well.

Hint: reduce to the case when one of the two fields doesn't vanish at x and use the straightening theorem.

If both fields vanish at x, then both sides of the equation are equal to x (nothing moves). Otherwise, assume ξ_1 doesn't vanish, take U'' a neighborhood at x non-vanishing, and transform to a situation where ξ_1 is a constant vector field with, say, value e_1 (in $\tilde{V}$). Then this amounts to showing that

$$\gamma_{\xi_2}^{\text{master}}(x + e_1 t_1, t_2) = \gamma_{\xi_2}^{\text{master}}(x, t_2) + e_1 t_1.$$

According to Problem 4(e), $[\xi_1, \xi_2] = 0$ implies that ξ_2 is independent of the first coordinate, so this follows.

9. Let ξ be a vector field on a domain $U \subset V$, continuously differentiable twice. Let $U' \subset U$ and $(-d, d) \subset \mathbb{R}$ be such that

$$\gamma^{\text{master}} : U' \times (-d, d) \to U$$

is defined.

For another continuously differentiable vector field η on U, consider the (time-dependent) vector field η_t on U', equal to $\phi_t^*(\eta)$ (see Problem 6). I.e., we regard η_t as a function

$$U_x \times (-d, d) \to V.$$

The ϕ_t is reversible by 7(c). So we want to think of a vector field

$$(\phi_t)^*(\eta) : U' \times (-d, d) \to V.$$

(b, (mandatory, but still bonus 2pts)) Consider the vector field $D_{\text{time}}\eta_t(x, 0)$ on U'. Show that

$$D_{\text{time}}\eta_t(x, 0) = [\xi, \eta](x).$$

Hint: use your knowledge of $D_{\text{time}} D_{\text{space}} \gamma^{\text{master}}(x, t)$.

Chain Rule. The left-hand side is

$$D_{\text{time}}\left((D\phi_t)^{-1} (\eta \phi_t x) \right).$$

Here $D\phi_t$ is the matrix and $\eta\phi_t x$ is a vector. Using the general product rule at $t = 0$, we get

$$D_{\text{time}}\left((D\phi_t)^{-1}\right)(\eta\phi_0 x) + (D\phi_t)^{-1} D_{\text{time}}(\eta\phi_t x) = D_{\text{time}}\left((D\phi_t)^{-1}\right)(\eta x) + D_{\text{time}}(\eta\phi_t x).$$

Recall that we want the answer to be $D_\xi\eta(x) - D_\eta\xi(x)$.

First, the Chain Rule gives

$$D_{\text{time}}(\eta\phi_t x). = D\eta(x) \circ \left(D_{\text{time}}\phi_t(x)|_{t=0}\right) = D\eta(x) \circ \xi(x)$$

the last step following from the differential equation. By definition, this equals $D_\xi\eta(x)$.

Now, we want to show

$$D_{\text{time}}\left((D\phi_t)^{-1}\right)\Big|_{t=0} = -D\xi(x).$$

Once this is done we can substitute in directly. We use the trick that

$$D_{\text{time}}\left(A(t)^{-1}\right) = -D_{\text{time}}\left(A(t)\right).$$

This follows from the fact that

$$0 = D_{\text{time}}\left(A(t) \cdot A(t)^{-1}\right)$$

and applying the Leibniz rule. So, it only remains to compute that

$$\begin{aligned} D_{\text{time}}\left(D\phi_t\right)|_{t=0} &= D_{\text{time}}D_{\text{space}}\gamma|_{t=0} \\ &= D\xi(\gamma(x,t)) \cdot D_{\text{space}}\gamma(x,t)|_{t=0} \\ &= D\xi(\gamma^{\text{master}}(x,0)) \cdot D_{\text{space}}\text{id}_V \\ &= D\xi(x). \end{aligned}$$

5. Let ξ_1, ξ_2, ξ_3 be vector fields, each continuously differentiable twice. Prove the Jacobi identity

$$[[\xi_1,\xi_2],\xi_3] + [[\xi_3,\xi_1],\xi_2] + [[\xi_2,\xi_3],\xi_1] = 0$$

(preferably, without long formulas).

Appeal to part (f) previously. We have

$$\begin{aligned} [[\xi_1,\xi_2,],\xi_3] &= D_{[\xi_1,x_2]}\xi_3 - D_{\xi_3}[\xi_1,\xi_2] \\ &= [D_{\xi_1}, D_{\xi_2}](\xi_3) - D_{\xi_3}(D_{\xi_1}\xi_2 - D_{\xi_2}\xi_1) \\ &= D_{\xi_1}D_{\xi_2}\xi_3 - D_{\xi_2}D_{\xi_1}\xi_3 - D_{\xi_3}D_{\xi_1}\xi_2 + D_{\xi_3}D_{\xi_2}\xi_1. \end{aligned}$$

Cyclically summing yields the conclusion.

§13.3 Review of Exterior Products

Recall the definition of $\Lambda^k(W)$ in the "quot" form and define

$$\Lambda^\bullet(W) = \bigoplus_k \Lambda^k(W).$$

Moreover, given $w_1, \dots, w_n \in W$ we let

$$w_1 \wedge \dots \wedge w_n$$

denote the image of $w_1 \otimes \dots \otimes w_n$.

In addition, given $\alpha \in \Lambda^{n_1}(W)$ and $\beta \in \Lambda^{n_2}(W)$, we may put

$$\alpha \wedge \beta = (-1)^{n_1 n_2}\beta \wedge \alpha \in \Lambda^{n_1+n_2}(W).$$

Finally, recall that if $e_1, \dots, e_m$ is a basis of W then a basis of $\Lambda^n(W)$ is

$$e_{i_1} \wedge \dots \wedge e_{i_n} \; i_1 < \dots < i_n.$$

§13.4 Differential Forms

Let $U \subseteq V$ be a domain. Let $\Omega^n(U)$ denote the set

$$C^\infty(U, \Lambda^n(V^\vee)).$$

Define $\Omega^\bullet(U) = \bigoplus_n \Omega^n(U)$.

Now suppose we have a map

$$\phi : U_1 \to U_2$$

of domains. We define a map

$$\phi^* : \Omega^n(U_2) \to \Omega^n(U_1)$$

by sending each $\alpha \in \Omega^n(U_2)$ to

$$(\phi^*(\alpha))(x) \overset{\text{def}}{=} \Lambda^n \left[((D\phi)(x))^\vee (\alpha(\phi(x))) \right] = \Lambda^n \left[(\alpha \circ \phi)(x) \circ (D\phi)(x) \right]$$

Here, given $T : W_1 \to W_2$ we have $\Lambda^n(T) = T \wedge \cdots \wedge T$.

In particular, if α is a 0-form, then a simpler definition is given by

$$\phi^*(\alpha) = \alpha \circ \phi.$$

Also,

Theorem 13.2 (de Rham Derivative)

There exists a unique map $\Omega^n \xrightarrow{d} \Omega^{n+1}(U)$ (for all n) with the following properties.

- For $f \in \Omega^0(U)$, $df = Df$.
- $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge (d\beta)$.
- $d \circ d(\alpha) = 0$.

§14 March 26, 2015

I managed to oversleep class today, so again these are from W.

> "Today I'm going to do something pretty reckless..." –Gaitsgory

§14.1 More on Differential Forms

Let $U \subseteq V$; recall we set $\Omega^0(U) = C^\infty(U)$, $\Omega^1(U) = C^\infty(U, V^\vee)$, and in general, $\Omega^k(U) = C^\infty(U, \Lambda^k V^\vee)$. Given two wedges of functions, we can define $(\alpha \wedge \beta)(x) = \alpha(x) \wedge \beta(x)$, and $(f \cdot \alpha)(x) = f(x) \cdot \alpha(x)$.

Problem 1 on this week's PSet is saying that there exists a $d : \Omega^k(U) \to \Omega^{k+1}(U)$ that satisfies

- $d(f) = Df$
- $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$.
- $d(d(\alpha)) = 0$.

So for instance, on 1-forms: take $f, g \in \Omega^1(U)$. Then $d(fdg) = df \wedge dg + f \wedge d(d(g)) = df \wedge dg$.

Lemma 14.1

Let α be a 1-form.

(a) The form α is the sum of 1-forms of the shape fdg. If $V = \mathbb{R}^n$, α is uniquely of the shape fdg.

(b) If $V = \mathbb{R}^n$, α is uniquely of the shape $\sum f_i dx_i$.

Proof. (b) clearly implies (a). Let $e_1^\vee, ..., e_n^\vee$ be a dual basis for $V^\vee$, then let $\alpha = \sum f_i e_i$. Note the 1-form dx_i acts by $dx_i(x) = e_i^\vee$. More generally, if $V^\vee \ni \psi$, consider the 1-form $\{\alpha_\psi, f_\psi(x) = \psi\}$. The we claim that for the function f_ψ, $df_\psi = \alpha_\psi$. Note that $df_\psi = Df_\psi$ since it's a 0-form. Thus $Df_\psi(x) : V \to \mathbb{R}$. But then for any linear map $T : V \to V'$, we have $DT(x) = T$, hence $Df_\psi(x) = \psi$. $\square$

Given

$$\begin{tikzcd} V_1 & V_2 \\ U_1 \arrow[u, hook] \arrow[r, "\varphi"] & U_2 \arrow[u, hook] \end{tikzcd}$$

we have a map $\varphi^* : C^\infty(U_2) \to C^\infty(U_1)$. Moreover, for a k-form α, $(\varphi^*(\alpha))(x) \in \Lambda^k(V_1^\vee)$. This is defined sensibly: we have for $x \in U_1$ that

$$V_1 \xrightarrow{D\varphi(x)} V_2$$
$$V_1^\vee \xleftarrow{(D\varphi(x))^\vee} V_2^\vee$$
$$\Lambda^k(V_1^\vee) \xleftarrow{\Lambda^k(D\varphi(x)^\vee)} \Lambda^k(V_2^\vee)$$

and of course, $\alpha(\varphi(x)) \in \Lambda^k(V_2^\vee)$.

§14.2 Topological Manifolds

> "Now, prepare for the worst." –Gaitsgory

Definition 14.2. A **topological manifold** of dimension n is a Hausdorff topological space such that for every $x \in X$, there is an open set $x \in U_x \subseteq X$ such that there is a homeomorphism from U_x to $U \subseteq \mathbb{R}^n$.

Example 14.3

Any $U \subseteq \mathbb{R}^n$.

Example 14.4

Take $S^n \subseteq \mathbb{R}^{n+1}$, with a homeomorphism given by projections from the north pole. (To be argued and clarified later.)

We'll now define C^∞ manifolds. Apparently, we aren't risking losing any generality by ignoring the C^k case.

Definition 14.5. A **C^∞ manifold** is a topological space equipped with a collection

$$\begin{tikzcd} U_\alpha \arrow[r, "\varphi"] \arrow[d, hook] & U \arrow[r, hook] & X \\ V & & \end{tikzcd}$$

with φ a homeomorphism such that

1. For every $x \in X$, there is a chart that contains it.
2. Given $V_\alpha \supseteq U_\alpha \overset{\varphi_\alpha}{\simeq} U' \hookrightarrow X \hookleftarrow U'' \overset{\varphi_\beta}{\simeq} U_\beta \subseteq V_\beta$, consider $U' \cap U'' \subseteq X$. Then $\varphi_\alpha^{-1}(U' \cap U'') \subseteq U_\alpha$ and $\varphi_\beta^{-1}(U' \cap U'') \subseteq U_\beta$, so $\varphi_\alpha^{-1}(U' \cap U'') \overset{\varphi_\alpha}{\approx} U' \cap U'' \overset{\varphi_\beta}{\approx} \varphi_\beta^{-1}(U' \cap U'')$. We require that this homeomorphism is C^∞. Terminologically, we say that charts intersect in C^∞, or are smoothly compatible. It is worth noting that $\varphi_\alpha^{-1} \circ \varphi_\beta : U_\beta \to U_\alpha$ is a map from domains of vector spaces, so it makes sense to ask whether the homeomorphism is C^∞.
3. This is just a completeness condition: suppose we have $U_\gamma \subseteq V_\gamma$ equipped with a homeomorphism γ between it and $U \subseteq X$, and this is smoothly compatible with all of the charts, then U_γ is itself a chart.

Conditions (1) and (2) define an **atlas**; it is a **maximal atlas** if (3) holds.

> "If you pick up a piece of paper, and it looks like a chart and fits in your folder, you can put it in your folder since it is."

Lemma 14.6

Let X be a Hausdorff topological space, equipped with a set of charts satisfying (1) and (2). Then X admits a unique structure of C^∞ manifold such that $(U_\alpha, \varphi_\alpha)$ are charts.

Proof. Take any

$$\begin{tikzcd} U_\gamma \arrow[r, dash, "\varphi_\gamma"] \arrow[d, hook] & U \arrow[r, hook] & X \\ V_\gamma & & \end{tikzcd}$$

We want to include it if and only if it's compatible with current charts. Then we declare $(U_\gamma, \varphi_\gamma)$ to be a chart if it's C^∞ against our current charts. Then completing the construction is more or less tautological, and therefore on the PSet. □

§14.3 Smoothness of Manifolds

Now, back to our example of S^n, we can just specify an atlas. Take planes both above and below the sphere, and lines from both the north and the south pole. Then we have charts for $S^n - \{\text{south pole}\}$ ("No polar bears. Sad." –Gaitsgory) and $S^n - \{\text{north pole}\}$, each of which are homeomorphic to $\mathbb{R}^n$. Then we get $\mathbb{R}^n - \{0\} \simeq S^n - \{\text{N.P \& S.P.}\} \simeq \mathbb{R}^n - \{0\}$ with respect to both projections.

> "No polar bears or penguins, now."

Then we want to claim that these homeomorphisms are C^∞. You can just do it, but it's ugly. There's a slicker way.

Let X be a C^∞ manifold.

Definition 14.7. A function f on X is said to be C^∞ if it's C^∞ on each chart: $f|_U \circ \varphi_\alpha : U_\alpha \to \mathbb{R}$.

Definition 14.8. A ***k*-form on *X***, $\alpha \in \Omega^k(X)$, is a datum of a k-form on every chart such that for $U', U'' \subseteq X$, $U' \overset{\varphi'}{\simeq} \widetilde{U}' \subseteq V'$ and $U'' \overset{\varphi'}{\simeq} \widetilde{U}'' \subseteq V''$, with α' a k-form on $\widetilde{U}'$ and α'' a k-form on $\widetilde{U}''$, φ' and φ'' are compatible in the sense that

$$\begin{tikzcd} \varphi'^{-1}(U' \cap U'') \arrow[r, "\varphi'(\varphi'')^{-1}"] \arrow[d, hook] & (\varphi'')^{-1}(U' \cap U'') \arrow[d, hook] \\ \widetilde{U}' & \widetilde{U}'' \end{tikzcd}$$

Then we want to require

$$(\varphi' \varphi''^{-1})^* (\alpha'|_{\varphi^{-1}(U' \cap U'')}) \simeq \alpha''|_{\varphi''^{-1}(U' \cap U'')}.$$

A vector field on a manifold is is the same thing: We have $\xi(x) \in V$, $\alpha(x) \in \Lambda^k(V^\vee)$. But our definition doesn't really give this–it gives different things on each chart. If X is a differentiable manifold, $x \in X$, we'll introduce $T_x X$, the tangent space at x, and $T_x^\vee X$, the cotangent space.

§14.4 Cotangent space

Take $x \in U \subseteq X$, f a C^∞ function. $Df(x)$ will live in the cotangent space, once we define it.

Definition 14.9. We set $Df(x) = 0$ if it's 0 on some/every chart.

Definition 14.10. Fix $x \in X$. A **germ** is a pair (U, f), where $f : U \to \mathbb{R}$ is smooth. We say two germs (U_1, f_1) and (U_2, f_2) are equal if f_1 and f_2 agree on some neighborhood of $U_1 \cap U_2$: that is, a germ only cares about what$f$ does on arbitrarily small neighborhoods of x.

One can endow this with a vector space structure in the obvious way, and observe that it makes sense to talk about fx and $(Df)x$ for a $U \subseteq V$.

Definition 14.11. Define $T_x^\vee X$ to be the set of germs vanish at x modulo the set of germs whose differential vanish at x. This is the **cotangent** space of x at V. We want that this is $V^\vee$ on each chart.

Lemma 14.12

Let U be contained in a chart. If $U \subseteq V$, then $T_x^\vee X \simeq V^\vee$.

Proof. Define this map by $f \mapsto Df(x)$. This is well defined and exactly kills the quotient. Thus injectivity is by definition. Then to verify that we have every element, for any functional $f \in V^\vee$, define a function $g(v) = f(v) - f(x)$. Then this is in $T_X^\vee$, and $Dg = f \in V^\vee$ since that's true for all linear functions. $\square$

We can then define the **tangent space** $T_x X = (T_x^\vee X)^\vee$. More concretely, consider paths $(-\epsilon, \epsilon) \xrightarrow{\gamma} X$ so that $\gamma(0) = x$. Then we'll mod out by an equivalence relation $\sim$ when two paths are tangent to each other at x.

Definition 14.13. Let X and Y be C^∞ manifolds. A C^∞ map $f : X \to Y$ is a continuous map such that for every x, we can consider the function $f(x) \in U_y \subseteq Y$, with U_y homeomorphic to U_y'. Then we can set $\widetilde{U}_x = f^{-1}(U_y)$, and then U_x to be $\widetilde{U}_x$ intersected with some chart around x, possibly completely containing U_x', so that we have

$$\begin{tikzcd} U_x \arrow[r, "f"] \arrow[d, leftrightarrow, "\sim"] & U_y \arrow[d, leftrightarrow, "\sim"] \\ U_x' \arrow[r, dashed, "f'"] & U_y' \end{tikzcd}$$

with $U_x' \subseteq V_x'$ and $U_y' \subseteq V_y'$. Then we ask for f' to be C^∞.

Now, we say $\gamma_1 \sim \gamma_2$ if "$D\gamma_1(0) = D\gamma_2(0)$." We don't know what either side of the equation is because we don't even know where they lie. However, we know what it means for them to be equal, since that just means they're equal on some/all charts, so we're OK. Now, finally, we can claim Paths $/ \sim \simeq T_x X$. "This is easy to prove, but it will overflow your minds if I do it... Too much for now." –Gaitsgory

> (In an email about the PSet) "The problems on manifolds are all more or less tautological, once you figure out what they are talking about." – Gaitsgory

§15 March 31, 2015

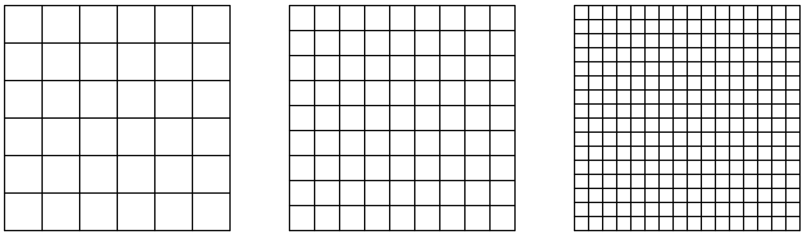
Here is a joke.

> Old McDonald had a form
> $e_i \wedge e_i = 0$.

Today Aaron Landesman is teaching integration. He says: "today we'll make a mess and just and define things without checking that they are well-defined. On Thursday Ashwin will get to clean up the mess and check things are well-defined".

§15.1 Boxes

Let $S = [a_1, b_1] \times \cdots \times [a_n, b_n]$. Let $f : S \to \mathbb{R}$ be uniformly continuous[1]. Take partitions $\mathfrak{p}_i$ with increasingly small meshes.



For each partition $\mathfrak{p}$ into hypercubes $\{S_\alpha\}$ we can take sample points in the boxes x_α, and consider

$$\Sigma_{\mathfrak{p}} = \sum_\alpha f(x_\alpha) \operatorname{vol}(S_\alpha)$$

and define

$$\int_S f = \lim_{\text{mesh } \mathfrak{p} \to 0} \Sigma_{\mathfrak{p}}.$$

Proposition 15.1
This is well-defined, like the Riemann integral case.

Proof. Left for Ashwin on Thursday. □

> "Left as an exercise to the Ashwin." – James Tao

§15.2 Supports

Let $f : U \to \mathbb{R}$ where $U \subseteq \mathbb{R}^n$.

Definition 15.2. We say the **support** of f, denoted $\operatorname{supp}(f)$, is the closure in $\mathbb{R}^n$ of the set $\{x \mid f(x) \neq 0\}$.

Note that the support may lie outside of U.

Definition 15.3. Suppose $f : U \to \mathbb{R}$ is uniformly continuous and the support of f is a subset of $U \subseteq \mathbb{R}^n$, we define $\overline{f} : \operatorname{supp}(U) \to \mathbb{R}$ by

$$\overline{f}(x) = \begin{cases} f(x) & x \in U \\ 0 & x \notin U. \end{cases}$$

[1] Or just continuous, since S is compact. But in general today our functions are going to be uniformly continuous.

Lemma 15.4

In the above notation, $\overline{f}$ is continuous.

Proof. It's trivial if $x \in U$,and for $x \notin U$ use the fact that $U \setminus \text{supp}(f)$ is closed. $\square$

Definition 15.5. Assume U is bounded, and $\text{supp}(f) \subseteq U$. Let S be a box containing U. Then we define

$$\int_U f = \int_S \overline{f}.$$

§15.3 Partitions of Unity

Definition 15.6. Let $A \subseteq \mathbb{R}^n$, and let $\{U_\alpha\}$ be an open cover of A such that U_α is bounded. Then a **partition of unity** subordinate to $\{U_\alpha\}$ is defined as follows. We pick an open set $U \supseteq A$ an open set, and a set of smooth functions $U \to [0,1]$, denoted Φ. It must satisfy the following properties.

(a) For all $x \in A$, there is a neighborhood $V_x \ni x$, such that only finitely many $\phi \in \Phi$ are nonzero when restricted to V_x.

(b) $\sum_{\phi \in \Phi} \phi(x) = 1$ for each $x \in A$ (this sum is finite by the preceding condition).

(c) For any $\phi \in \Phi$ there exists an α such that $\text{supp}(\phi) \subseteq U_\alpha$.

In a moment we'll show these exist. Once we do we can defie the following.

Definition 15.7. Let $A \subseteq \mathbb{R}^n$, and $f : A \to \mathbb{R}$ be uniformly continuous. Then we define

$$\int_A f = \sum_{\phi \in \Phi} \int_A \phi \cdot f$$

where $\{U_\alpha\}$ is an open cover of A and Φ is a partition of unity subordinate to $\{U_\alpha\}$.

Theorem 15.8

This definition does not depend on the choice of $\{U_\alpha\}$ and Φ. It also does not depend on the order of the infinite sum.

§15.4 Existence of Partitions of Unity

Theorem 15.9

Let $A \subseteq \mathbb{R}^n$ and let U_α be an open cover of A such that each U_α is bounded. Then there exists a partition of unity subordinate to it.

The remainder of the lecture is devoted to the proof of this.

§15.5 Case 1: A is Compact

Let's first solve the case if A is a compact set.

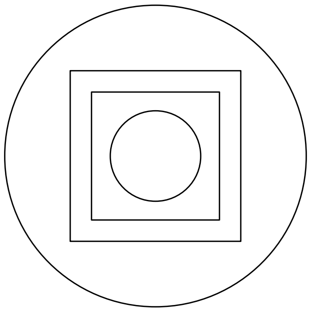
Lemma 15.10

Given $B(p,r)$ a ball then we may select $s < r$ and $f : \mathbb{R}^n \to [0,1]$ a smooth function such that

$$f(x) = \begin{cases} 1 & p \in B(x,s) \\ 0 & p \notin B(x,r). \end{cases}$$

Proof. Choose s sufficiently small so that there are cubes R and S centered at x such that

$$B(x,r) \supset R \supset S \supset B(x,s).$$



(Actually I think if you equip $\mathbb{R}^n$ with the max metric this is clear.) Now for $n = 1$ we can use a bump function to get the desired result. For $n > 1$, just take a product of these smooth functions: if $R = \prod_i [a_i, b_i]$ and $S = \prod_i [c_i, d_i]$, define $g_i : \mathbb{R} \to \mathbb{R}$ to work on the ith dimension, and let

$$f(x_1, \dots, x_n) = \prod_i g_i(x_i).$$ □

We're given U_α a cover of A. Take a ball $B_x \subseteq U_\alpha$ for each $x \in X$, and let B_x' be a ball within each B_x as described by the lemma: that means we have a *smooth* function f_x for each X such that

- f_x is one on the set B_x', and
- f_x is zero outside of B_x.

Now $\bigcup B_x' = A$, so we can take a finite subcover of size n. Denote the corresponding balls and functions by B_i' and f_i.

Then, for each i we set

$$\phi_i = \frac{f_i}{f_1 + \dots + f_n}$$

as a function $A \to \mathbb{R}$.

Proposition 15.11

The ϕ_i are a partition of unity on $\bigcup_i U_i \supset A$ which is subordinate to U_α.

Proof. First, observe that $\sum_i f_i > 0$ for every $x \in X$, since $x \in B_i'$ for some i and thus $f_i(x) = 1 > 0$. Hence the denominator never vanishes, and so ϕ_i is indeed a function to $[0,1]$.

Moreover, $\phi_i = \frac{f_i}{f_1+\cdots+f_n} < 1$ for every x.
Finally, we see that the support of each ϕ_i lives within U_i by construction. □

§15.6 Case 2: A is a certain union

We have thus solved the problem when A is compact. Now, we will prove the result in the case

$$A = \bigcup_{i \ge 1} A_i$$

with A_i compact, and so that the interior of A_{i+1} contains A_i for which each A_i. This is the hardest case, so one might want to skip to the next subsection first.

Define

$$B_i = \operatorname{int}(A_{i+1}) \setminus A_{i-2}$$

and

$$C_i = A_i \setminus \operatorname{int}(A_{i-1}).$$

Hence C_i are compact and $B_i \supset C_i$. Take for each i an open cover

$$\left\{U_\alpha^i\right\}_\alpha = \{U_\alpha \cap B_i\}_\alpha$$

an open cover of B_i. (At this point we can forget about the B_i now; there's just an open set for the C_i to live in).

Then there exists a partition of unity Φ^i of C_i which is subordinate to U_α^i. Now we set

$$\Psi = \bigcup \Phi^i$$

where all funcitons are now extended by zero to all of $\mathbb{R}^n$. Moreover, we define

$$\sigma(x) = \sum_{\psi \in \Psi} \psi(x).$$

Lemma 15.12

The above σ is well-defined in the sense that only finitely many $\psi(x)$ terms have nonzero contribution. Moreover, $\sigma(x)$ is never zero.

Proof. We wish to check there are only finitely many $\psi \in \Psi$ such that $\psi(x) \neq 0$. Fix x, $x \in A_i \setminus A_{i-1}$. Observe that if $\psi \in \Phi^k$ for some $k > i+2$, we have $\psi(x) = 0$. So we only need to consider $k < i+2$, each of which there are finitely many, so we're done.

The fact that $\sigma(x) \neq 0$ for any given $x \in A$, we know that $x \in A_i$ for some i, at which point the Ψ^i contributes a nonzero term. □

Now, we define

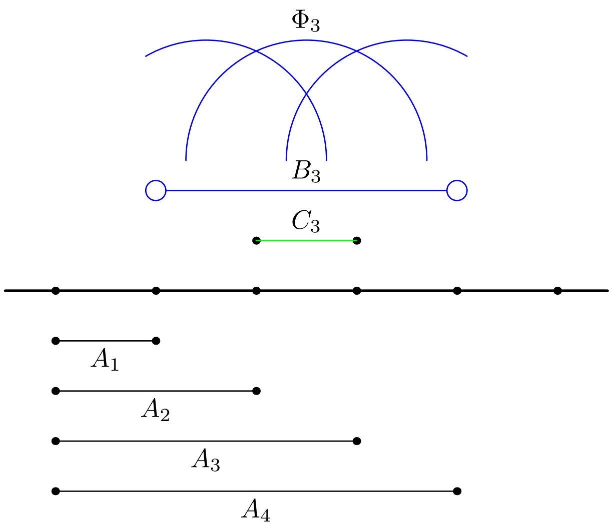
$$\phi_i(x) = \frac{\Psi_i(x)}{\sigma(x)}$$

In this way, $\sum_{\phi\in\Phi} \phi = 1$. Moreover, the support condition follows from the previous case as usual.

Example 15.13

Work in $\mathbb{R}$. If $A_1 = [0,1]$, $A_2 = [0,2]$, $A_3 = [0,3]$, and $A_4 = [0,4]$, then

$$C_3 = [2,3] \subseteq B_3 = (1,4).$$



NOT TO SCALE sheesh (The blue guys are supposed to be a unity partition and they're not.)

The "onion ring" guarantees that there are finitely many nonzero ϕ's at a given point, and the σ is just normalizing.

§15.7 Case 3: A open

Definition 15.14. The **boundary** of a set A is defined as the difference of the closure and the interior:

$$\overline{A} \setminus \operatorname{int} A.$$

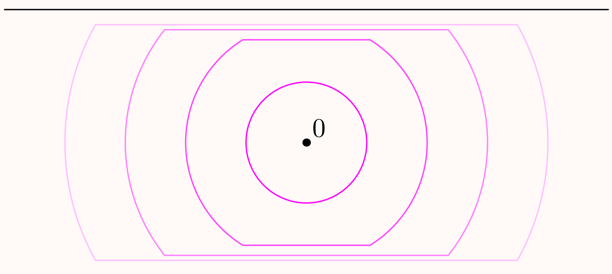
We show we can reduce the previous case now.

Define

$$A_i = \left\{ x \mid d(x, \partial A) \ge \frac{1}{i}, |x| \le i \right\}.$$

Example 15.15

Take $\mathbb{R}^2$ and let $U = \{(x,y) \mid |y| < 2.2\}$ be a horizontal strip. Then this might look the following.



§15.8 Case 4: General A

Given a general A, consider an open cover U_α of A. We know there exists a partition of unity subordinate to U_α of $U = \bigcup_\alpha U_\alpha$. Then Φ is a partition of unity for A. (Check this.)

This completes the proof.

> "Now we know that partitions of unity exists. You basically don't need to know this proof."

§16 April 2, 2015

Ashvin gets to clean up the mess now.

§16.1 Integration on Bounded Open Sets

First, consider a rectangle $S = [a_1, b_1] \times \cdots \times [a_n, b_n]$, and let $f : S \to \mathbb{R}$ be continuous (and hence uniformly so).

We want to show the definition we gave last time is well-defined. Recall that for each partition $\mathfrak{p}$ into hypercubes $\{S_\alpha\}$ we can take sample points in the boxes x_α, and consider

$$\Sigma_{\mathfrak{p}} = \sum_\alpha f(x_\alpha)\,\mathrm{vol}(S_\alpha)$$

and define

$$\int_S f = \lim_{\text{mesh } \mathfrak{p} \to 0} \Sigma_{\mathfrak{p}}.$$

We need to show that

- this limit exists,
- it's independent of the choice of sample points, and
- it's independent of the choice of partitions.

Note that f is uniformly continuous, as S is compact.

Proof that the limit exists. We show that it's Cauchy.

Pick $\varepsilon > 0$. Since f is uniformly continuous, there is a $\delta > 0$ such that whenever $\|x - x'\| < \delta$ we have $|f(x) - f(x')| < \frac{\varepsilon}{\mathrm{vol}(S)}$. Note that there exists N such that for all $n > N$ and $\alpha \in P_n$, there exists a ball B of diameter δ which completely contains the square containing α in $\mathfrak{p}_n$.

We claim this N works. Let $m, n > N$. Let $\mathfrak{p}'$ be the partition obtained by superposing $\mathfrak{p}_m$ and $\mathfrak{p}_n$, and let

$$\Sigma' = \sum_{\alpha \in \mathfrak{p}'} f(x_\alpha)\,\mathrm{vol}(S_\alpha).$$

We claim that Σ' differs from Σ_m and Σ_n by less than ε, which will give $|\Sigma_m - \Sigma_n| < 2\varepsilon$ which is certainly sufficient.

This is some direct computation:

$$\begin{aligned}
\left|\Sigma' - \Sigma_m\right| &= \left| \sum_{\alpha \in \mathfrak{p}'} f(x_\alpha)\,\mathrm{vol}(S_\alpha) - \sum_{\beta \in \mathfrak{p}_m} f(x^m_\beta)\,\mathrm{vol}(S^m_\beta) \right| \\
&= \left| \sum_{\beta \in \mathfrak{p}_m} \left[\left(\sum_{\alpha \in \beta} f(x_\alpha)\,\mathrm{vol}(S_\alpha) \right) - f(x^m_\beta)\,\mathrm{vol}(S^m_B) \right] \right| \\
&= \left| \sum_{\beta \in \mathfrak{p}_m} \left[\left(\sum_{\alpha \in \beta} f(x_\alpha)\,\mathrm{vol}(S_\alpha) \right) - f(x^m_\beta) \sum_{\alpha \in \beta} \mathrm{vol}(S_\alpha) \right] \right| \\
&= \left| \sum_{\beta \in \mathfrak{p}_m} \sum_{\alpha \in \beta} \left(\left(f(x_\alpha) - f(x^m_\beta) \right) \mathrm{vol}(S_\alpha) \right) \right|
\end{aligned}$$

Applying the triangle inequality,

$$\begin{aligned}
&\le \sum_{\beta \in \mathfrak{p}_m} \sum_{\alpha \in \beta} \left| f(x_\alpha) - f(x_\beta^m) \right| \operatorname{vol}(S_\alpha) \\
&< \sum_{\beta \in \mathfrak{p}_m} \sum_{\alpha \in \beta} \frac{\varepsilon}{\operatorname{vol}(S)} \operatorname{vol}(S_\alpha) \\
&= \varepsilon.
\end{aligned}$$ □

.

Proof of independence of sample points. Fix a choice of partitions. Suppose we pick sample points x_α and x'_α to obtain Σ_n and $\Sigma_{n'}$. We show that $|\Sigma_n - \Sigma'_n| \to 0$. Fix $\varepsilon > 0$ and select $\delta > 0$ so that points within δ have f-output within ε (uniform continuity of f). Let the diameter of the boxes be less than δ for $n \ge N$. Then for such n we obtain

$$\left| \Sigma_n - \Sigma'_n \right| \le \sum_\alpha \left| f(x_\alpha) - f(x_\alpha)' \right| \operatorname{vol}(S_\alpha) \le \sum_\alpha \varepsilon \cdot \operatorname{vol}(S_\alpha) = \varepsilon \operatorname{vol}(S).$$ □

Proof of Independence of Partition Choice. There's actually a nice trick to show independence of the choice of partitions. Suppose that $\{\mathfrak{p}_n\}$ and $\{\mathfrak{q}_n\}$ are two such partitions. Interlace the sequences

$$\mathfrak{p}_1, \ \mathfrak{q}_1, \ \mathfrak{p}_2, \ \mathfrak{q}_2, \ \dots$$

Then the limit exists to some limit L, and hence the subsequences $\mathfrak{p}_n$ and $\mathfrak{q}_n$ both converge to L. □

Hence it makes sense to define $\int_S f = \lim_{n \to \infty} \Sigma_n$.

§16.2 This Integral is Continuous

For now on, for simplicity, let

$$S = [a_1, b_1] \times [a_2, b_2] \subseteq \mathbb{R}^2.$$

In particular we're just working in two dimensions now, so that my LaTeX-ing is easier.

Let $f : S \to \mathbb{R}$ be (uniformly) continuous. Define $g : [a_1, b_1] \to \mathbb{R}$ by

$$g(x) = \int_{y \in [a_2, b_2]} f(x, y).$$

Analogously, define $[a_2, b_2] \to \mathbb{R}$ by

$$h(y) = \int_{x \in [a_1, b_1]} f(x, y).$$

Lemma 16.1

The functions g and h are continuous.

Proof. We'll just do g since h is analogous... Pick $x \in [a_1, b_1]$. Consider an interval $I = [x - r, x + r] \subseteq [a_1, b_1]$ (though for the n-dimensional case I is a closed ball of radius r). Then f is uniformly continuous on $I \times [a_2, b_2]$.

Given $\varepsilon > 0$, pick $\delta > 0$ such that

$$\left|x - x'\right| < \delta$$

implies that

$$\left|f(x, y) - f(x', y)\right| < \varepsilon \quad \forall y \in [a_2, b_2]$$

where $x' \in I$. Then, we see that

$$\left|g(x') - g(x)\right| = \left|\int_{y \in [a,b]} f(x', y) - f(x, y)\right| \le \int_{y \in [a_2, b_2]} \left|f(x', y) - f(x, y)\right| < \varepsilon(b_2 - a_2)$$

which is enough. □

Now we claim the following.

Theorem 16.2 (Switching Order of Summation)
We have $\int_S f = \int_{x \in [a_1, b_1]} g(x) = \int_{y \in [a_2, b_2]} h(y)$.

Proof. Let

$$F_1(t) = \int_{x \in [a_1, t]} g(x).$$

Let

$$F_2(t) = \int_{x \in [a_1, t] \times [a_2, b_2]} g(x).$$

If $t = a_1$, $F_1(t) = F_2(t) = 0$. So it suffices to show that F_1 and F_2 are differentiable and have equal derivatives.

By the Fundamental Theorem of Calculus, $F_1'(t)$ exists and coincides with $g(t)$.

For F_2 it's trickier. Recall that f is uniformly continuous, so for any $\varepsilon > 0$ we can find $h > 0$ such that

$$\forall y \in [a_2, b_2] : \left|f(t + h', y) - f(t, y)\right| < \varepsilon$$

where $0 \le h' < h$.

We have

$$\int_{(x,y) \in [t, t+h] \times [a_2, b_2]} f(t, y) = h \int_{t \in [a_2, b_2]} f(t, y)$$

for obvious reasons (it's just a slice of width h). To compute $F_2'(t)$, we put

$$\begin{aligned}
F_2'(t) &= \lim_{h \to 0} \frac{1}{h} \int_{(x,y) \in [t, t+h] \times [a_2, b_2]} f(x, y) \\
&= \lim_{h \to 0} \frac{1}{h} \left[\int_{(x,y) \in [t, t+h] \times [a_2, b_2]} f(t, y) + \int_{(x,y) \in [t, t+h] \times [a_2, b_2]} (f(x, y) - f(t, y))\right] \\
&= \int_{y \in [a_2, b_2]} f(t, y) \lim_{h \to 0} \frac{1}{h} \int_{(x,y) \in [t, t+h] \times [a_2, b_2]} (f(x, y) - f(t, y)) \\
&= g(t) + \lim_{h \to 0} \frac{1}{h} \int_{(x,y) \in [t, t+h] \times [a_2, b_2]} (f(x, y) - f(t, y)).
\end{aligned}$$

Hence we wish to check that the latter limit is zero. The following computation is sufficient:

$$\begin{aligned}
&\lim_{h\to 0}\frac{1}{|h|}\left|\int_{(x,y)\in[t,t+h]\times[a_2,b_2]}(f(x,y)-f(t,y))\right| \\
&\le \lim_{h\to 0}\frac{1}{|h|}\int_{(x,y)\in[t,t+h]\times[a_2,b_2]}|(f(x,y)-f(t,y))| \\
&< \lim_{h\to 0}\frac{1}{|h|}|h|(b_2-a_2)\varepsilon \\
&= \varepsilon(b_2-a_2).
\end{aligned}$$

□

The result that we can switch summations is called **Fubini's Theorem**.

§16.3 Open Sets

We're done with boxes. Now let U be a *bounded* open set in $\mathbb{R}^n$, and let $f : U \to \mathbb{R}$ be a uniformly continuous function; we wish to define $\int_U f$.

Recall that the support of f, $\operatorname{supp}(f)$, is the closure of the nonvanishing points of f.

Assume first that $\operatorname{supp}(f)$ is compact, and hence contained in some box $S \subseteq \mathbb{R}^n$. Then we define $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ by setting $\tilde{f}(x)$ to be fx on U and zero outside it. Then we define

$$\int_U f = \int_S \tilde{f}.$$

more generally, if $\operatorname{supp}(f)$ is not compact, we take a partition of unity $\{\phi_k\}$ as in last lecture and let

$$\int_U f = \sum_k \int_U f \cdot \phi_k.$$

We now need to verify that

- Re-ordering the terms of sum shouldn't matter.
- If f has compact support, these two definitions should coincide.
- This doesn't depend on the choice of $\{\phi_k\}$

Well, the third bullet kind of implies the first and second... Although if partitions of unity are indexed by a set rather than a sequence, then the first bullet point is a valid concern and we get around it by following.

Definition 16.3. We say f is **integrable**, that is, the series $\sum_k \int f \cdot \phi_k$ is absolutely convergent.

This follows, for example, that since f is bounded by M, since in that case

$$\sum_k \left|\int f\cdot\phi_k\right| \le \sum_k \int |f|\,|\phi_k| \le \sum_k \int M\cdot\phi_k < \infty.$$

The second concern, again, follows by taking a trivial partition of unity with constant value 1 on $\operatorname{supp}(U)$. So let's just do the third part.

Lemma 16.4

If $\{\phi_k\}$, $\{\psi_j\}$ are two partitions of unity, then $\{\phi_k \psi_j\}$ is a partition of unity.

Proof. Exercise. OK fine I'll write it out:

- Smoothness is clear.
- $\operatorname{supp}(\phi_k \cdot \psi_j) = \operatorname{supp}(\phi_k) \cap \operatorname{supp}(\psi_j)$.
- Given neighborhoods V_x for the ϕ and W_x for the ψ, take the intersection.
- To show the sum is 1, realize there are only finitely many nonzero terms; the sum is $\sum_{k,j} \phi_k \psi_j = \sum_k \phi_k \sum_j \psi_j = 1$. $\square$

Before we can get the final result, we need one more lemma.

Lemma 16.5 (Discrete Fubini)

Let $\sum_{k,j} b_{k,j}$ be absolutely convergent biseries. Then

$$\sum_{k,j} b_{k,j} = \sum_k \sum_j b_{k,j} = \sum_j \sum_k b_{k,j}.$$

This resolves the main result.

Theorem 16.6

If $\{\phi_i\}$, $\{\psi_j\}$ are partitions of unity, then

$$\sum_k \int f\phi_k = \sum_j \int f\psi_j.$$

Proof. We show both are equal to $\sum_{j,k} \int f\phi_k\psi_j$. Observe that

$$\sum_{k,j} \int f\phi_k\psi_j = \sum_k \left(\sum_j \int f\phi_k\psi_j \right) = \sum_k \left(\int f\phi_k \right).$$

Similarly flipping the order of summation gives

$$\sum_{k,j} \int f\phi_k\psi_j = \sum_j \sum_k \int f\phi_k\psi_j = \sum_j \int f\psi_j.$$ $\square$

§17 April 7, 2015

Last time we saw how to define $\int_S f$ for a box S. Also, for $S = S_1 \times S_2$, we saw Fubini's Theorem that

$$\int_{S_1 \times S_2} f = \int_{S_1} \int_{S_2} f = \int_{S_2} \int_{S_1} f.$$

Then, using the partitions of unity, we defined

$$\int_U f$$

for $U \subseteq \mathbb{R}^n$ bounded, by summing over partitions of unity.

§17.1 Integration Over Vector Spaces

We want to define a notion

$$\int_U f$$

for $U \subseteq V$. But this is not well-defined. For example, if $V = \mathbb{R}^1$ and $f \equiv 1$ then there's a change of basis called "multiply by V" and we have

$$\int_{(a,b)} 1 = b - a \quad \text{and} \quad \int_{(ca,cb)} 1 = c(b-a).$$

Lemma 17.1
Let $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n \xrightarrow{f} \mathbb{R}$. Then

$$\int_U f = |\det T| \int_{T^{-1}(U)} f \circ T.$$

Sketch of Proof. Suffices to show it for boxes, then note that boxes go to parallelpipeds. □

What if it's not linear?

§17.2 Change of Variables

The main theorem of today is the following.

Theorem 17.2 (Change of Variables Formula)
Let U_1 and U_2 be domains in $\mathbb{R}^n$ and suppose we have a setup

$$U_1 \xrightarrow{\phi} U_2 \xrightarrow{f} \mathbb{R}$$

where ϕ is a diffeomorphism. Then

$$\int_{U_2} f = \int_{U_1} |\det D_\phi| \, (f \circ \phi).$$

Perhaps I should write this as

$$\int_{U_2} f(x) = \int_{U_1} |\det D_\phi(x)| \, (f \circ \phi(x))$$

to emphasize the determine varies per point.

A quick remark. By using partitions of unity, we may assume that $\operatorname{supp} f$ is compact. Also, by picking a proper cover, we can make nice assumptions on our set (since a proper cover can be made subordinate to any cover).

Proof when $n = 1$. Using partitions of unity, we may assume $U_2 = (a_2, b_2)$. Define

$$F_2 : (a_2, b_2) \to \mathbb{R} \quad \text{by} \quad F_2(x) = \int_{[*,x]} f$$

where $* = a_2$ (or anything else), and let $F_1 = F_2 \circ \phi$. Thus $F_1' = \phi' \cdot (F_2' \circ \phi)$.

Since ϕ is a diffeomorphism, ϕ' never vanishes. So WLOG we may assume $\phi' > 0$ (*orientation-preserving*); in that case $\phi(a_1) = a_2$ and $\phi(b_1) = b_2$. (In the other case, a_2 and b_2 swap introducing a sign.)

By the Fundamental Theorem of Calculus,

$$\int_{(a_2,b_2)} f = F_2(b_2) - F_2(a_2) = F_1(b_1) - F_1(a_1) = \int_{(a_1,b_1)} F_1'.$$

Since $F_1' = \phi' \cdot (F_2' \circ \phi)$, meaning

$$\int_{(a_2,b_2)} f = \int_{(a_1,b_1)} \phi' \cdot (f \circ \phi)$$

as required. $\square$

Proof for $n > 1$ By Induction. Let $0 < m < n$, so $n = m + (n - m)$. We want to throw Fubini at this.

Suppose we're lucky enough that the following diagram commutes:

$$\begin{tikzcd} U_1 \arrow[rr, "\phi"] \arrow[dr, "p_1"'] & & U_2 \arrow[dl, "p_2"] \\ & \mathbb{R}^m & \end{tikzcd}$$

where each p_i is a projection

$$p_i : U_i \hookrightarrow \mathbb{R}^n \twoheadrightarrow \mathbb{R}^m$$

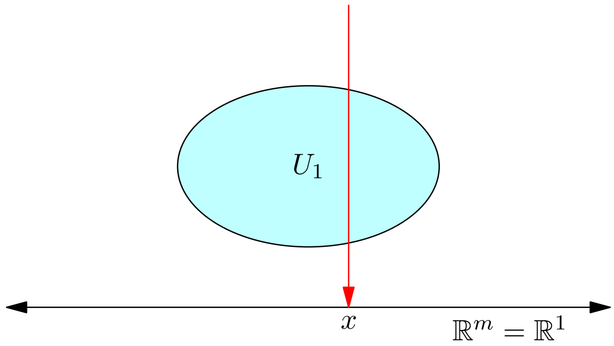
Define $U_m \subseteq \mathbb{R}^m$ a domain. By Fubini's Theorem, we may define $F_1, F_2 : U_m \to \mathbb{R}$ by

$$F_2(x) = \int_{y \in p_2^{-1}(x) \cap U_2} f(x, y)$$

and

$$F_1(x) = \int_{y \in p_1^{-1}(x) \cap U_1} (f \circ \phi)(x, y) \cdot |\det D\phi(x, y)| \, .$$

Picture this as integrating over "slices".



Claim 17.3. $F_1 = F_2$.

Proof. For all x, the map ϕ defines a diffeomorphism

$$\phi_x = p_1^{-1}(x) \cap U_1 \to p_2^{-1}(x) \cap U_2.$$

By induction, we know that

$$F_2 = \int_{p_1^{-1}(x) \cap U_1} (f \circ \phi)(x, y) \left| \det D\phi_x(x, y) \right|.$$

So, we just have to show

$$\left| \det D\phi \right| = \left| \det D\phi_x \right|.$$

This follows from the fact that we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathbb{R}^{n-m} \arrow[r, "D\phi_x"] \arrow[d, hook] & \mathbb{R}^{n-m} \arrow[d, hook] \\
\mathbb{R}^n \arrow[r, "D\phi"] \arrow[d] & \mathbb{R}^n \arrow[d] \\
\mathbb{R}^m \arrow[r, leftrightarrow, "\text{id}"] \arrow[d] & \mathbb{R}^m \arrow[d] \\
0 & 0
\end{tikzcd}$$

Hence $\det D\phi = \det \text{id} \cdot \det D\phi_x = \det D\phi_x$ as needed. ∎

So we're done in that very fortunate case. More generally, since it suffices to doing things locally, we claim we can reduce to this case as follows.

Claim 17.4. Let $\phi_1, \phi_2, \dots, \phi_n$ be the n projections of ϕ. For all $x \in U_1$ there exists an index i such that the map

$$\tilde{\phi}_i(\overline{x}) = (\phi_1(\overline{x}), \dots, \phi_{i-1}(\overline{x}), \pi_i(x), \phi_{i+1}(\overline{x}), \dots, \phi_n(\overline{x}))$$

is a diffeomorphism on a neighborhood of x.

Proof. We wish to use the Inverse Function Theorem. It suffices that

$$D\tilde{\phi}_i(x) : \mathbb{R}^n \to \mathbb{R}^n$$

is invertible. So we want to find an i so that the map

$$(D\phi_1(x), \dots, e_i, \dots, D\phi_n(x))$$

is linearly independent. Such an i must exist since the $D\phi_j$ are all linearly independent, meaning it's impossible that all the basis elements are linearly dependent with it. ■

This completes the proof. □

§17.3 Integration Over Manifolds

Recall from PSet 8 that we have

5. For a real vector ℓ of dimension 1, we define an *orientation* on ℓ to be the choice of one of the two cosets in $\ell \setminus \{0\}$ with respect to the equivalence relation

$$l_1 \sim l_2 \iff \exists c \in \mathbb{R}^{>0} \text{ such that } c \cdot l_1 = l_2.$$

For a finite-dimensional real vector space V, we define an **orientation** on V to be an orientation on the 1-dim vector space $\Lambda^n(V)$, where $n = \dim(V)$.

Let $U \subseteq V$ be a domain. Let ω be a top-dimensional (meaning $\dim V$-form, as $\dim V + 1$ vanishes) bounded differential form on U, and set ε as an orientation of V. First, pick a basis of V using $\mathbb{R}^n \xrightarrow{T} V$. Then we define

$$\int_{u,\varepsilon} \omega = \pm \int_{T^{-1}(U)} T^* \omega$$

where the sign is $+1$ if T preserves the orientation and -1 otherwise. Here I mean to look at

$$\Lambda^n T : \Lambda^n \mathbb{R}^n \to \Lambda^n V$$

and decide whether it preserves the orientation ε.

Lemma 17.5

This definition doesn't depend on the choice of basis.

Proof. Equivalent to the lemma earlier about linear transformations. □

> I don't want to bore you all to sleep, so I won't prove it. I'll put it on your PSet. – Gaitsgory

Now let $U \subseteq V$.

Definition 17.6. An **orientation** of U is a continuous map

$$U \to \{\text{North}_V, \text{South}_V\}$$

as topological spaces, where the range is a discrete space and its elements are the two orientations of V. (Hence we assign an orientation to each connected component of U.)

Corollary 17.7

Suppose $U_1 \xrightarrow{\phi} U_2$ is a diffeomophism of domains $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$. Let ω be a top-dimensional form on U_2 with compact support. Let ε_i be an orientation on U_i for $i = 1, 2$. If ε_1 and ε_2 agree, then

$$\int_{U_2, \varepsilon_2} \omega = \int_{U_1, \varepsilon_1} \phi^*(\omega_2).$$

Here we say ε_1 and ε_2 **agree** under ϕ if for all $x \in U_1$, the map

$$\Lambda^n D\phi(x) : \Lambda^n V_1 \xrightarrow{\sim} \Lambda^n V_2$$

agrees with ε_1 and ε_2.

§18 April 9, 2015

§18.1 Digression: Orientations on Complex Vector Spaces

Let's try to put orientations on complex vector spaces, because why not.

First, we put an orientation on the one-dimensional space $\mathbb{C}$ by regarding it as a real vector space with basis $(e_1, i \cdot e_1)$. We claim that for any choice of e_1, we can put $(e_1, i \cdot e_1)$ as a positive guy. To show that this is consistent, we need to show that the linear transformation $(e_1, i \cdot e_1) \mapsto (e_1', i \cdot e_1')$ has positive determinant. One can check this.

§18.2 Setup

Take our setup

$$U_1 \xrightarrow{\phi} U_2 \xrightarrow{f} \mathbb{R}$$

and U_2 domains in $\mathbb{R}^n$ and ϕ a diffeomorphism. We will assume that

$$(f \circ \phi) \cdot |\det D\phi|$$

which is true, for example, if we have compact supports. Then the change of variables theorem stated that

$$\int_{U_1} (f \circ \phi) \, |\det D\phi| = \int_{U_2} f.$$

Recall that an orientation is a continuous map sending each $x \in U$ to be ε_x an orientation on U. Note that as an edge case, if U is a single point, then $\dim V = 0$. We define

$$\Lambda^0(V) = \mathbb{R}.$$

The reason for doing this is so that

$$\Lambda^{\text{top}}(V_1 \oplus V_2) \simeq \Lambda^{\text{top}}(V_1) \oplus \Lambda^{\text{top}}(V_2)$$

holds. But $\mathbb{R}$ has a preferred orientation already, so there is nothing to do in this edge case.

Now recall how we defined

$$\int_{U,\varepsilon} \omega = \int_{T^{-1}(U)} g$$

for a domain U, a linear map $T : \mathbb{R}^n \to V$ with pullback $T^*(\omega) = f dx_1 \wedge \dots dx_n$, and $g = \pm f$ depending on whether T preserves orientation. On the homework we show that

(a) this is independent of T, and

(b) if $\phi : U_1 \to U_2$ is a diffeomorphism and ϕ maps ε_1 to ε_2, then

$$\int_{U_1,\varepsilon_1} \phi^*(\omega) = \int_{U_2,\varepsilon_2} \omega$$

(with no constant factor!).

Let me try to explain the pullback a little more. Suppose $\phi : U_1 \to U_2$ is a diffeomorphism, and observe that we can put

$$\phi = (\phi_1, \dots, \phi_n).$$

Suppose

$$\omega = f dy_1 \wedge \dots \wedge dy_n$$

(different f here), where we're using y for the coordinates in U_2. Then we calculate

$$\begin{aligned}\phi^*(\omega) &= \phi^* f \phi^*(dy_1) \dots \wedge \phi^*(dy_n) \\ &= (f \circ \phi) d(y_1 \circ \phi) \wedge \dots \wedge d(y_n \circ \phi) \\ &= (f \circ \phi) d\phi_1 \wedge \dots \wedge d\phi_n\end{aligned}$$

by definition; observe that we may expand

$$d\phi_1 = \sum_j \frac{\partial \phi_j}{y_j} dy_i.$$

At the end of the day, we arrive at

$$\phi^*(\omega) = (f \circ \phi) \det \begin{pmatrix} \frac{\partial \phi_1}{\partial y_1} & \dots & \frac{\partial \phi_n}{\partial y_1} \\ \ddots & \vdots & \ddots \\ \frac{\partial \phi_1}{\partial y_n} & \dots & \frac{\partial \phi_n}{\partial y_n} \end{pmatrix} = (f \circ \phi) \cdot \det D\phi.$$

In the special case where T is a linear map, we have the nice property that $\det T$ can be interpreted as the constant in $\Lambda^n T : \mathbb{R} \to \mathbb{R}$.

Observe $D\phi : \mathbb{R}^n \to \mathbb{R}^n$ when viewed as a matrix in the standard basis has $\frac{\partial \phi_i}{\partial x_i}$. Finally, I remark that in case you haven't noticed yet,

$$\frac{\partial g}{\partial x_i} \overset{\text{def}}{=} D_{e_i} g.$$

You can think of this as saying that these pullbacks *absorb* the determinant; that's what they are for.

§18.3 Integration on Manifolds

Let X denote a differentiable manifold now, with an atlas

$$V_\alpha \supseteq U_\alpha \xrightarrow{\phi_\alpha} U'_\alpha \subseteq X$$

of charts.

Definition 18.1. An **orientation** on X is the choice of an orientation ε_α on U_α for every α with the following property: for all α or β, the diffeomorphism

$$U_\alpha \supseteq \phi_\alpha^{-1}(U'_\alpha \cap U'_\beta) \xrightarrow{\phi_\beta^{-1} \circ \phi_\alpha} \phi_\beta^{-1}(U'_\alpha \cap U'_\beta) \subseteq U_\beta$$

preserves orientation.

As usual, we can run this definition on an atlas (a set of charts which cover every point), and it will extend uniquely to an orientation on the maximal atlas of X.

A second definition is as follows.

An orientation amounts to a choice of an orientation ε_x on $T_x X$, for each $x \in X$. This assignment needs to be "continuous". To talk about continuity, we resort to charts: we want it to be continuous on every chart via

$$U'_\alpha \xrightarrow{\phi_\alpha^{-1}} U_\alpha \hookrightarrow V_\alpha \xrightarrow{(D\phi_\alpha)^{-1}} T_x X.$$

Finally we can define the integral (ack).

Definition 18.2. Let X be a differentiable manifold, and ω a compactly supported top-dimensional differential form on X, and ε an orientation on X. Select a partition of unity Ψ_α for each chart (U_α, ϕ_α). (Since the support of ω is compact, there are only finitely many α's such that the support of ϕ_α intersects the support of ω). Then we define

$$\int_{X,\varepsilon} \omega \overset{\text{def}}{=} \sum_\alpha \int_{U_\alpha, \varepsilon_\alpha} \phi_\alpha^*(\omega \cdot \psi_\alpha).$$

Proposition 18.3

This is independent of the choice of atlas.

Proof. Let $(U_\alpha \xrightarrow{\phi}_\alpha U'_\alpha)$ and $(U_\beta \xrightarrow{\phi}_\beta U'_\beta)$ be two totally unrelated atlases equipped with partitions of unity ψ_α and ψ_β. We compute

$$\begin{aligned}\sum_\alpha \int_{U_\alpha,\varepsilon_\alpha} \phi_\alpha^*(\psi_\alpha \cdot \omega) &= \sum_\alpha \int_{U_\alpha,\varepsilon_\alpha} \phi_\alpha^*(\psi_\alpha \cdot \omega \cdot \sum_\beta) \\ &= \sum_\alpha \sum_\beta \int_{U_\alpha,\varepsilon_\alpha} \phi_\alpha^*(\psi_\alpha \cdot \omega)\end{aligned}$$

where the summation is okay due to there being finitely many zero terms. So now we wish to check that

$$\int_{U_\alpha,\varepsilon_\alpha} \phi_\alpha^*(\psi_\alpha \cdot \omega) = \int_{U_\beta,\varepsilon_\beta} \phi_\beta^*(\psi_\beta \cdot \omega)$$

in which case we'll be done by symmetry.

$$U_\alpha \xrightarrow{\phi_\alpha} U'_\alpha \subseteq X \supseteq U'_\beta \xleftarrow{\phi_\beta} U_\beta.$$

Notice that the support of $\omega\psi_\alpha\psi_\beta$ is contained in $U'_\alpha \cap U'_\beta$, since ψ_α has support inside U'_α and ψ_β has support inside U'_β. Effectively we're integrating over $\phi_\alpha^{-1}(U_\beta)$ and $\phi_\beta^{-1}(U_\alpha)$. This is done using the transition function

$$\phi_\alpha^{-1}(U_\beta) \xrightarrow{\phi_\beta^{-1}\circ\phi_\alpha} \phi_\beta^{-1}(U_\alpha).$$ □

§18.4 Stoke's Theorem

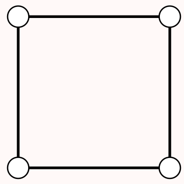
Let $S \subseteq \mathbb{R}^n$ be a box contained inside some domain U. Let $\omega \in \Omega^{n-1}(U)$ be the set of continuously differentiable $n-1$ forms.

Now we state Stoke's Theorem, which is the generalization of the Fundamental Theorem of Calculus.

We can consider $d\omega$, which is now a top-dimensional differentiable form. On the other hand, we can consider ∂S, which is the "boundary" of the box, and then take its interior to get $\overset{\circ}{\partial S}$. (In other words, take the union of the interiors of the faces.) This is a manifold.

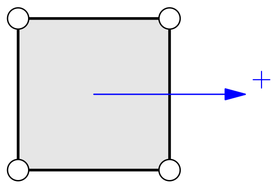
Example 18.4

If S is a square, then $\partial \mathring{S}$ consists of four open segments.



We would now like to assert that $\int_{\mathring{S}} d\omega$ is equal to $\int_{\partial \mathring{S}} \omega$. But to do this we need orientations. For $d\omega$ this is no problem as we have a standard orientation on $\mathbb{R}^n$. But it remains to orient $\partial \mathring{S}$.

Well, given a vector space $V = V_1 \oplus V_2$, an orientation of V_i can be specified from an orientation of V_2 and V (the order of this matters). Thus, we orient $\partial \mathring{S}$ as follows: let V_1 be the orthogonal one-dimensional vector out of a face, and V_2 the rest of the face. We take the orthogonal guy *out* of the face in V_1 as the positive direction, plus the standard orientation on $\mathbb{R}^n$, to get the orientation on each face.



Theorem 18.5 (Baby Stoke's Theorem)

With the orientations above,

$$\int_{\mathring{S}} d\omega = \int_{\partial \mathring{S}} \omega.$$

Proof. First, we handle the case $n = 1$. Thus, we have $S = [a, b]$. Then, $\omega \in \Omega^0(\mathbb{R})$ is represented by a single function f, and

$$\int_{\mathring{S}} \omega = \int_{[a,b]} f' \, dx = f(b) - f(a)$$

by the Fundamental Theorem of Calculus.

Hence, we just need to check $\int_{\mathring{S}} \omega$ equals this. Observe that $\int \mathring{S} = \{a, b\}$. The point b has orientation agreeing with that of $\mathbb{R}$, and the point a has orientation against that of $\mathbb{R}$. So we get $f(b)$ and $f(a)$, yay.

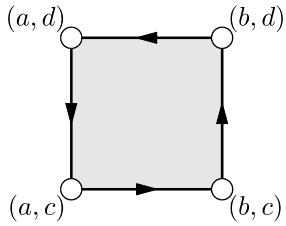
> "Do you want me to jump straight to the inductive case or do $n = 2$ first?" – Matt
> "Which do you want" – Gaitsgory
> "$n = 6$." – Matt
> (laughter)
> "Do $n = 120$!" – James

We'll do $n = 2$; the case $n \geq 2$ is homework.

We have $\omega \in \Omega^1(\mathbb{R}^2)$, so we may put

$$\omega = f_1 dx_1 + f_2 dx_2.$$

The positive orientations are given as follows.



First, consider $\omega = f dx_1$ for simplicity (the other guy is analogous and we'll be done by adding). We compute

$$d\omega = \frac{\partial f}{\partial x_2} dx_2 \wedge dx_1 = -\frac{\partial f}{\partial x_2} dx_1 \wedge dx_2.$$

Now we compute this using Fubini's Theorem:

$$\begin{aligned}
\int_{\mathring{S}} d\omega &= -\int_{\mathring{S}} \frac{\partial f}{\partial x_2} \\
&= -\int_{s\in[a,b]} \int_{t\in[a,b]} \frac{\partial f}{\partial x_2} \\
&= -\int_{s\in[a,b]} \left(f(s,d) - f(s,c) \right) \\
&= \int_{s\in[a,b]} f(s,c) - \int_{s\in[a,b]} f(s,c)
\end{aligned}$$

where the last step is by the Fundamental Theorem of Calculus. This gives us the two vertical segments, and the horizontal segments contributes nothing since x_1 is constant on them, and $dx_1 = 0$.

Adding in $f dx_2$ gives the other contribution. Let's make sure we didn't botch the signs. Put $\omega = f dx_2$, and

$$d\omega = \frac{\partial f}{\partial x_1} dx_1 \wedge dx_2.$$

$$\begin{aligned}
\int_{\mathring{S}} d\omega &= \int_{\mathring{S}} \frac{\partial f}{\partial x_1} \\
&= \int_{t\in[c,d]} \int_{s\in[a,b]} \frac{\partial f}{\partial x_1} \\
&= \int_{t\in[c,d]} f(b,t) - \int_{t\in[c,d]} f(a,t).
\end{aligned}$$

This matches our orientation sanity checks. $\square$

§19 April 14, 2015

Theorem 19.1

Let $\omega \in \Omega^{n-1}(U)$ such that $\overline{B(x,r)} \subseteq U$, then

$$\int_{B(x,r)} d\omega = \int_{\partial \overline{B(x,r)}} \omega.$$

Let $X \supseteq \partial X$ be Hausdorff, second countable, and paracompact.

Definition 19.2. A structure on $(X, \partial X)$ of an n-dimensional manifold with boundary is a supply of charts $\mathbb{R}^{\ge 0} \times \mathbb{R}^{n-1} \supseteq U_\alpha \overset{\varphi_\alpha}{\approx} U'_\alpha \subseteq X$ such that $\varphi_\alpha(\partial X) = U_\alpha \cap (\{0\}) \times \mathbb{R}^{n-1}$ and a compatibility condition.

We must first define what it means for a map of half spaces to be C^∞. Take φ a homeomorphism.

$$\begin{tikzcd}
\partial U_1 \arrow[d] & \subseteq & U_1 \arrow[d, "\varphi"] & \subseteq & \mathbb{R}^{\ge 0} \times \mathbb{R}^{n-1} \\
\partial U_2 & \subseteq & U_2 & \subseteq & \mathbb{R}^{\ge 0} \times \mathbb{R}^{n-1}
\end{tikzcd}$$

We may decompose $\partial U \subseteq U \supseteq \mathring{U}$, with $\partial U = U \cap (\{0\} \times \mathbb{R}^{n-1})$, $\mathring{U} = U \cap (\mathbb{R}^{>0} \times \mathbb{R}^{n-1})$. Then we want φ to be C^∞ on the interior, and for every $x \in U_1$, there exists $r > 0$ and a C^∞ extension of φ to $B(x,r)$. From here, we have all of our old theorems from manifolds. The only difference is that there is a boundary.

Example 19.3

Consider $\overline{B(x,1)}$. We claim that we can construct a chart between the strictly upper half sphere and $(0,1] \times \mathbb{R}^{n-1}$ by projecting in the usual manner, with a proportion of depth related to the distance of each point in the circle from the center. (Aaron says the formula is dividing by the secant of the angle.) We can then grab the center point just by considering the open ball.

Let $(X, \partial X)$ be a manifold with boundary, $f : X \to \mathbb{R}$.

Definition 19.4. f is C^∞ if for some/any chart, $f|_{\mathring{X}}$ is C^∞ and for every x there is r such that f extends to a C^∞ function on $B(x,r)$.

Definition 19.5. A k-differential form on $(X, \partial X)$ is (on some/any chart)

- ω on $\mathring{X}$
- For every x there exists r such that ω extends to a C^∞ function on $B(x,r)$.

We then want to define a restriction to the boundary, $\Omega^k(X) \to \Omega^k(\partial X)$. Fix $x \in \partial X$, and a chart that contains X. Extend $\phi^*(\omega)$ to $B(x,r)$ and then we can just restrict to $B(x,r) \cap \mathbb{R}^{n-1}$. We could have different extensions of $\phi^*(\omega)$, however, the restriction will be the same by continuity. That is, for $\omega_1, \omega_2 : B(x,r) \to \Lambda^k(\mathbb{R}^n)$, $\omega_1|_{B(x,r)\times(\mathbb{R}^{>0}\times\mathbb{R}^{n-1})} = \omega_2|_{B(x,r)\times(\mathbb{R}^{>0}\times\mathbb{R}^{n-1})}$, so by continuity, they are also equal on the boundary.

Definition 19.6. An orientation on $(X, \partial X)$ is an orientation on the interior $\overset{\circ}{X}$.

Given $(X, \partial X)$, compactly supported ω, and an orientation ϵ, we want to define $\int_{X,\epsilon} \omega$. Using partitions of unity we can assume $\operatorname{supp}(\omega) \subseteq U_\alpha$. Then

$$\int_{U_{\alpha},\epsilon} \omega := \int_{\overset{\circ}{U}_{\alpha},\epsilon} \phi_\alpha^*(\omega).$$

Given an orientation on $\overset{\circ}{X}$, we then must define an orientation on the boundary. Let $x \in \partial U$, we have the short exact sequence

$$0 \to \mathbb{R}^{n-1} \to \mathbb{R}^n \to \mathbb{R}^n/\mathbb{R}^{n-1}$$

splits so

$$\Lambda^n(\mathbb{R}^n) \simeq \mathbb{R}^n/\mathbb{R}^{n-1} \otimes \Lambda^{n-1}\mathbb{R}^{n-1}.$$

Thus is suffices to orient $\Lambda^n(\mathbb{R}^n)$ and $\mathbb{R}^n/\mathbb{R}^{n-1}$. We just need to orient $\mathbb{R}^n/\mathbb{R}^{n-1}$ since we already have the former, and here we just take out to be positive.

Digression on "out": Let $\gamma : (-\epsilon, \epsilon) \to U$ with $\gamma(0) = x$ a path such that $D\gamma \in \mathbb{R}^n$, that projects to a positive vector in $\mathbb{R}^n/\mathbb{R}^{n-1}$.Then for $t > 0$ sufficiently small, $\gamma(t) \notin U$.

Theorem 19.7 (Stoke's Theorem)

Let $(X, \partial X)$ be a manifold with boundary, $\omega \in \Omega^{n-1}(X)$ with compact support, and orientation ϵ on X. Then

$$\int_{X,\varepsilon} d\omega = \int_{\partial X,\varepsilon} \omega.$$

Proof. Take a partition of unity, $\omega = \sum_\alpha \omega\psi_\alpha$. We can therefore assume $\operatorname{supp}(\omega) \subseteq U_\alpha$. We want to show $\int_{\overset{\circ}{U}_{\alpha},\epsilon} \phi^*(d\omega) = \int_{\partial U_\alpha,\epsilon} \phi^*\omega$. Note that the LHS is equal to $\int_{\overset{\circ}{U}_{\alpha},\epsilon} d\phi^*(\omega)$. Then we can just draw a box around our function and declare it to be 0 on the outside, and we're done by last class. □

Note that $\operatorname{supp}(\omega) \cap \partial X = \emptyset \implies \int_{X,\epsilon} d\omega = 0$ immediately by Stoke's Theorem. Dealing with the case of manifolds with corners will be on the homework.

> [Gaitsgory is saying that the case of manifolds with corners is not class level difficulty, it's only homework level]
> Stefan: "I feel like the 'homework level' stuff is getting harder and harder over time."
> Gaitsgory: "Well... You're getting older."

§20 April 16, 2015

Finally complex analysis today. Most of these notes are excerpted from my *Napkin* project, see `http://web.evanchen.cc/napkin.html`.

§20.1 Complex Differentiation

Let $f : U \to \mathbb{C}$ be a complex function. Then for some $z \in U$, we define the **derivative** at z_0 to be

$$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}.$$

Note that this limit may not exist; when it does we say f is **differentiable** at z_0.

This is not the same as being differentiable $\mathbb{R}^2 \to \mathbb{R}^2$, since the latter is

$$\frac{f(x+v)-f(x)-T(v)}{\|v\|} \to 0.$$

Complex multiplication is a special case of T, rather than vice-versa. In fact, T "comes from λ" if and only if $T \circ (i \cdot -) = (i \cdot -) \circ T$.

Let $f : U \to \mathbb{C}$ be differentiable, and write $f(x+iy) = f_R(x+iy) + i \cdot f_I(x+iy)$. Then we see the total derivative of f ought to be

$$\begin{pmatrix} \frac{\partial f_R}{\partial x} & \frac{\partial f_R}{\partial y} \\ \frac{\partial f_I}{\partial x} & \frac{\partial f_I}{\partial y} \end{pmatrix}.$$

The fact that it commutes with $i \circ -$ means that

$$\begin{pmatrix} \frac{\partial f_R}{\partial x} & \frac{\partial f_R}{\partial y} \\ \frac{\partial f_I}{\partial x} & \frac{\partial f_I}{\partial y} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial f_R}{\partial x} & \frac{\partial f_R}{\partial y} \\ \frac{\partial f_I}{\partial x} & \frac{\partial f_I}{\partial y} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Now we multiply the matrices.

If you fail to do it, I'll really know I taught you well.

This gives:

Corollary 20.1 (Cauchy-Riemann Equations)

If f is complex differentiable it satisfies

$$\frac{\partial f_R}{\partial y} = -\frac{\partial f_I}{\partial x}$$

and

$$\frac{\partial f_R}{\partial x} = \frac{\partial f_I}{\partial y}.$$

Moreover if the partials are continuously differentiable then the converse is true as well.

These are *really* strong.

But note that having a complex differentiability is actually much stronger than a real function having a derivative. In the real line, h can only approach zero from below and above. and for the limit to exist we need the "left limit" to equal the "right limit". But the complex numbers form a *plane*: h can approach zero from many directions, and we need all the limits to be equal.

So we restrict our attention to differentiable functions called *holomorphic functions*. It turns out that the multiplication on $\mathbb{C}$ makes all the difference. The primary theme in what follows is that holomorphic functions are *really, really nice*, and that knowing tiny amounts of data about the function can deterimne all its values. Some highlights:

- It'll turn out that having a first derivative is enough to guarantee having *all* derivatives (which is unlike the real case, in which no number of derivatives will guarantee the rest).
- Now that you have all derivatives, you can take the Taylor series, (unlike the real case, in which even if you have all derivatives it's possible the Taylor series is totally wrong).
- It'll turn out that knowing just the values of a holomorphic function on the boundary of the unit circle will tell you all the values everywhere.
- Even knowing just the values of the function at $1, \frac{1}{2}, \frac{1}{3}, \dots$ is enough to determine the whole function!
- Integrals in the complex plane will just work like magic.

If a function $f : U \to \mathbb{C}$ is complex differentiable at all the points in its domain it is called **holomorphic**. In the case $U = \mathbb{C}$, we sometimes call the function **entire**.

§20.2 Examples

Example 20.2 (Not Differentiable Function)

Complex conjugation, $z \mapsto \overline{z}$, is an example of a map which is not differentiable at zero. As we approach 0 from various directions, we get different values of the ratio $\frac{f(h)}{h}$ vary.

For example, $f(z) = 6$, $f(z) = z^6$ work.

"It's not just 'like 1', it *is* 1" – Gaitsgory

Example 20.3 (Examples of Holomorphic Functions)

In all the examples below, the derivative of the function is the same as in their real analogues (e.g. the derivative of e^z is e^z).

(a) Any polynomial $z \mapsto z^n + c_{n-1}z^{n-1} + \dots + c_0$ is holomorphic.

(b) The complex exponential $\exp : x + yi \mapsto e^x(\cos y + i \sin y)$ can be shown to be holomorphic.

(c) sin and cos are holomorphic when extended to the complex plane by $\cos z = \frac{e^{iz}+e^{-iz}}{2}$ and $\sin z = \frac{e^{iz}-e^{-iz}}{2}$.

(d) As usual, the sum, product, chain rules and so on apply, and hence **sums, products, nonzero quotients, and compositions of holomorphic functions are also holomorphic**.

You are welcome to try and prove these results, but I won't bother to do so.

Example 20.4
$f(z) = 1/z$ is complex differentiable for $f : U \to \mathbb{C} \to 0$.

> "Oh, it still says theorem. It's not 'theorem', that's a joke. It's '*examples*'." – Gaitsgory

Note $e^z = \sum_k \frac{z^k}{k!}$ works here.

> "Do you remember when we did applied math between those two big snows?" – Gaitsgory

§20.3 Inverse Function Theorem

Still applies since the inverse of a nonzero $\lambda \cdot - : \mathbb{C} \to \mathbb{C}$ is still just complex multiplication.

§20.4 Complex Differentiable Forms

For $U \subseteq \mathbb{R}^n$, we can consider

$$\Omega^n(U) \otimes_{\mathbb{R}} \mathbb{C}$$

and view it as a formal pair $\omega_R + i\omega_I$. Thus

$$\Omega^\bullet_{\mathbb{C}}(U)$$

is a ring, and we can speak of the differential $d : \Omega^n_{\mathbb{C}}(U) \to \Omega^{n+1}_{\mathbb{C}}(U)$ as before. Also we can take a pullback as before.

From now on, we will assume that all our holomorphic functions are *continuously differentiable*. This is not needed, but it will make life easier.

§20.5 Cauchy Integral Formula

Let $z_0 \in U$; then we can consider disks in the plane.

Take $z \in \Omega^0_{\mathbb{C}}(U)$, say $z = x + iy$. Then

$$dz = dx + idy \in \Omega^1_{\mathbb{C}}(U).$$

> "Now I'm going to multiply them."
> "You can write formal symbols without understanding what they mean."
> – Gaitsgory

So the point of $dz = dx + idy$ means that to integrate, we just sum the real and imaginary parts separately.

Now, suppose we want to integrate

$$\int_{S(z_0,r)} \frac{f(z)}{z - z_0} dz$$

where we pick the counterclockwise orientation; thus it should give us some complex number.

Theorem 20.5 (Cauchy's Integral Formula)

For a holomorphic function f, let S be a circle of any radius such that z_0 is inside S. Then

$$\int_S \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0).$$

Everything about holomorphic functions is going to follow from this.

Lemma 20.6

Let f be a complex continuously differentiable function. Then f is holomorphic if and only if

$$d(f dz) = 0.$$

Proof. Let $f = f_R + i f_I$. Then quick computation gives

$$d\left((f_R + i f_I)(dx + idy)\right) = \left(\frac{\partial f_R}{\partial y} d_y \wedge d_x - \frac{\partial f_I}{\partial x} d_x \wedge d_y\right) + \left(\frac{\partial f_I}{\partial y} d_y \wedge d_x - \frac{\partial f_R}{\partial x} d_x \wedge d_y\right) i.$$

This is zero if and only if the Cauchy-Riemann equations. $\square$

> "How many terms are there altogether? Eight. That's a large number." – Gaitsgory

Corollary 20.7

The value of the integral in the problem in Cauchy's Integral Formula doesn't depend on the circle chosen.

Proof. Consider two circles as described. Let

$$\omega = \frac{f(z)}{z - z_0} dz.$$

We invoke Stoke's Theorem, where the manifold in question is the annulus X cut out by these two circles. This gives

$$\int_{X,\varepsilon} d\omega = \int_{\partial X} \omega.$$

But ω is holomorphic, hence $d\omega = 0$. Also, ∂X is the two circles, so

$$0 = \int_{\partial X} \omega = \int_{\text{outer}} \omega - \int_{\text{inner}} \omega. \qquad \square$$

Of course, these don't actually have to be circles.

§20.6 Computation when $f = 1$

We still want to prove Cauchy's Integral Formula. There's a π, so we need to do something.

> "Rachit, what's π?" –Gaitsgory

Now we prove Cauchy's Integral Formula. First, we prove it in the case $z_0 = 0$, $f(z) = 1$, and S the circle of radius 1.

> "In the past weeks we talked about integrals and partitions of unity and whatnot... So now we have to actually do it." – Gaitsgory

We have the function

$$\mathbb{R} \to S \subseteq \mathbb{R}^2 \text{ by } t \mapsto \exp(it).$$

Let $\phi(t) = \exp(it)$.

Claim 20.8. $\phi^*(\frac{dz}{z}) = i\, dt$.

Proof. We have

$$\frac{\phi^* dz}{z \circ \phi} = \frac{d(z \circ \phi)}{z \circ \phi} = \frac{(z \circ \phi)' dt}{z \circ \phi}.$$

Since $z \circ \phi = \exp(it)$, its derivative is just $i \exp(it)$, so we're done. $\square$

Claim 20.9.

$$\int_{S^1, \varepsilon} \frac{dz}{z} = \int_{[0,2\pi]} \phi^* \left(\frac{dz}{z} \right).$$

Proof. Clear, modulo an issue of $[0, 2\pi)$ instead of $[0, 2\pi]$ $\square$

Then

$$\int_{[0,2\pi]} \phi^* \left(\frac{dz}{z} \right) = \int_{[0,2\pi]} i dt = 2\pi i.$$

§20.7 General Proof of Cauchy's Formula

Now we want to show the general case

$$\int_{S_r} \frac{f(z)}{z} = 2\pi i \cdot f(0) = \int_{S_r} \frac{f(0)}{z}.$$

This amounts to

$$\int_{S_r} \frac{f(z) - f(0)}{z} dz = 0.$$

Taking the pullback again, the left-hand side equals

$$\int_{[0,2\pi]} \phi^* \left(\frac{dz}{z} \right) ((f(z) - f(0)) \circ \phi) = i \cdot \int_{[0,2\pi]} f(r \exp(it)) - f(0).$$

But r is arbitrary, and we can make $f(r \exp(it)) - f(0)$ as small as we want by continuity. More explicitly, for any $\varepsilon > 0$ we can take r small enough so that

$$|f(r \exp(it)) - f(0)| < \varepsilon$$

whence

$$\left| \int_{[0,2\pi]} f(r \exp(it)) - f(0) \right| < 2\pi\varepsilon \xrightarrow{\varepsilon \to 0} 0.$$

§21 April 21, 2015

Let z_0 be a point in a domain U. Then we showed last class that

$$f(z_0) = \frac{1}{2\pi i}\int_S \frac{f(z)dz}{z - z_0},$$

where f is complex differentiable and C^1.

Theorem 21.1 (Maximum Modulus Principle)
Let S be a circle centered at z_0. Then $|f(z_0)| \le \max_{z\in S} |f(z)|$.

Proof. For simplicity, set $z_0 = 0$, and use the Cauchy Integral Formula. Then we can take $[0, 2\pi] \to S$ by $t \mapsto \exp(it)$, so $\int_S \frac{f(z)}{z} dz = i \int_{[0,2\pi]} f(\exp(it))$. Then $|\int_S \frac{f(z)}{z} dz| = |\int_{[0,2\pi]} f(\exp(it))| \le \int_{[0,2\pi]} |f(\exp(it))| \le \int_{[0,2\pi]} \max |f(z)| = 2\pi \max_{z\in S} |f(z)|$. □

Theorem 21.2 (Liouville)
Let f be holomorphic on all of $\mathbb{C}$ (that is, f is *entire*), and for every ϵ there is R such that $|f(z)| < \epsilon$ for $|z| > R$. Then $f = 0$.

Proof. Fix z_0, by Theorem 21.1 we can bound $f(z_0) < \epsilon$ for every ϵ, thus $f = 0$. □

Theorem 21.3 (Fundamental Theorem of Algebra)
Every complex polynomial has a root.

Proof. Suppose there is not a z such that $p(z) = 0$. Then consider $\frac{1}{p(z)}$ is well defined on $\mathbb{C}$. Moreover, at large z, the highest term of the polynomial dominates the function, so the $\frac{1}{p(z)}$ satisfies the conditions of Liouville's Theorem. Thus $\frac{1}{p(z)} = 0$, contradiction. □

Theorem 21.4 (Holomorphic Functions are Smooth)
Let f be holomorphic and C^1 on U. Then f is C^∞.

Proof. It suffices to show f' is C^1 by problem 8 on the PSet. Take any ball $B \subseteq U$ with $z_0 \in B$, then $f(z_0) = \int_S \frac{f(z)dz}{z-z_0}$. Partials exist by the midterm, as we have $\frac{\partial f_R}{\partial x_0} = \frac{1}{2\pi i}\int \frac{\partial}{\partial x_0}(\frac{1}{z-z_0}) f(z)dz$, and the inner term is clearly differentiable with respect to z_0. Then it continues to satisfy the Caucy-Riemann equations after you do the integration, which finishes the problem. □

> "We'll now state the following false theorem." –Gaitsgory

> Let f be C^∞ on (a, b) such that for some $x \in (a, b)$, $f^{(n)} = 0$. Then $f = 0$.

We've already seen this is false–consider for instance $e^{-\frac{1}{x}}$ at $x = 0$.

> "This is completely false in the real world." – Gaitsgory

However, it is true for holomorphic functions.

Theorem 21.5 (Holomorphic Functions are Analytic)
Let f be holomorphic on U and $\overline{B} \subseteq U$ a disk of radius r, with z_0 the center. Then

$$f(z) = \sum \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

uniformly and absolutely on $\overline{B}$.

Proof. By the Cauchy Integral Formula, $f(z) = \frac{1}{2\pi i}\int_S \frac{f(\widetilde{z})}{(\widetilde{z}-z)}$, where S is a circle slightly larger than B.

> "No, z' isn't good, what should we use... Ah, $\widetilde{z}$." –Gaitsgory (paraphrased)

Then

$$\frac{1}{\widetilde{z} - z} = \frac{1}{\widetilde{z}}\left(1 + \frac{z}{\widetilde{z}} + \frac{z^2}{\widetilde{z}^2} + ...\right)$$

converges uniformly and absolutely, by taking the radius of S to be r_0, so that

$$\left|\frac{z}{\widetilde{z}}\right| < \frac{r}{r_0} < 1.$$

Then $g_n \to g$ uniformly implies $\int_S g_n \to \int_S g$, so we are left with

$$\int \frac{f(\widetilde{z})d\widetilde{z}}{\widetilde{z}}\left(\sum_{n=m}^{\infty} \frac{z^n}{\widetilde{z}^n}\right),$$

which we want to bound $< \epsilon$, which is clear. This will show that the sequence $\sum_{i=0}^{n}(\int_S \frac{f(\widetilde{z})}{\widetilde{z}^{i+1}})z^i$ converges to $f(z)$.

Now $\frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i}\int f(\widetilde{z})d\widetilde{z} \cdot (\frac{1}{\widetilde{z}-z_0})^{(n)}|_{z_0=0}$, which follows by the Cauchy formula on $f^{(n)}(0)$. $\square$

Theorem 21.6
Let a_i be a sequence of complex numbers such that $\limsup \sqrt[n]{|a_n|} \le \frac{1}{r_0}$. Then the series $\sum a_n z^n$ converges pointwise to a holomorphic function on B_r, and the convergence is uniform on $\overline{B}_r$, for any $r < r_0$.

Proof. First, uniform convergence on $\overline{B}_r$: Take $r < r' < r_0$, then for all but finitely many n, $\sqrt[n]{|a_n|} < \frac{1}{r'} \implies |a_n| < \frac{1}{(r')^n}$, so $|a_n z^n| < (\frac{r}{r'})^n$, which we know how to estimate. Then the tails are $\le \sum_{n=m}^{\infty}(\frac{r}{r'})^n = (\frac{r}{r'})^m(\frac{1}{1-\frac{r}{r'}})$.

We now need a lemma. In particular, if $f_n \to f$ uniformly on some domain in U, with f_i holomorphic, then we'd like to claim f is as well. This will be presented below. Then we can apply this to our case to get that our function is holomorphic, and then by writing $B_{r_0} = \cup_{r<r_0} B_r$, we get our theorem. (Note that our functions are holomorphic on $\overline{B}_r$, so *a fortiori* they are holomorphic on B_r.) $\square$

We used the following lemma in our theorem:

Lemma 21.7

Let $f_n \to f$ uniformly on U be holomorphic. Then f is holomorphic.

Proof. We have $f(z_0) = \lim f_n(z_0) = \lim \frac{1}{2\pi i} \int_S \frac{dz}{z-z_0} f_n(z) = \frac{1}{2\pi i} \int \frac{dz}{z-z_0} f(z)$, hence the limit is a holomorphic function. $\square$

Theorem 21.8 (Identity Theorem)

Let $U_0 \subseteq U$ and f_0 a holomorphic function on U_0. Assume U is connected. Then there is at most one f on U such that $f|_{U_0} = f_0$. (We say that there is at most one *analytic continuation* of f_0.)

Proof. Let f_1, f_2 be two holomorphic extensions, then $(f_1 - f_2)|_{U_0} = 0$. We may therefore assume that $f_0 = 0$, and we wish to show $f = 0$. Set $V^n = \{z \in U \mid f^{(n)}(z) = 0\}$. Note these are closed sets. Then write $V = \cap_n V^n$, and the intersection of closed sets is closed, so V is closed as well.

We claim that V is also open in U. Given $z \in V$, there is a disk $\overline{B} \subseteq U$ around z. On B, $f(z) = \sum \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = f = 0$ on all of B, so V contains $B \implies V$ is open. Then U connected implies $V = U$. $\square$

On Thursday we will study the geometry of holomorphic mappings. Suppose U_1, U_2 are domains, and $U_1 \xrightarrow{f} U_2$. Then f is biholomorphic if f, f^{-1} are both holomorphic.

Example 21.9

Take $U_1 = \mathbb{C}$ and $U_2 = B$. There are diffeomorphisms between $\mathbb{C}$ and B through stereographic projections. However, we'll prove that there is not a biholomorphic map between the two.

Theorem 21.10 (Liouville)

A bounded holomorphic function on $\mathbb{C}$ is constant.

From this, we can conclude $\mathbb{C} \neq B$.

Example 21.11

Take $U_1 = \mathbb{C}$ and

$$U_2 = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}.$$

Here U_2 is holomorphic to B by the map $\exp(iz)$, so $U_2 = B \neq \mathbb{C}$ holomorphically.

We will also show that all transforms of $\mathbb{C}$ are of the form $az + b$. Similarly, U_2 from Example 21.11 (the upper half plane) may be manipulated exclusively by maps of the form $z \mapsto \frac{az+b}{\overline{b}z+\overline{a}}$.

§22 April 28, 2015

Additional office hours tomorrow, Wednesday, 4PM-5PM.

The final will have

1. Differential equations (vector fields, commutator)

A. A bonus from the preceding pset

2. Orientations

3. Complex Analysis

4. Complex Analysis

B. Complex Analysis

§22.1 Riemann Mapping Theorem

Today we prove the Riemann mapping theorem.

Theorem 22.1 (Riemann Mapping Theorem)
Let U be a simply connected domain of $\mathbb{C}$ which is not $\mathbb{C}$. Then there exists a biolomorphic function

$$f : U \to B(0,1).$$

Proof. First, consider the set S of injective functions $f : U \to B(0,1)$. Pick any $z_0 \in U$. Consider the map

$$S \to \mathbb{R}_{\ge 0} \quad \text{by} s \mapsto \left|f'(z_0)\right|.$$

Let F be the element of S such that

$$\left|F'(z_0)\right| \ge \left|f(z_0)\right|.$$

Note that we haven't checked that F exists or even that S is nonempty, but taking that on faith for now, we have:

Claim 22.2. $F(z_0) = 0$ and F is surjective.

> "Those of you who have not lost the ability to differentiate as a result of 55. . . " – Gaitsgory
> "Will we have to differentiate on the final exam?"
> "OH YEAH"

Proof. For the first part, assume for contradiction $F(z_0) \neq 0$. Then set

$$G(z) = \frac{F(z) - F(z_0)}{1 - \overline{F(z_0)}F(z)}$$

and verify that $G \in S$ still. Then

$$G'(z_0) = \frac{F'(z_0)}{1 - \left|F(z_0)\right|^2}.$$

But $|G'(z_0)| > |F'(z_0)|$, contradicting the fact that F is maximal.

Now we show it's surjective. Assume w isn't in the image.

> "What letters of the alphabet do we know? We know F, we've used it, we've used G..."

Let $H_0(z) = \frac{F(z)-w}{1-\overline{w}\cdot F(z)}$. Since this function is non-vanishing, and the domain is simply connected, we can find a function H_1 such that $H_1(z)^2 = H_0(z)$ ("take square roots"). Since H_0 is injective so is H_1, thus $H_1 \in S$ too. Then let

$$H_2(z) = \frac{H_1(z) - H_1(z_0)}{1 - \overline{H_1(z_0)} \cdot H_1(z)}.$$

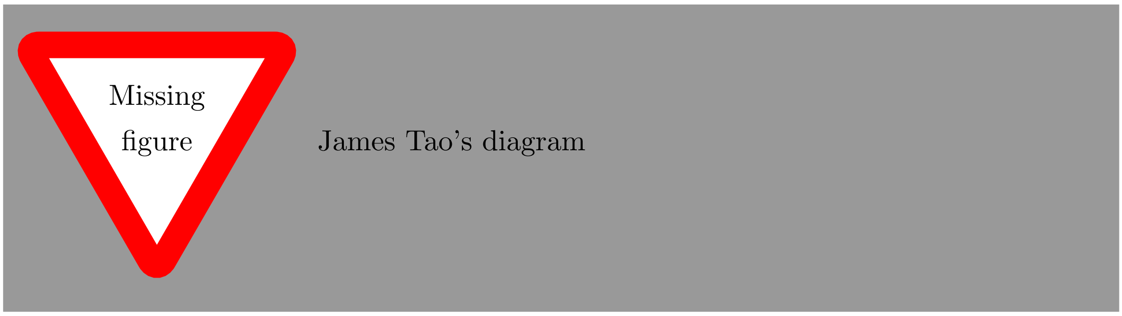
With a *very fun* computation we get

$$\left|H_2'(z_0)\right| = \frac{1+|w|}{2\sqrt{|w|}} \cdot F'(z_0)$$

By AM-GM, we have

$$\frac{1+|w|}{2\sqrt{|w|}} > 1$$

which is the desired contradiction. ■



Next, we will show the following lemma.

Lemma 22.3
Let $f : U \to \mathbb{C}$ be holomorphic and injective. Then $f'(z) \neq 0$ for all $z \in U$.

Not true in $\mathbb{R}$, take $f(x) = x^3$ for example.

Proof. Assume not. By shifting, we have $f(0) = f'(0) = 0$. Write $f(z) = z^n g(z)$ for some nonvanishing g. Restricted to a small disk, we can extract an nth root of g (according to the last pset), so we may put $f(z) = (z \cdot h(z))^n$ for some on some small neighborhood of 0. We claim $z \cdot h(z)$ is biholomorphic on a small disk; indeed its derivative is $h(0) \neq 0$ on a small disk.

So on a small disk, we have

$$\text{disk} \xrightarrow{z\cdot h(z)} \text{disk} \xrightarrow{(-)^n} \mathbb{C}$$

is injective, but the first arrow is an isomorphism and the second arrow is not injective, which is impossible. ■

Now, non-vanishing differential is sufficient to get that f is biholomorphic. That's all.

Oops no actually so we need to show the F exists in the first place. First, we show that $S \neq \varnothing$. We have a function

$$U \to B(0,1).$$

Since $U \neq \mathbb{C}$, there exists $a \notin \mathbb{C}$. So we can take

$$\sqrt{z-a}$$

which is nonvanishing and holomorphic on U. Let ϕ_1 and ϕ_2 be these two square roots.

Claim 22.4. Let $i \in \{1,2\}$. If $\phi_i(z') = \phi_i(z'')$ then $z' = z''$.

Proof. Trivial, do it yourself. ■

Now ϕ_1 and ϕ_2 have nonvanishing derivative, so they have open images. Let w be in the image of ϕ_1, arbitrarily. Then there exists r such that $\overline{B}(w,r)$ is contained inside the image of ϕ_i, but does not intersect the image of ϕ_2. Set

$$f(z) = \frac{r}{w - \phi_2(z)}.$$

This f works.

Finally, we wish to show that S indeed has a maximum as we claimed. For this we appeal to the following.

Theorem 22.5

Let S be any set of holomorphic functions on U such that there exists a Λ such that $|f(z)| \le \Lambda$ such that for all $f \in S$ and $z \in U$. Then any sequence $f_n \in S$ has a subsequence that converges uniformly for all $K \subseteq U$.

By Morera's Theorem, the limit of the uniform convergence is actually holomorphic.

Theorem 22.6 (Hurwitz)

Let $f_n \to f$ uniformly on every compact, where f is nonconstant. If all the f_i are injective then so is the limit f.

Together these imply the result. □

§22.2 Proofs of First Theorem

First, we prove the first theorem. Note that since an open set can be written as a union of nestned compacts, it suffices to do this for a given compact K. Check $\exists r$ such that $\forall x \in K$, $\overline{B(x,r)} \subseteq U$.

Claim 22.7. f_n are equicontinuous when restricted to K.

Proof. Blah. □

§22.3 Proof of Hurwitz's Theorem

It suffices to show the following:

Theorem 22.8 (Hurwitz')

If $f_n \to f$ uniformly on $U \ni z_0$ and $f(z_0) = 0$ but $f \not\equiv 0$, then for all r there exists N such that for all $n \ge N$, f_n will have a zero on $B(z_0, r)$.

"'holomolomorphic"

Theorem 22.9

Let g be holomorphic on U and let $\overline{B}(z_0, r) \subseteq U$. Assume g has no zeros on $\partial B(z_0, r)$. If

$$\int_{\partial B(z_0,r)} \frac{g'(z)}{g(z)}\,dz \neq 0$$

then g has zeros in the ball.

Proof. If g has no zeros then Cauchy's Theorem shows that the integral is zero. $\square$

Then we can just note that $f_n \to f$ means $f_n' \to f'$ and then

$$\int \frac{f_n'}{f_n}\,dz \to \int \frac{f'}{f}\,dz.$$

This establishes Hurwitz' and hence Hurwitz.

§22.4 A Theorem We Tried to Use But Couldn't

We deduce the above from

Theorem 22.10

Let f and g be holomorphic on U such that $|g| < |f|$. Then if f has a zero in $B(z_0, r)$ then so does $f + g$.

Proof. Write $f + g = f \cdot (1 + \frac{g}{f})$. $\square$