Math 55a Lecture Notes

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This is Harvard College's famous *Math 55a*, instructed by Dennis Gaitsgory. The formal name for this class is "Honors Abstract and Linear Algebra" but it generally goes by simply "Math 55a".

The permanent URL is `http://web.evanchen.cc/~evanchen/coursework.html`, along with all my other course notes.

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§1 September 2, 2014

§1.1 Boring stuff

Sets include $\mathbb{R}$, $\mathbb{Z}$, et cetera. A subset $Y \subseteq X$ is exactly what you think it is. $\mathbb{Q}, \{0\}, \{1\}, \varnothing, \mathbb{R} \subseteq \mathbb{R}$. Yay.

$X_1 \cup X_2$, $X_1 \cap X_2$.

...Gaitsgory what are you doing

For a fixed universe X, we write $\overline{Y}$, $X \setminus Y$, $X - Y$ for $\{x \in X \mid x \notin Y\}$.

Lemma 1.1

For $Y \subset X$,

$$\overline{(\overline{Y})} = Y.$$

Proof. Trivial.

darn this is being written out?

$$x \in \overline{(\overline{Y})} \iff x \notin \overline{Y} \iff x \in Y.$$

Hence $\overline{(\overline{Y})} = Y$. $\square$

Lemma 1.2

$\overline{(X_1 \cap X_2)} = \overline{X_1} \cup \overline{X_2}$.

Proof. Compute

$$x \in \overline{X_1 \cap X_2} \iff x \notin X_1 \cap X_2 \iff x \notin X_1 \vee x \notin X_2 \iff x \in \overline{X_1} \vee x \in \overline{X_2} \iff x \in \overline{X_1} \cup \overline{X_2}.$$

$\square$

Lemma 1.3

$\overline{X_1 \cup X_2} = \overline{X_1} \cap \overline{X_2}$.

Proof. HW. But this is trivial and follows either from calculation or from applying the previous two lemmas. $\square$

Given a set X we can consider its power set $\mathcal{P}(X)$. It has 2^n elements.

§1.2 Functions

Given two sets X and Y a map (or function) $X \xrightarrow{f} Y$ is an assignment $\forall x \in X$ to an element $fx \in Y$.

Examples: $X = \{55 \text{ students}\}$, $Y = \mathbb{Z}$. Then $f(x) = \$$ in cents (which can be negative).

Definition 1.4. A function f is injective (or a monomorphism) if $x \neq y \implies fx \neq fy$.

Definition 1.5. A function f is surjective (or an epimorphism) if $\forall y \in Y \exists x \in X : fx = y$.

Composition;

$$X \xrightarrow{f} Y \xrightarrow{g} Z.$$

§1.3 Equivalence relations

An equivalence relation $\sim$ must be symmetric, reflexive, and transitive. A relation $\sim$ will partition its set X into cosets or equivalence classes. (The empty set is not a coset.)

Lemma 1.6

Let X be a set and $\sim$ an equivalence relation. Then for any $x \in X$ there exists a unique coset $\overline{x}$ with $x \in \overline{x}$.

Proof. Very tedious manual work. $\square$

Now we can take quotients $X/\sim$, and we have projections $\pi : X \to X/\sim$.

§2 September 4, 2014

§2.1 Review of equivalence relations go here

meow

§2.2 Universal property of a quotient

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & Y \\ X/\sim \arrow[ur, "\tilde{f}"] & \end{tikzcd}$$

Proposition 2.1

Let X and Y be sets and $\sim$ an equivalence relation on X. Let $f : X \to Y$ be a function which preserves $\sim$, and let π denote the projection $X \to X/\sim$. Prove that there exists a unique function $\tilde{f}$ such that $f = \tilde{f} \circ \pi$.

The uniqueness follows from the following obvious lemma.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \\ X' \arrow[ur, "f_1", "f_2"'] & \end{tikzcd}$$

Lemma 2.2

In the above commutative diagram, if g is surjective then $f_1 = f_2$.

Proof. Just use g to get everything equal. Yay. □

§2.3 Groups

Definition 2.3. A **semi-group** is a set G endowed with an associative[1] binary operation $* : G^2 \to G$.

Lots of groups work.

Example 2.4 (Starfish)

Let G be an arbitrary set and fix a $g_0 \in G$. Then let $ab = g_0$ for any $a, b \in G$. This is a semigroup.

A "baby starfish" has $|G| = 1$.

[1] $(ab)c = a(bc)$

Definition 2.5. A semi-group G is a **monoid** if there exists an identity $1 \in G$ such that $\forall g \in G$, $g \cdot 1 = 1 \cdot g = g$.

Proposition 2.6

The identity of any semi-group G is unique.

Proof. Let 1, $1'$ be identities. Then

$$1 = 1 \cdot 1' = 1'.$$

□

Definition 2.7. A **group** is a monoid G with inverses: for any $g \in G$ there exists g^{-1} with

$$gg^{-1} = g^{-1}g = 1.$$

Proposition 2.8

Inverses are unique.

Proof. Suppose x_1, x_2 are both inverses of g. Then

$$x_1 = x_1 g x_2 = x_2.$$

□

Definition 2.9. A group is **abelian** if it is commutative.

§2.4 Homomorphisms

Definition 2.10. Let G and H be groups. A **group homomorphism** is a map $f : G \to H$ that preserves multiplication.

§3 September 9, 2014

§3.1 Direct products

Given two sets X and Y we can define the direct product

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}.$$

For example, $\mathbb{R}^2$ is the Euclidean plane.

Hence the operation of a semigroup should be thought of as

$$G \times G \xrightarrow{\text{mult}} G.$$

Given $f : Y_1 \to Y_2$ we define $\mathrm{id}_f \times f : X \times Y_1 \to X \times Y_2$ by

$$(x, y_1) \mapsto (x, fy_1).$$

§3.2 Commutative diagrams

We can then rephrase associativity using the following commutative diagram.

$$\begin{tikzcd}
G \times G \times G \arrow[r, "\mathrm{id}_G \times *"] \arrow[d, "* \times \mathrm{id}_G"] & G \times G \arrow[d, "*"] \\
G \times G \arrow[r, "*"] & G
\end{tikzcd}$$

We can also rephrase homomorphisms as follows: given $\varphi : G \to H$ we require the following diagram to commute.

$$\begin{tikzcd}
G \times G \arrow[r, "\varphi \times \varphi"] \arrow[d, "*_G"] & H \times H \arrow[d, "*_H"] \\
G \arrow[r, "\varphi"] & H
\end{tikzcd}$$

§3.3 Sub-things

Definition 3.1. Let G be a semigroup / monoid / group. We say $H \le G$ is a **sub-semigroup** if $h_1, h_2 \in H \implies h_1 h_2 \in H$. Moreover, if G is a monoid and $1 \in H$ then H is a **sub-monoid**. Finally, if H is closed under inverses as well then H is a **subgroup**.

Example 3.2

Let G be the additive group of integers. Then $\mathbb{N}$ is a sub-semigroup, $\mathbb{Z}_{\ge 0}$ is a sub-monoid, and $2\mathbb{Z}$ is a subgroup.

Lemma 3.3

If $H_1 \le G$ and $H_2 \le G$ are subgroups, then $H_1 \cap H_2$ is a subgroup.

Proof. Obvious. $\square$

§3.4 Let's play Guess the BS!

"In what follows I'll state some false statements. You will be asked to prove them at your own risk."

Lemma 3.4
Given $\varphi : G \to H$ a homomorphism, $\varphi(G)$ is a subgroup of H.

Proof. Given any $\varphi(a)$, $\varphi(b)$ in $\varphi(G)$ we have

$$\varphi(a)\varphi(b) = \varphi(ab) \in \varphi(G).$$

Then use $\varphi(1) = 1$ to get the rest of the conditions. □

Lemma 3.5
If $\varphi : G \to H$ and $H' \le H$, then $\varphi^{-1}(H')$ is a subgroup of G.

Proof. This one turns out to be true. □

Fact 3.6. Given H_1, H_2 subgroups of G, $H_1 \cup H_2$ need not be a subgroup of G.

Proof. Take $G = \mathbb{Z}$, $H_1 = 100\mathbb{Z}$, $H_2 = 101\mathbb{Z}$. □

§3.5 Kernels

Definition 3.7. Given a homomorphism $\varphi : G \to H$, the **kernel** $\ker \varphi$ is defined by

$$\ker \varphi = \varphi^{-1}(\{1\}).$$

Proposition 3.8
Let $\varphi : G \to H$ be a homomorphism. Then φ is injective as a map of sets if and only if $\ker \varphi = \{1\}$.

Proof. In all cases $1 \in \ker \varphi$. If $|\ker \varphi| \ne 1$ then clearly φ is not injective. On the other hand, suppose $\ker \varphi = \{1\}$. If $\varphi a = \varphi b$ we get $\varphi(ab^{-1}) = 1$, so if we must have $ab^{-1} = 1$ or $a = b$. □

Definition 3.9. Let G be a group and let $H \le G$ be a subgroup. We define the right equivalence relation on G with respect to H $\sim^r$ as follows: $g_1 \sim^r g_2$ if $\exists h \in H$ such that $g_2 = g_1 h$.

Define $\sim^\ell$ similarly.

To check this is actually an equivalence relation, note that

$$1 \in H \implies g \sim^r g$$

and

$$g_1 \sim g_2 \implies g_1 = g_2 h \implies g_1 h^{-1} = g_2 \implies g_2 \sim^r g_1.$$

Finally, if $g_1 = g_2 h'$ and $g_2 = g_3 h''$ then $g_1 = g_3(h'h'')$, so transitivity works as well.

Note that $g_1 \sim^r g_2 \iff g_1^{-1} g_2 \in H$.

Definition 3.10. Let G/H be the set of equivalence classes of G with respect to $\sim^r$.

§3.6 Normality

Definition 3.11. Let H be a subgroup of G. We say H is **normal** in G if $\forall g \in G$ and $\forall h \in H$ we have $ghg^{-1} \in H$. (This called the conjugation of h by g.)

Theorem 3.12

Let H be a normal subgroup of G. Consider the canonical projection $\pi : G \to G/H$. Then we can place a unique group structure on G/H for which π is a homomorphism.

Proof. For uniqueness, apply the lifting lemma to the following commutative diagram

$$\begin{tikzcd}
G \times G \arrow[r, "*"] \arrow[d, "\pi \times \pi"] & G \arrow[r, "\pi"] & G/H \\
G/H \times G/H \arrow[urr, dotted, "\exists"] & &
\end{tikzcd}$$

Now we claim existence when H is normal.

Let's ignore normality for now. Given $\overline{x}, \overline{y} \in G/H$, we choose $x'y' = \pi(xy)$ for $x \in \pi^{-1}(x')$ and $y \in \pi^{-1}(y')$. Now to check this is well-defined (and makes the diagram commutative).

Given $\pi(x_1) = \pi(x_2) = x'$ and $\pi(y_1) = \pi(y_2) = y'$, we want to check that

$$\pi(x_1 \cdot y_1) = \pi(x_2 \cdot y_2).$$

Evidently $x_2 = x_1 h'$ and $y_2 = y_1 h''$, so the above would be equivalent to

$$\pi(x_1 \cdot y_1) = \pi(x_1 h' y_1 h'').$$

You can check this is equivalent to

$$y_1^{-1} h' y_1 \in H$$

for all choices of $y_1 \in G$ and $h' \in H$.

That's where (and only where) the normal condition comes in. It implies that our map is indeed well-defined, and we win.

Finally, we need to show associativity and inverses. We want the following diagram to commute.

$$\begin{tikzcd}
G^3 \arrow[dr, "\pi^3"] & & \\
& G/H \times G/H \times G/H \arrow[r, "\mathrm{id} \times *"] \arrow[d, "* \times \mathrm{id}"] & G/H \times G/H \arrow[d, "*"] \\
& G/H \times G/H \arrow[r, "*"'] & G/H
\end{tikzcd}$$

Note that the G^3 has been added in. We use associativity of G to do cool things. OK the rest of the details are left as an exercise. $\square$

§3.7 Examples of normal groups

Example 3.13

Any subgroup of an abelian group is normal.

Example 3.14

Let $G = S_3$ be the permutations of three elements. If $H = \{1, (1\ 2)\}$ then this subgroup is not normal.

Here are some other useful examples of non-normal subgroups.

- In a dihedral group $D_{2n} = \left\langle r^n = s^2 = 1 \right\rangle$, the subgroup $\langle s \rangle$ is not normal.
- Take the free group F_2 on two letters. Plenty of subgroups are not normal here, for example $\langle a \rangle$.

Lemma 3.15

$\ker \varphi$ is normal.

Proof. $\varphi(ghg^{-1}) = \varphi(g)\varphi(h)\varphi(g^{-1}) = \varphi(g)\varphi(g^{-1}) = 1$. □

It is not true that $\varphi(G)$ is not normal in general. Take any non-normal $H \le G$, then we can build $\varphi : H \to G$ be the identity, so $\varphi(H) = H$.

§4 September 11, 2014

Happy birthday to Max Schindler!

§4.1 Rings

Definition 4.1. A **ring** R is a set endowed with two binary operations $+$ and $\cdot$, addition and multiplication, such that

(i) R is an abelian group with respect to addition. The additive identity is 0.

(ii) R is a monoid[2] with respect to multiplication, whose identity is denoted 1.

(iii) Multiplication distributes over addition.

The ring R is commutative if multiplication is commutative as well.

Example 4.2

Here are examples of commutative rings:

- $\mathbb{Z}$, $\mathbb{R}$ are rings.
- $\mathbb{R}[t_1, t_2, \dots, t_n]$ are rings.
- The integers modulo n are rings.

Example 4.3

Square matrices are the standard example of non-commutative rings.

Lemma 4.4

Let R be a ring. Then $r \cdot 0 = 0$.

Proof. Compute

$$r \cdot 0 = r \cdot (0 + 0) = r \cdot 0 + r \cdot 0.$$

Hence $r \cdot 0 = 0$. □

Lemma 4.5

In a ring, $r \cdot (-1) = -r$.

Proof. Compute

$$r \cdot (-1) + r = r \cdot (-1) + r \cdot 1 = r \cdot (-1 + 1) = r \cdot 0 = 0.$$ □

A little sidenote made at the end of class.

Definition 4.6. A commutative ring R is a **field** if $R - \{0\}$ is a group with respect to multiplication.

[2] Some sources do not require a 1 to exist, but in practice no one cares about rings without 1 anyway.

§4.2 Ring homomorphisms

As usual, we figure out how two rings talk to each other.

Definition 4.7. Given rings R and S, a **ring homomorphism** is a function $\varphi : R \to S$ which respects addition and multiplication such that $\varphi(1) = 1$.

Example 4.8

We can embed $\mathbb{Z}$ in $\mathbb{R}$, $\mathbb{Q}$ in $\mathbb{R}$, and so on. These maps will all be homomorphisms. Moreover, we can compose ring homomorphisms.

Example 4.9

We can construct a homomorphisms from $\mathbb{R}[t]$ into $\mathbb{R}$ by sending $p(t)$ to $p(2014)$.

§4.3 Modules, and examples of modules

In this section, fix a ring R. The addition is $+$ and has identity 0; the multiplication has identity 1 and is written $r_1 r_2$.

Definition 4.10. A **left R-module** is an additive abelian group M (meaning M is an abelian group with operation $+$) equipped with an additional multiplication: for each $r \in R$ and $m \in M$ we define an $r \cdot m \in M$. This multiplication must satisfy the following properties for every $r_1, r_2 \in R$ and $m \in M$:

(i) $r_1 \cdot (r_2 m) = (r_1 r_2) \cdot m$.[3]

(ii) Multiplication is distributive, meaning $(r_1+r_2)\cdot m = r_1\cdot m+r_2\cdot m$. and $r\cdot(m_1+m_2) = r \cdot m_1 + r \cdot m_2$.

(iii) $1 \cdot m = m$.

A module generalizes the idea of vector spaces.

Example 4.11

A trivial example of a module is $M = \{0\}$. All the axioms of modules are identities inside M, so there is nothing to verify.

Example 4.12

Additionally, we can let M be additive abelian group underlying R. The action of M on R is just left multiplication.

[3]Note that if R is not commutative, then it is not necessarily possible to define a right R-module by simply setting $m \cdot r = r \cdot m$. In other words, left and right modules are different beasts.

Example 4.13

The following module, denoted $R^{\oplus 2}$, R^2, $R \oplus R$, is the module M whose elements are $R \times R$ and whose addition is done componentwise. The action $r \cdot m$ is given by

$$r \cdot (r_1, r_2) = (rr_1, rr_2).$$

Obviously we can generalize to $R \oplus R \oplus \dots \oplus R = R^{\oplus n}$.

Example 4.14

Let R be the $n \times n$ matrices with real coefficients and let $M = \mathbb{R}^n$. Then we get our standard linear algebra thing.

§4.4 Abelian groups are $\mathbb{Z}$-modules

Lemma 4.15

Let A be an abelian group. The structure of an abelian group can be uniquely extended to a structure of a $\mathbb{Z}$-module.

Proof. Suppose $\cdot$ is the action of $\mathbb{Z}$ on A. We must have, for every positive integer n,

$$n \cdot a = (1 + 1 + \dots + 1)\, a = a + a + \dots + a.$$

Thus there is at most one possible $\cdot : \mathbb{Z} \times A \to A$, the form given above. (Negative integers are not hard to grab.)

Now you can check that since A is abelian, all the axioms hold for this $\cdot$, and we are done. $\square$

§4.5 Homomorphisms of R-modules

Definition 4.16. An **R-module homomorphism** is a map $\varphi : M \to N$ which respects the addition and action, meaning $\varphi(m_1 + m_2) = \varphi(m_1) + \varphi(m_2)$ and $\varphi(rm) = r \cdot \varphi(m)$.

Notice that homomorphisms can, once again, be composed.

§4.6 Matrices

Let m and n be positive integers, and consider a map of sets

$$R^m \xrightarrow{T} R^n.$$

Define

$$e_1 = (1, 0, \dots, 0) \in R^m.$$

Now consider the column vector

$$T(e_1) \in R^n.$$

Now we define a matrix

$$M = \underbrace{\begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \dots & T(e_m) \\ | & | & & | \end{pmatrix}}_{m} \Big\} n.$$

Proposition 4.17

The above map from $\text{Hom}_R(R^m, R^n)$ to $\text{Mat}_{n\times m}(R)$ is a bijection of sets.

Here $\text{Hom}_R(R^m, R^n)$ is the set of ring homomorphisms. In other words, with T a homomorphism, our T is determined uniquely by $T(e_1)$, $T(e_2)$, ..., $T(e_n)$.

Proof. First, suppose M is given. We will construct T from M. Obviously we will need to have

$$T(e_i) = \begin{pmatrix} m_{i1} \\ m_{i2} \\ \vdots \\ m_{in} \end{pmatrix}$$

in other words, the ith column of M. If T is going to be a homomorphism, we had better have

$$T(\langle r_1, r_2, \dots, r_m\rangle) = T(r_1e_1 + r_2e_2 + \dots + r_me_m) = \sum_{i=1}^{m} r_iT(e_i).$$

Hence we know exactly what T needs to be based on M. Hence, we just have to show it is actually a homomorphism, which is not hard.

Conversely, it is trivial to produce M given T. $\square$

§4.7 Sub-modules and Ideals

Let M be an R-module.

Definition 4.18. A **left R-submodule** of M is a subset $M' \subseteq M$ such that

- M' is a subgroup (with respect to inherited addition).
- If $m' \in M'$ then $\forall r \in R$, we have $rm' \in M'$. In other words, M' absorbs left multiplication.

Definition 4.19. A **left ideal** in I is a subset of R which is also a left R-submodule under the natural interpretation. Explicitly, I is a left ideal if and only if

(i) I is closed under addition, meaning $i_1 + i_2 \in I$ for all $i_1, i_2 \in I$,

(ii) $0 \in I$, and $i \in I \implies -i \in I$,

(iii) For any $r \in R$ and $i \in I$, $ri \in I$.

A **right ideal** is defined similarly, where the last axiom has $ir \in I$ (as opposed to $ri \in I$). If I is both a left and right ideal, it is a **two-sided ideal** or just **ideal**.

Example 4.20

If $R = \mathbb{Z}$, then $\{0\}$, $2\mathbb{Z}$, $999\mathbb{Z}$, and $\mathbb{Z}$ are all submodules.

§5 September 16, 2015

§5.1 Review

Below is problem 10 on the previous problem set, which is apparently important.

Proposition 5.1

For any R-module M, the map from $\operatorname{Hom}_R(R, M)$ to M by $\alpha \mapsto \alpha(1)$ is an isomorphism. In other words, a homomorphism of R to an R-module is determined by its value at 1.

Also we have the following obvious bijections. We have a map

$$\operatorname{Hom}(X_1 \cup X_2, Y) \to \operatorname{Hom}(X_1, Y) \times \operatorname{Hom}(X,Y)$$

by

$$\alpha \mapsto (\alpha \circ i_1, \alpha \circ i_2)\,.$$

$$\begin{tikzcd}
& X_1 \sqcup X_2 & \\
X_1 \arrow[ur, "i_1"] & & X_2 \arrow[ul, "i_2"']
\end{tikzcd}$$

We have another map

$$\operatorname{Hom}(X, Y_1 \times Y_2) \to \operatorname{Hom}(X, Y_1) \times \operatorname{Hom}(X, Y_2)$$

by

$$\beta \mapsto (p_1 \circ \beta, p_2 \circ \beta)\,.$$

$$\begin{tikzcd}
& Y_1 \times Y_2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
Y_1 & & Y_2
\end{tikzcd}$$

§5.2 Direct Sums of Modules

The analog of these two maps in the world of modules is the direct set.

Definition 5.2. Let M_1 and M_2 be R-modules. We define the **direct sum**, denoted $M_1 \oplus M_2$, as the module whose underlying group is $M_1 \times M_2$. The action of R on $M_1 \oplus M_2$ is

$$r \cdot (m_1, m_2) = (rm_1, rm_2)\,.$$

Definition 5.3. If M is an R-module, then $M^{\oplus n}$ represents the direct product of M n times.

$$\begin{tikzcd}
M_1 & & M_2 \\
& M_1 \sqcup M_2 \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] & \\
M_1 \arrow[ur, "i_1"] & & M_2 \arrow[ul, "i_2"']
\end{tikzcd}$$

Proposition 5.4

For R-modules M_1 and M_2 composition with π_1 and π_2 defines a bijection

$$\operatorname{Hom}_R(N, M_1 \oplus M_2) \to \operatorname{Hom}_R(N, M_1) \times \operatorname{Hom}(N, M_2).$$

Moreover, there is a bijection by pre-composition with i_1 and i_2:

$$\operatorname{Hom}_R(M_1 \oplus M_2, N) \to \operatorname{Hom}_R(M_1, N) \times \operatorname{Hom}(M_2, N).$$

We can consider larger direct sums $M_1 \oplus M_2 \cdots \oplus M_n$. In particular, we have a bijection

$$\operatorname{Hom}_R(R^{\oplus n}, M) \to \operatorname{Hom}_R(R, M)^n \simeq M^n.$$

As a corollary, we can re-derive something we showed last week, namely

$$\operatorname{Hom}_R\left(R^{\oplus n}, R^{\oplus m}\right) \simeq \operatorname{Mat}_{m \times n}(R).$$

Let's now consider three integers

$$R^{\oplus n} \xrightarrow{T} R^{\oplus m} \xrightarrow{S} R^{\oplus k}.$$

Of course we can take the composition of these two maps.

Proposition 5.5

Let us take the matrix M which corresponds to composing the two maps

$$R^{\oplus n} \xrightarrow{T} R^{\oplus m} \xrightarrow{S} R^{\oplus k}.$$

Then $M = ST$.

§5.3 Direct Products of Modules

Let A be an indexing set, meaning we consider a sequence $(M_a)_{a \in A}$ of modules. We claim that

$$M = \prod_{a \in A} M_a$$

also has a module structure.

We have the following analogous theorem.

Proposition 5.6

There is a bijection

$$\operatorname{Hom}_R\left(N, \prod_a M_a\right) \to \prod_a \operatorname{Hom}(N, M_a)$$

by composition with p_a.

However, surprisingly enough, it is NOT true that we have a bijection

$$\operatorname{Hom}_R\left(\prod_a M_a, N\right) \to \prod_a \operatorname{Hom}_R(M_a, N).$$

We have to make the following modification.

Definition 5.7. We define the infinite direct sum $\bigoplus_a M_a$, a subset of $\prod_a M_a$, in which at most finitely many indices a have a nonzero value.

Proposition 5.8

There is a bijection

$$\operatorname{Hom}_R\left(\bigoplus_a M_a, N\right) \to \prod_a \operatorname{Hom}_R(M_a, N).$$

by pre-composition with inclusion.

Note that $\prod_a M_a = \bigoplus_a M_a$ when we have finitely many indices.

§5.4 Sub-Modules

Let M' be a submodule of a module M. Then we have a projection $\pi : M \to M/M'$.

Lemma 5.9

The set M/M' acquires a unique structure of an R-module so that π is a homomorphism of R-modules.

Proof. We already know we can put the necessary group structure. So we first have to show there is a map act'_r such that the following diagram commutes.

$$\begin{tikzcd}
M \arrow[r, "\pi"] \arrow[d, "\text{act}_r"'] \arrow[dr, "\pi \circ \text{act}_r"] & M/M' \arrow[d, dotted, "\text{act}'_r"] \\
M \arrow[r, "\pi"'] & M/M'
\end{tikzcd}$$

The universal property (Proposition 2.1) implies that act'_r exists and is unique if and only if $\pi \circ \text{act}_r$ vanishes on M'. Note that

$$\pi \circ \text{act}_r(m') = \pi(r \cdot m') = 0$$

since $m' \in M' \implies r \cdot m' \in M'$, end of story.

The axioms on act'_r follow by inheritance from M because we commutative diagram are magic. For example, if you want to show associativity, that's equivalent to

$$\text{act}_{r_1 r_2} = \text{act}_{r_2} \circ \text{act}_{r_1}.$$

If I plug these two things into the dotted arrow, both make the diagram commute. That's enough to make them equal. (This was Lemma 2.2.) ☐

Lemma 5.10

Let M' be a R-submodule of the R-module M. Let $T : M \to N$ be an R-module homomorphism such that $T(M') = 0$. Then there is a unique $T' : M/M' \to N$ such that $T' \circ \pi = T$.

$$\begin{tikzcd} M \arrow[r, "T"] \arrow[d, "\pi"'] & N \\ M/M' \arrow[ur, "T'"'] & \end{tikzcd}$$

Proof. We already know T' is a homomorphism of groups. So M has an act_R, while M/M' also has an action act'_R. We wish to show that

$$\text{act}'_R \circ T' = T \circ \text{act}_R.$$

Consider the following commutative diagram.

$$\begin{tikzcd} & M/M' \arrow[dr, "T'"] & \\ M/M' \arrow[ur, "\text{act}'_R"] \arrow[dr, "T'"'] & M \arrow[l, "\pi"'] \arrow[r, "T"] & N \\ & N \arrow[ur, "\text{act}_R"'] & \end{tikzcd}$$

Do magic. Again, verifying axioms is just a matter of the "both satisfy the dotted arrow" trick in the previous proof. ☐

§5.5 Free Modules

Definition 5.11. Let $R^{\oplus A}$ be the free module, which is the set of maps

$$\left\{ A \xrightarrow{\varphi} R \mid \varphi(a) = 0 \text{ except for finitely many } a \right\}.$$

Theorem 5.12

Any R-module M is isomorphic to the quotient of a free module by the image of a map from another free module. That is, we can find A, B and

$$R^{\oplus B} \xrightarrow{T} R^{\oplus A}$$

such that M is isomorphic to $R^{\oplus A} / \operatorname{Im}(T)$.

The proof uses the following lemma.

Lemma 5.13

For any R-module M there exists a free module $R^{\oplus A}$ equipped with a surjection $R^{\oplus A} \to M$.

Proof of Lemma. Specifying a map $R^{\oplus A} \to M$ is equivalent to specifying a map $R \to M$ for all $a \in A$, which by Proposition 5.1 is specifying an element $m_a \in M$ for each $a \in A$. So a brutal way to do this is to select $A = M$, and $m_a = a \in M = A$. Also, this is obviously surjective. □

Proof of Theorem. First, we have a surjection $S : R^{\oplus A} \to M$ and consider $\ker(S)$. We have another surjection

$$T : R^{\oplus B} \to \ker(S) \subseteq R^{\oplus A}.$$

Now $\operatorname{Im}(T) = \ker S$. By the first isomorphism theorem,

$$M = R^{\oplus A} / \ker S = R^{\oplus A} / \operatorname{Im}(T).$$

□

§5.6 Return to the Finite

Recall that $m_1, \dots, m_n \in M$ specify a $R^{\oplus n} \to M$ by Proposition 5.1. Call this map $S \in \operatorname{Hom}_R(R^{\oplus n}, M)$.

Note: None of this actually needs to be infinite.

Lemma 5.14

The following are equivalent.

(i) The map S is surjective

(ii) If $\sum r_i m_i = 0$ then $r_i = 0 \quad \forall i$; that is, there are no nontrivial linear combinations of the m_i which yield 0.

Proof. Let e_i be the vector with 1 in the ith coordinate and zero elsewhere. Note that

$$\sum r_i m_i = \sum r_i S(e_i) == S\left(\sum r_i e_i\right) = S\left(\langle r_1, \dots, r_n \rangle\right).$$

So $\sum r_i \cdot m_i = 0$ if and only if $S(\langle r_1, \dots, r_n \rangle) = 0$. That is equivalent to S being surjective. □

Definition 5.15. In the above lemma, if the conditions hold, we say that $m_1, \dots, m_n$ are **linearly independent**.

Lemma 5.16

S is surjective if and only if $\forall m \in M$, $\exists r_1, \dots, r_n \in R$ such that $m = \sum_{i=1}^{n} r_i m_i$.

Definition 5.17. In the above lemma, if the conditions hold, we say that $m_1, \dots, m_n$ **span** M.

Lemma 5.18

The following are equivalent.

(a) For all $m \in M$, there is a *unique* choice of $r_1, \dots, r_n$ such that $m = \sum r_i m_I$.

(b) S is an isomorphism.

(c) The (m_i) are both linearly indepndent and span M.

Definition 5.19. If (m_i) are linearly independent and span M then we say that (m_i) is a **basis**.

§6 September 18, 2014

Recall that

$$\mathrm{Hom}_R(R^n, M) \sim M^{\times n}$$

is a bijection by

$$\alpha \mapsto (\alpha(e_1), \dots, \alpha(e_n)) .$$

Fix a $T : R^n \to M$ by

$$\langle r_1, \dots, r_n \rangle \mapsto m_1 r_1 + \cdots + m_n r_n.$$

Note that T is injective, surjective, bijective if and only if the r_i are linearly independent, span, and a basis, respectively.

Henceforth, assume R is a field. We also denote this with k.

Definition 6.1. A k-module (that is, a module of a field), is called a vector space.

§6.1 Linearly independent, basis, span

Proposition 6.2

Let V be a vector space and $v_1, \dots, v_n \in V$. The following are equivalent.

(1) $v_1, \dots, v_n$ are a basis.

(2) $v_1, \dots, v_n$ are a span, but no proper subset of them is a span.

(3) $v_1, \dots, v_n$ are linearly independent, but for any $w \in V$, the set $\{v_1, \dots, v_n, w\}$ is not linearly independent.

This is the only point that we will use the fact that R is a field. Afterwards, we will basically ignore this condition. So all our other work will rely on this proposition.

Proof. To show (1) implies (2), assume for contradiction that $v_2, \dots, v_n$ span, then write v_1 as a linear combination of these. Notice that this does not use the condition that R is a field.

To show (1) implies (3), just write w with the v_i. Again we do not use the fact that R is a field.

To show (2) implies (1) now we need the condition. Suppose there is a $\sum a_i v_i = 0$. WLOG $a_1 \neq 0$, now we have $v_1 = \sum_{i=2}^n \frac{-a_i}{a_1} v_i$, and now we can trash v_1 in any of our spans.

To show (3) implies (1), suppose for contradiction that

$$a_0 w + \sum a_i v_i = 0.$$

Since $v_1, \dots, v_n$ are linearly independent, $a_0 \neq 0$ and division gives that w is the span of the $\{v_i\}$, which is a contradiction. $\square$

In the proof that (2) implies (1) we isolate the following lemma that does not require the field condition.

Lemma 6.3

Let R be a ring (not even necessarily commutative). If $m_1, \dots, m_n$ span M and $m_1 = \sum_{i=2}^{n} r_i m_i$. Then $m_2, \dots, m_n$ span M.

The proof is immediate.

Corollary 6.4

If $v_1, \dots, v_n$ span V, there exists a subset of these v_i's which is a basis.

Proof. Take the minimal subset which spans V. □

§6.2 Dimensions and bases

Definition 6.5. The vector space V is finite-dimensional if and only if it has a finite spanning collection.

Corollary 6.6

A finite dimensional vector space admits a basis.

Corollary 6.7

Any finite dimensional vector space is isomorphic to k^n for some n.

Proof. Just look at the basis and do the natural projection. □

Now we have the following good theorem.

Theorem 6.8

Let $v_1, \dots, v_n$ span V and let $w_1, \dots, w_m$ be linearly independent. Then $m \le n$.

Proof in end of this lecture. Note that at this point, we use k-linear map to refer to maps of k-modules.

§6.3 Corollary Party

This theorem has the following corollaries.

Corollary 6.9

Any two bases of a vector space have the same size.

Proof. Let the sizes be m and n. Use the lemma $m \le n$ and $n \le m$ gives $m = n$. □

Definition 6.10. The size of the basis of a vector space V is the **dimension** of that space.

Corollary 6.11

Let $T : k^n \to k^m$ be a k-linear map.

(1) If T is surjective, then $n \geq m$.

(2) If T is injective, then $n \leq m$.

(3) If T is bijective, then $n = m$.

Proof. Let $\mathcal{B} = \{e_i\}$ be a basis of k^n.

(1) $T(\mathcal{B})$ spans k^m by surjectivity.

(2) $T(\mathcal{B})$ is linearly independent k^m by injectivity.

(3) Combine (1) and (2). □

Corollary 6.12 (Completion to a Basis)

Let V be a finite-dimensional basis and consider a subspace $V' \subseteq V$. Then any linearly independent subset W of V' can be completed to a basis in that subspace.

Proof. Let $W = \{w_1, \dots, w_k\}$. Construct $w_{k+1}, \dots$ inductively.

Suppose $w_{k+1}, \dots, w_{k+i}$ has been added but we're not done. Then we can add in w_{k+1+i} so that all the guys are linearly independent (if this isn't possible then our proposition implies that what we had was already a basis).

We can only get up to n, since we cannot have a linearly independent set of size $n+1$ within V, much less V'. □

Corollary 6.13

If V is finite dimensional then any linearly independent collection can be completed to a basis.

Proof. Put $V' = V$ in the above. □

Corollary 6.14

Let V be a finite-dimensional vector space and V' a subspace. Then V and V/V' are finite dimensional, and

$$\dim V = \dim V' + \dim V/V'.$$

Proof. By previous corollary, V' has a basis $w_1, \dots, w_k$. Moreover V/V' has a spanning set (just project a basis of V down). Let $\overline{v_1}, \dots, \overline{v_m}$, be a basis of $V = V'$. Then our homework shows that

$$w_1, \dots, w_k, v_1, \dots, v_m$$

are a basis of V. □

Definition 6.15. Let $T : V \to W$ be a k-linear map. If V and W are finite dimensional define

$$\operatorname{rank}(T) = \dim(T(V)).$$

Moreover, set

$$\operatorname{coker}(T) = W/T(V).$$

Corollary 6.16

$\dim V = \operatorname{rank}(T) + \operatorname{rank}(\ker T)$.

Proof. $V/\ker(T) \cong \Im(T)$ by First Isomorphism Theorem. □

Lemma 6.17

If $T : V \to W$ is k-linear, then

$$\dim V - \dim \ker(T) = \dim W - \dim \operatorname{coker}(\mathrm{T}).$$

Proof. Both are equal to $\dim \operatorname{Im}(T)$ by the previous corollary. □

Proposition 6.18

Let $T : V \to V$. Then T is injective if and only if it is surjective.

Proof. Injectivity is equivalent to

$$\dim \ker T = 0$$

and surjectivity is equivalent to

$$\dim \operatorname{coker}(T) = 0.$$

But $\dim \ker T = \dim \operatorname{coker}(T)$ by the previous corollary. □

§6.4 Proof of Theorem

Our goal is to show that if $v_1, \dots, v_n$ are spanning, and $w_1, \dots, w_m$ are linearly independent, then $m \le n$.

The proof is by induction on m. If $m = 1$ then clearly $n \ge 1$. Now assume the theorem holds for $m - 1$.

Our first step is to reduce to the case $v_1 = w_1$. Here we use for the last time that R is a field. WLOG we can write

$$w_1 = \sum_{i=1}^{n} a_i v_i$$

with $a_1 \ne 0$. Then

$$v_1 = \frac{1}{a_1} w_1 + \sum_{i \ge 2} -\frac{a_i}{a_i} v_i.$$

Here we have a rare occasion where we use that R is a field. Since $v_1, \dots, v_n$ span, we can throw in w and get a span $v_1, \dots, v_n, w$, but then we can delete v_1 and get that

$$v_2, \dots, v_n, w_1$$

is a span. Thus this reduces us to the case $v_1 = w_1$.

Now, set $V' = k \cdot w_1$ and consider $\overline{V} = V/V'$ (in other words, we look mod w_1). Since $w_1, v_2 \dots, v_n$ span V, we have $\overline{w_1}, \overline{v_2}, \dots, \overline{v_n}$ span $\overline{V}$. Since $\overline{w_1} = 0$, we can ignore it. Hence $\overline{v_2}, \dots, \overline{v_n}$ span $\overline{V}$.

Now we claim that $\overline{w_2}, \dots, \overline{w_n}$ are still linearly independent. Suppose $a_2w_2 + \cdots + a_nw_n \equiv 0 \pmod{w_1}$. (Gaitsgory is going to write this as $\overline{a_2w_2} + \cdots + \overline{a_nw_n} = \overline{0}$, but why would you do that...) By definition, there is some a with

$$aw_1 = a_2w_2 + \cdots + a_nw_n$$

holds in V. This contradicting the fact that the $\{w_i\}$ were linearly independent.

By applying the inductive hypothesis on $\overline{V}$, we get $m - 1 \le n - 1$ or $m \le n$.

§7 September 23, 2014

§7.1 Midterm Solutions

1, 5 pts Let $V \to W$ be a surjection of vector spaces with W finite-dimensional. Show that it admits a right inverse.

2, 5 pts Let

$$0 \to V_1 \to V \to V_2 \to 0$$

be a short exact sequence of vector spaces with V_2 finite-dimensional. Show that it admits a splitting.

3, 5 pts Let $T : V \to W$ be a map of finite-dimensional vector spaces. Show that V admits a basis $v_1, ..., v_k, v_{k+1}, ..., v_n$ and W admits a basis $w_1, ..., w_k, w_{k+1}, ..., w_m$ such that

$$\begin{cases} T(v_i) = w_i & \text{for } i \le k \\ T(v_i) = 0 & \text{for } i > k. \end{cases}$$

Solution 1

Take $T : V \to W$. Pick v_i such that $T(v_i) = w_i$ for each i. Set $S(w_i) = v_i$.

Solution 2

Refer to PSet3, Problem 3b.

Solution 3

Applying Problem 2, consider the short exact sequences

$$0 \to \ker T \to V \to \Im(T) \to 0$$

and

$$0 \to \operatorname{Im}(T) \to W \to \operatorname{coker}(T) \to 0.$$

Then $V = \ker T \oplus \operatorname{Im}(T)$ and $W = \operatorname{Im}(T) \oplus \operatorname{coker}(T)$. Pick a basis in $\ker(T)$, $\operatorname{Im}(T)$, $\operatorname{coker}(T)$.

§7.2 Endomorphisms

The third midterm problem suggests that maps between vector spaces are boring. In this section we discuss endomorphisms.

A linear map $T : V \to V$ is an **endomorphism** of vector spaces. Throughout this section, all vector spaces are finite-dimensional, and T is such an endomorphism on the finite-dimensional vector space V.

There are two extreme cases, a nilpotent endomorphism and an invertible endomorphism.

Definition 7.1. The map T is **nilpotent** if $T^n \equiv 0$ for some nonnegative integer n.

Gaitsgory: "Oh I forgot, it's for me to start learning names".

The following lemma is only valid in the finite-dimensional case.

Lemma 7.2

T is nilpotent if and only if for every vector v there exists an m such that $T^m(v) = 0$.

Proof. One direction is trivial. For the other direction, pick a spanning set, and take an m which makes everything on the spanning set vanish. Then all elements of T vanish. □

Definition 7.3. A subspace $V' \subseteq V$ is T-invariant if

$$v \in V' \implies T(v) \in V'.$$

We'll write $T|_{V'} : V' \to V'$ in this case. For our purposes, this notation only makes sense if V' is T-invariant.

Lemma 7.4

If T is nilpotent then $T|_{V'}$ is nilpotent.

Proof. Obvious. □

Theorem 7.5

T is nilpotent if and only if and only if V has a basis in which the matrix of T is strictly upper-triangular; i.e. there exists a basis $e_1, \dots, e_n$ such that

$$T(e_i) = \sum_{j<i} a_{ij} e_j.$$

Proof. Homework. □

Definition 7.6. The map $T : V \to V$ is **invertible** if it admits a two-sided inverse.

Lemma 7.7

If $V' \subseteq V$ is T-invariant, and T is invertible, then $T|_{V'}$ is invertible.

Proof. The map $T|_{V'} : V' \to V'$ is injective. Since it's an endomorphism, and V' is finite-dimensional, that forces injectivity (look at ranks). □

A homework problem is to show that this fails when V is not finite-dimensional.

§7.3 Given a map we can split into invertible and nilpotent parts

Theorem 7.8

Take $T : V \to V$. Then there exist $V^{\text{nilp}}, V^{\text{inv}} \subseteq V$ such that

- both are T-invariant
- $T|_{V^{\text{nilp}}}$ is nilpotent,
- $T|_{V^{\text{inv}}}$ is invertible

and

$$V^{\text{nilp}} \oplus V^{\text{inv}} \cong V.$$

Proof. Consider

$$\ker(T) \subseteq \ker(T^2) \subseteq \dots$$

The dimension of these is bounded by $\dim V$, so $\exists N$ such that

$$\ker(T^N) = \ker(T^{N+1}) = \dots.$$

We call this $\ker(T^\infty)$; this is called the **eventual kernel**.

For brevity, denote the eventual kernel by K.

Claim. K is T-invariant and $T|_K$ is nilpotent.

Proof of Claim. It is not hard to see that T maps $\ker(T^i)$ into $\ker(T^{i-1})$. The second claim is obvious. ∎

Next, consider the decreasing sequence

$$\operatorname{Im}(T) \supseteq \operatorname{Im}(T^2) \supseteq \dots$$

So there exists an N such that

$$\operatorname{Im}(T^N) = \operatorname{Im}(T^{N+1}) = \dots$$

Set $I = T^\infty(V)$ to be this **eventual image**.

Claim. I is T-invariant and $T|_{\operatorname{Im}(T^\infty)}$ is invertible.

Proof of Claim. Note that T surjects $\operatorname{Im}(T^i)$ onto $\operatorname{Im}(T^{i+1})$ Moreover, T is a surjection from $\operatorname{Im}(T^N)$ to $\operatorname{Im}(T^{N+1}) = \operatorname{Im}(T^N)$, so it is an isomorphism. ∎

We wish to show that the map

$$T : K \oplus I \to V$$

is an isomorphism.

Proof of injectivity. The map $M' \oplus M'' \to M$ is injective if and only if $m' + m'' \neq 0$ for any nonzero m', m''; hence we just want to show $K \cap I = 0$.

Let $W = I \cap K$. Then one can check that $T|_W$ is invertible since $T|_I$ is invertible. Similarly, $T|_K$ is nilpotent since $T|_K$ is nilpotent. Now $T|W$ is a map which is both invertible and nilpotent; this can only occur if $W = \{0\}$. ∎

However, for a sufficiently large positive integer N we have $K = \ker(T^N)$ and $I = \operatorname{Im}(T^N)$, and hence

$$\dim K + \dim I = \dim V.$$

So T is an injective map between two spaces of the same dimension, and hence the proof is complete. $\square$

Remark. One cannot replace the eventual image and eventual kernel with just $\operatorname{Im}(T)$ and $\ker(T)$. For a counterexample, put $V = K^2$, $T(e_1) = 0$, $T(e_2) = e_1$.

§7.4 Eigen-blah

Fix $T : V \to V$ again, with the field F underlying V.

Definition 7.9. An **eigenvector** of T in V is a vector $v \in V$ with $v \neq 0$ such that $T(v) = \lambda v$ for some $\lambda \in F$. Over all eigenvectors v, the set of λ that can be achieved are the **eigenvalues**. The **spectrum** $\operatorname{Spec}(T)$ is the set of all of the eigenvalues.

Can we bound the number of eigenvalues from below? The answer is no, other than trivial bounds.

Example 7.10

If $V = \mathbb{R}^2$ and T is rotation by 90° then $\operatorname{Spec}(T)$ is the empty set. No eigenvalues or eigenvectors exist at all.

Now, we search from bounds from above.

Proposition 7.11

For any T,

$$|\operatorname{Spec} T| \le \dim V.$$

This follows from the following proposition, which actually does not require the fact that $\dim V$ is finite.

Proposition 7.12

Let $v_1, \dots, v_m$ be eigenvectors with distinct eigenvalues. Then they are linearly independent.

Proof. Suppose

$$a_1 v_1 + a_2 v_2 + \cdots + a_m v_m = 0$$

and moreover assume m is minimal (so that $a_i \neq 0$ for any i). Actually, we can also assume $a_i = 1$ since $a_i v_i$ is also an eigenvector.

Hence we have

$$\sum v_i = 0.$$

Then

$$0 = T(v_i) = \sum \lambda_i v_i = 0.$$

Moreover,

$$\lambda_m \sum v_i = 0.$$

Subtracting, we now have that

$$\sum (\lambda_i - \lambda_m) v_i = 0.$$

But this is a smaller combination since it has only $m-1$ nonzero terms. □

§7.5 Diagonalization

Let $T : V \to V$ be a matrix map on the basis $e_1, \dots, e_n$. Observe that T is a diagonal matrix precisely when every e_i is an eigenvector.

Definition 7.13. We say T is **diagonalizable** if V has a basis of T-eigenvectors.

The goal of spectral theory is to make everything into a diagonal form.
To be continued Thursday.

§8 September 25, 2014

Recall the definition of Spec(T) last time.

§8.1 Eigenspaces

Definition 8.1. Set an eigenspace V^λ as

$$V^\lambda = \ker(T - \lambda I) = \{v \in V \mid Tv = \lambda v\}.$$

Here I is the identity matrix. Notice that $V^\lambda \neq 0 \iff \lambda \in \operatorname{Spec}(T)$.

Corollary 8.2

$\bigoplus_\lambda V^\lambda \hookrightarrow V$ is injective.

Proof. We have $(v_1, \dots, v_m) \mapsto v_1 + \dots + v_m$; WLOG the v_i are nonzero. Suppose the latter is zero. The guys have distinct eigenvalues, so they are linearly independent. □

Recall the definition of a diagonalizable matrix.

Lemma 8.3

If T is both diagonalizable and nilpotent then $T = 0$.

Proof. Just view T as a matrix, then T^k is just the kth power of each diagonal entry (and zeros elsewhere). □

Proposition 8.4

Let $T : V \to V$. Then the following are equivalent.

(a) T is diagonalizable

(b) $\exists V_\lambda \subseteq V$ (where $\lambda \in \operatorname{Spec}(T)$) which is T-invariant such that

$$T|_{V_\lambda} = \lambda \cdot \mathrm{id}_{V_\lambda}$$

and $\bigoplus V_\lambda \cong V$.

(c) $\bigoplus_{\lambda \in \operatorname{Spec}(T)} V^\lambda \to V$ is an isomorphism.

Proof. To show (a) implies (b), let V_λ be the span of all basis elements e_i of T which give the eigenvalue λ.

For (c) to (a), just take a basis of each V^λ.

For (b) to (c), observe that $V_\lambda \subset V^\lambda$. now

$$\bigoplus_\lambda V_\lambda \to \bigoplus_\lambda V^\lambda \hookrightarrow V$$

where the last arrow denotes an injection. □

§8.2 Generalized eigenspaces

Definition 8.5. For any λ in our field, set $V^{(\lambda)}$ to be the eventual kernel of $T - \lambda I$. In other words

$$V^{(\lambda)} = \ker\left((T-\lambda I)^\infty\right) = \left\{v \in V \mid \exists n : (T-\lambda I)^n v = 0\right\}.$$

This is called the **generalized eigenspace for T with eigenvalue λ.**

Observe that $V^\lambda \subseteq V^{(\lambda)}$.

Example 8.6

$V^{(0)} = V$ if and only if T is nilpotent, but $V^0 = V$ if and only if $T = 0$.

Lemma 8.7

The following are equivalent.

(a) $V^\lambda \neq 0$, meaning there exists some eigenvector with eigenvalue λ.

(b) $V^{(\lambda)} \neq 0$

(c) $T - \lambda I$ is not invertible.

Proof. Let $S = T - \lambda I$. Since we are finite-dimensional, $\ker S = 0$ and $\ker S^\infty = 0$ if and only if S is invertible. □

§8.3 Spectral Theorem

Definition 8.8. Let

$$V^{\text{non-spec}} = \bigcap_{\lambda \in \operatorname{Spec}(T)} \operatorname{Im}\left((T-\lambda I)^\infty\right).$$

We can let λ run over the entire field since if $\lambda \notin \operatorname{Spec}(T)$ implies $T - \lambda I$ is invertible, which means that the eventual image is all of V. Think of these as "invisible". That is, $V^{\text{non-spec}}$ is precisely the set of elements which are not killed by any $T - \lambda I$. In other words, the component $V^{\text{non-spec}}$ has *empty* spectrum.

Remark. It's worth noting at this point that $V^{\text{non-spec}} = 0$ in the case that V is algebraically closed. For then we can necessarily find a λ which is a root of $\det(T - \lambda I)$, forcing it to be non-invertible. (Just expand the determinant as a polynomial).

Theorem 8.9 (Spectral Theorem)

For any $T : V \to V$

$$\left(\bigoplus_{\lambda \in \operatorname{Spec}(T)} V^{(\lambda)}\right) \oplus V^{\text{non-spec}} \to V$$

is an isomorphism.

Next week we will show $V^{\text{non-spec}} = 0$ given some assumptions on V.

Here are some corollaries.

Proposition 8.10

The vector subspaces

$$V^\lambda, V^{(\lambda)}, \operatorname{Im}\left((T-\lambda I)^\infty\right)$$

are T-invariant.

Proof. Clear for V^λ, so focus on the latter two.

$$\begin{tikzcd} V_1 \arrow[r, "S"] \arrow[d, "T_1"] & V_2 \arrow[d, "T_2"] \\ V_1 \arrow[r, "S"] & V_2 \end{tikzcd}$$

In the PSet problem, S intertwines T_1 and T_2. This implies that S maps $\ker(T_1^\infty)$ into $\ker(T_2^\infty)$. Similarly for the eventual image.

Suppose we only have one space V and two maps $T, S : V \to V$ with $T \circ S = S \circ T$. Then S maps $\ker(T^\infty)$ and $\operatorname{Im}(T^\infty)$ to itself; i.e. this are S-invariant. Similarly for S and T.

Now let $S = T - \lambda I$. We claim that $TS = ST$. Check this. Now apply the above. Then we get $V^{(\lambda)}$ and $\operatorname{Im}((T-\lambda I)^\infty)$. $\square$

Corollary 8.11

If $v_1, \dots, v_n$ with $v_i \in V^{(\lambda_i)}$ and suppose all the λ_i are distinct. then $v_1, \dots, v_n$ are linearly independent.

In other words, generalized eigenvectors with distinct eigenvalues are linearly independent.

Proof. It's equivalent to $\bigoplus_\lambda V^{(\lambda)} \hookrightarrow V$ is an injection, which follows from the spectral theorem. $\square$

§8.4 Lemmata in building our proof

Definition 8.12. Let V be a vector space decomposed as a direct sum $V_1 \oplus V_2$. We shall say that a vector subspace $V' \subset V$ is **compatible with the given direct sum decomposition** if the map

$$(V' \cap V_1) \oplus (V' \cap V_2) \to V'$$

is an isomorphism.

Proposition 8.13 (PSet 4.5)

If $V' \subseteq V$ is T-invariant then it's compatible with the splitting of T into the eventual image and eventual kernel.

We combine these to prove the following lemma.

Lemma 8.14

Suppose $S \circ T = T \circ S$, then

$$\begin{gathered}(\ker(T^\infty) \cap \ker(S^\infty)) \oplus (\operatorname{Im}(T^\infty) \cap \ker(S^\infty)) \\ \oplus (\ker(T^\infty) \cap \operatorname{Im}(S^\infty)) \oplus (\operatorname{Im}(T^\infty) \cap \operatorname{Im}(S^\infty)) \\ \to V\end{gathered}$$

is an isomorphism.

Proof. First write

$$V \cong \ker(T^\infty) \oplus \operatorname{Im}(T^\infty).$$

Since $\ker(T^\infty)$ is S-invariant we note that

$$\ker(T^\infty) = \ker(T^\infty) \cap \ker(S^\infty) \oplus \ker(T^\infty) \cap \operatorname{Im}(S^\infty).$$

Similarly for the other guy. □

We like to now generalize this to n spaces.

Lemma 8.15

Let $T_1, \dots, T_n$ be pairwise commuting maps. Consider the set of all 2^n functions $\alpha : \{1, 2, \dots, n\} \to \{\mathrm{I}, \mathrm{K}\}$. Then

$$\bigoplus_\alpha V^{\alpha(1)} \cap \dots \cap V^{\alpha(n)}$$

is isomorphic to V, where $V^{\alpha(i)}$ is $\ker(T_i^\infty)$ or $\operatorname{Im}(T_i^\infty)$ depending on the $\alpha(i)$.

Proof. Induct on n. □

Lemma 8.16

Let $T_1, \dots, T_k$ be endomorphisms of V pairwise commuting, but assume that

$$\ker(T_i^\infty) \cap \ker(T_j^\infty) = 0$$

for all $i \neq j$. Then

$$\bigoplus_{i=1}^k \ker(T_i^\infty) \bigoplus \bigcap_{i=1}^k \operatorname{Im}(T_i^\infty) \cong V.$$

Proof. In the preceding lemma, any term which has $\alpha(i) = \alpha(j) = \mathrm{K}$ with $i \neq j$ immediately disappear by our hypothesis. So the only remaining terms are $\bigcap_{i=1}^k \operatorname{Im}(T_i^\infty)$ and

$$\ker\left(T_i^\infty\right) \bigcap_{j \neq i} \operatorname{Im}(T_j^\infty).$$

So it suffices to prove that for any $i \neq j$, we have

$$\ker(T_i^\infty) \cap \operatorname{Im}(T_j^\infty) \cong \ker(T_i^\infty).$$

Since $\ker(T_i^\infty)$ is a T_j-invariant subspace, we know that

$$\ker(T_i^\infty) \cong \left(\ker(T_i^\infty) \cap \operatorname{Im}(T_j^\infty)\right) \cap \left(\ker(T_i^\infty) \cap \ker(T_j^\infty)\right)$$

but the right term is zero. $\square$

§8.5 Proof of spectral theorem

We want to apply the previous lemma. Set $T_i = T - \lambda_i I$.

It is not hard to check that these pairwise commute. We'd like to apply the previous lemma (as it implies the Spectral Theorem), but we need to verify that pairs of kernels have trivial intersection; that is

$$V^{(\lambda)} \cap V^{(\mu)} = 0$$

for all $\lambda \neq \mu$. To see this, look at the operator $T - \mu I$ restricted to $V^{(\lambda)} \cap V^{(\mu)}$. Since it's nilpotent over $V^{(\mu)}$, it's nilpotent over the intersection. But

$$T - \mu I = (T - \lambda I) + (\lambda - \mu) I.$$

For this we just require the following lemma.

Lemma 8.17

Let $S : W \to W$ be nilpotent. Then for any $\nu \neq 0$, the map $S - \nu I$ is invertible.

Proof. If not, then $S - \nu I$ has a kernel, so $\exists w \in W$ such that $Sw = \lambda w$. But S is nilpotent, which is impossible since then $S^n w = \lambda^n w$. $\square$

Proof 2. Alternatively, suppose $S^N = 0$. It's equivalent to prove that $I - S$ is invertible, but the inverse explicitly is

$$(I - S)^{-1} = I + S + \cdots + S^{N-1}.$$

$\square$

Now we're done.

§8.6 Recap of Proof

We considered $T - \lambda_i I$. We looked at eventual kernels / images, showing that they're all compatible, and constructed a map. There are 2^n terms, but it turns out only $n+1$ survive, and we got the Spectral Theorem.

§9 September 30, 2014

Adjusted office hours: this week is Wed 3-4PM and next week is Monday 3-4PM (instead of Monday).

We will now carry on with spectral theory.

§9.1 Review

We have $T : V \to V$. Last time we showed that we had a decomposition

$$V = \bigoplus_\lambda V^{(\lambda)} \oplus V^{\text{non-spec}}.$$

See the last lecture for their definitions. Note that $T|_{V^{\text{non-spec}}}$ has an empty spectrum.

Lemma 9.1

Suppose we have the following commutative diagram.

$$\begin{tikzcd} V_1 \arrow[r, "T_1"] \arrow[d, "S"'] & V_1 \arrow[d, "S"] \\ V_2 \arrow[r, "T_2"'] & V_2 \end{tikzcd}$$

In that case S maps $V_1^{(\lambda)}$ to $V_2^{(\lambda)}$for each λ and $V_1^{\text{non-spec}}$ and $V_2^{\text{non-spec}}$.

Proof. The map S intertwines $(T_1 - \lambda I)^N$ with $(T_2 - \lambda I)^N$. Hence it sends their kernels and images to each other. This solves 9 and 10(a), which probably means I got them wrong. $\square$

§9.2 Taking polynomials of an endomorphism

Let V be a vector field over k. The set of $T : V \to V$ is denoted $\text{End}(V)$, and is a k-algebra. We can define a homomrphism of k-homomorphisms from $k[t] \to \text{End}(V)$ by mapping t to any particular endomorphism T.

$$\begin{tikzcd} k[t] \arrow[rr] & & \text{End}(V) \\ & k \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Evidently $\ker(\varphi)$ is an ideal in $k[t]$. Recall that in a polynomial ring over a field, any algebra is generated by a single monic polynomial

$$I = (p)$$

Now consider the following definition.

Definition 9.2. We have $\varphi : k[t] \to \text{End}(V)$, and φ sends t to T. Now suppose $\ker \varphi = (\min_T)$. We call $\min_T$ the **minimal polynomial** of T.

Now if $\ker\varphi = 0$, then $\min_T$ is zero. But we claim this never happens.

Remark 9.3. This is silly. All we're doing is look at "polynomials" in T, i.e. things like

$$T^3 + 2T^2 + 99T - 4$$

as an endomorphism in $\mathrm{End}(V)$. The stuff about φ is just unnecessary obfuscation.

Lemma 9.4

For any $\varphi : k[t] \to \mathrm{End}(V)$ a k-algebra homomorphism, $\ker\varphi \neq 0$.

Proof. The point is to show that φ cannot be injective. Indeed, note that $\mathrm{End}(V)$ is the set of $n \times n$ matrices whose entries are in k, so it has dimension n^2. But $k[t]$ has "infinite" dimension, and we can take, say, the polynomials of degree at most $n^2 + 1$ only; already the map cannot be injective. $\square$

Lemma 9.5

$\min_T = t^n$ if and only if T is nilpotent.

Proof. If $t^n = 0$ then $T^n = 0$. Conversely, if $T^N = 0$ then $t^n = 0$ and $\min_T \mid t^n$. $\square$

§9.3 Minimal Polynomials

Lemma 9.6

Consider $T : V \to V$. Suppose we have a T-invariant subspace V'. Let $T' : V' \to V'$ and $T'' : V/V' \to V/V'$ in the canonical manner.

(a) $\min_{T'}$ divides $\min_T$ and $\min_{T''}$ divides $\min_T$.

(b) $\min_T$ divides $\min_{T'} \cdot \min_{T''}$.

Proof. In this proof you should really ignore the φ's because they actually suck and just make it more confusing. Basically, $\min_{T'}$ divides $\min_T$ is equivalent to

$$(\min_{T'}) \subseteq (\min_T)$$

(these are ideals); in other words any polynomial p which makes T vanish also makes T' vanish. Of course this is obvious. The same goes for $\min_{T''}$, completing (a).

For (b) this is just saying that if p makes both T' and T'' vanish then p also makes T vanish. $\square$

Hi Luran.

Lemma 9.7

Given $T : V \to V$ over the field k, and consider $\lambda \in k$. Then $\lambda \in \mathrm{Spec}(T)$ if and only if λ is a root of $\min_T$.

Proof. First suppose $\lambda \in \operatorname{Spec}(T)$. Let $V' = \ker(T - \lambda I) \neq 0$ and let $T' : V' \to V'$ be the restriction to V' (observing that V' is T-invariant). Observe that $\min_{T'} = t - \lambda$ and $\min_{T'} \mid \min_T$.

Conversely, suppose λ is a root of $\min_T$. Then $(t - \lambda)$ divides $\min_T$. If $\lambda \notin \operatorname{Spec}(T)$ now, then $T - \lambda I$ is invertible, and this yields a contradiction to the minimality of $\min_T$. □

§9.4 Spectral Projector

Theorem 9.8

For all $\lambda \in \operatorname{Spec}(T)$ there exists $p_\lambda \in k[t]$ such that $p_\lambda(T)$ is the identity when restricted to $V^{(\lambda)}$ and is the zero polynomial on $\bigoplus_{\mu \neq \lambda} V^{(\mu)} \oplus V^{\text{non-spec}}$, aka "everything else".

Definition 9.9. The p_λ is called the **spectral projector**.

Proof. Let stuff $= \bigoplus_{\mu \neq \lambda} V^{(\mu)} \oplus V^{\text{non-spec}}$.

Let $\min_T = (t - \lambda)^n \cdot f$ with n maximal. Bezout's lemma states we may find $r, q \in k[t]$ such that

$$r(t - \lambda)^n + qf = 1.$$

We claim that $p_\lambda = qf$ works.

We need to show that

$$r(T)(T - \lambda)^n \equiv 0$$

is the zero operator for over $V^{(\lambda)}$ (since in that case $qf = 1$), and we also want

$$q(T)f(T) \equiv 0$$

when restricted to stuff.

To prove the first statement, let $V' = V^{(\lambda)}$ be a subspace and now $\min_{T'}$ divides $(t - \lambda(^n \cdot f$. But $\min_{T'} = (t - \lambda)^m$ for some m, and $m \leq n$, so $(T - \lambda)^n$ kills $V^{(\lambda)}$.

For the second statement, we wish to show that $f(T)$ kills stuff. But $\min_T = (T - \lambda)^n f(T)$ kills everything. So we just have to show $(T - \lambda)^n$ is invertible over all the $V^{(\mu)}$ and $V^{\text{non-spec}}$. We can replace $(T - \lambda)^n$ with $T - \lambda$. It's invertible over $V^{(\mu)}$ by Lemma 8.17 and it's invertible over $V^{\text{non-spec}}$ since T has empty spectrum over it. □

§9.5 Polynomials

Proposition 9.10

The following are equivalent over any field k.

1. Every polynomial factors into linear factors.
2. Every polynomial has a root.
3. Every polynomial irreducible polynomial is linear.

Proof. Clear. □

Definition 9.11. We say that a field k is **algebraically closed** if the above equivalent conditions hold.

Example 9.12

The complex numbers $\mathbb{C}$ are algebraically closed. Actually, you can show that any field k you can embed it into an algebraically closed field $\overline{k}$.

Proposition 9.13

Let k be an algebraically closed field. If $V \neq 0$ then $\operatorname{Spec}(T) \neq 0$ for any operator T.

Proof. Note that $\min_T$ has a root. □

Corollary 9.14

If k is algebraically closed, $V = \bigoplus_\lambda V^{(\lambda)}$.

Proof. We have $V^{\text{non-spec}} = 0$. □

Proposition 9.15

Let k be non-algebraically closed. Then there exists a V and $T : V \to V$ such that $\operatorname{Spec}(T)$ is empty.

Proof. Let p be an irreducible polynomial of degree $n > 1$, and set $V = k[t]/(p)$. You can check that this is indeed a field; it has a basis $t^0, t^1, \dots, t^{n-1}$. Now let $T : V \to V$ by $q \mapsto tq$ (all mod p). We claim it has empty spectrum.

Consider any $\lambda \in k$. Then $T - \lambda$, which is the map $q \mapsto (t - \lambda)q$. The map $t - \lambda$ is nonzero since $n > 1$, and it is invertible since we're in a field. □

§10 October 2, 2014

This lecture will discuss Jordan canonical form.

§10.1 Jordan Canonical Form

Let $T : V \to V$ be nilpotent.

Definition 10.1. The map T is **regular** if there exists a basis $e_1, \dots, e_n$ such that $T(e_i) = e_{i-1}$ for each $i \ge 2$ but $T(e_1) = 0$.

Example 10.2

This is a regular 4×4 matrix.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The goal of the lecture is to prove the following theorem.

Theorem 10.3

Given nilpotent $T : V \to V$, it is possible to write

$$V = \bigoplus_i V_i$$

such that for any i, V_i is T-invariant and T restricted to V_i is regular.

Moreover, the multiset

$$\{\dim V_1, \dim V_2, \dots\}$$

is unique.

In terms of the matrix, this looks like

$$\left(\begin{array}{ccc|cccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

§10.2 A big proposition

Proposition 10.4
Let V be n-dimensional and T nilpotent. The following are equivalent.

(i) The map T is regular.

(ii) There exists $v \in V$ such that v, Tv, T^2v, $\dots$ form a basis.

(iii) The map T^{n-1} is nonzero.

(iv) For each $1 \le i \le n$, then $\dim \ker(T^i) = i$.

(v) $\dim \ker(T) = 1$.

Proof. Go in a line.

(i) $\implies$ (ii) Pick $v = e_1$.

(ii) $\implies$ (iii) Otherwise $T^{n-1}(v) = 0$ is in a basis.

(iii) $\implies$ (iv) Apply problem 1 from PSet 4, which states that

$$\dim(\ker(T^{i+1})) - \dim(\ker(T^i)) \le \dim(\ker(T^i)) - \dim(\ker(T^{i-1})).$$

(iv) $\implies$ (v) Set $i = 1$.

(v) $\implies$ (i) Use PSet 4, Problem 1. □

Corollary 10.5
Let T be regular. The inclusion

$$\operatorname{im} T^i \subseteq \ker T^{n-i}$$

is an equality.

Proof. The inclusion follows from the fact that $T^{n-i} \circ T^i \equiv 0$. Now just compare the dimensions; both are $n - i$. □

It's worth pointing out the following consequence explicitly.

Lemma 10.6
If S is regular on W and $\dim W = m$ then

$$\dim \ker S^i = \begin{cases} i & \text{if } i \le m \\ m & \text{if } i \ge m. \end{cases}$$

§10.3 Young Diagrams

Consider a Jordan decomposition $V = \bigoplus_i V_i$. Let $n_i = \dim V_i$, and assume that

$$n_1 \ge n_2 \ge \cdots \ge n_k.$$

Moreover, $n = n_1 + \cdots + n_k$.

Then we look at Young diagrams[4]. For a partition $\mathfrak{p}$, let $(\mathfrak{p})^\checkmark$ denote the partition corresponding to flipping the Young diagram. That is, if $n = n_1 + \cdots + n_k$ is a partition $\mathfrak{p}$, then $\mathfrak{p}^\checkmark$ is

$$n = n_1^\checkmark + \cdots + n_\ell^\checkmark$$

where $n_i^\checkmark$ is the number of blocks in the ith row.

This proposition will imply the uniqueness.

Proposition 10.7

Let $V = \bigoplus_i V_i$ and let $n_i = \dim V_i$. Define $n_i^\checkmark$ as above. Then

$$n_i^\checkmark = \dim \ker T^i - \dim \ker T^{i-1}.$$

Proof. It's equivalent to show that the volume of the first i rows of the Young diagram is equal to $\dim \ker T^i$.

The value of the first i rows is

$$\sum_j \begin{cases} i & n_j \ge i \\ n_j & n_j \le i. \end{cases}$$

Moreover[5] we have

$$\ker T^i = \bigoplus_j \ker \left(T^i|_{V_j} \right).$$

Thus

$$\dim \ker T^i = \sum_j \dim \ker T^i|_{V_j}.$$

□

Remark 10.8. This is a refinement of "looking at the orders of T to distinguish". The pattern of the n_i that we have is dictated by the integers which cause the T^i to vanish on a V_i.

§10.4 Proof of Existence

We now prove existence of the Jordan decomposition. We go by induction on $\dim V$.

Let n be the minimal integer such that $T^n = 0$, so that $T^{n-1} \neq 0$. There exists $v \in V$ such that $T^{n-1}(v) \neq 0$.

Define

$$V' = \operatorname{Span}\left(v, Tv, T^2v, \dots, T^{n-1}v \right).$$

Evidently V' is T-invariant (cue laughter) and $T|_{V'}$ is regular. Moreover, $V'' = V/V'$ has lesser dimension.

Consider the short exact sequence

$$0 \to V' \to V \to V'' \to 0$$

By induction hypothesis ($\dim V'' < \dim V$), we may write

$$V'' = \bigoplus_i V_i''$$

[4] These apparently will show up again at the end of the semester.

[5] In general, given $T : V_1 \oplus V_2 \to W$ we have $\ker T = \ker T_{V_1} \oplus \ker T_{V_2}$.

where V_i'' is T'' invariant (where T'' is the induced operator).

Let's consider the commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & V' \arrow[r, "i"] \arrow[d, "T'"] & V \arrow[r, "\pi"] \arrow[d, "T"] & V'' \arrow[r] \arrow[d, "T''"] & 0 \\
0 \arrow[r] & V' \arrow[r, "i"] & V \arrow[r, "\pi"] & V'' \arrow[r] & 0
\end{tikzcd}$$

Needles to say p is a projection. We'll now attempt to construct a right inverse $j : V'' \to V$ of π so that

$$T \circ j = j \circ T''.$$

If we can construct j we are done since we can isomorph

$$V' \oplus V'' \xrightarrow{(i,j)} V.$$

The property that $T \circ j = j \circ T''$ is necessary!

To create a map $j : \bigoplus V'' \to V$, recall that to map out of a direct sum is to map out of each summand. Set $V_i = p^{\text{pre}}(V_i'')$ for each i. For each i we have a short exact sequence

$$0 \to V' \to V \xrightarrow{p_i} V_i'' \to 0.$$

So we can focus on splitting p_i, and **we will omit the indices i until further notice.** Thus there exists $w \in V''$ such that

$$w, (T'')w, (T'')^2 w, \dots, (T'')^{m-1} w$$

is a basis.

Say that we managed to select $j(w) = \tilde{w} \in V$. By construction,

$$p(\tilde{w}) = w.$$

Then,

$$\begin{aligned}
j(T''(\tilde{w})) &= T(\tilde{w}) \\
j((T'')^2(\tilde{w})) &= T^2(\tilde{w}) \\
&\vdots
\end{aligned}$$

Hence this lets us construct the entire function j which does the right thing w, $T''w$, up to $(T'')^{m-2}(w)$. However, we still need to check that it does the right thing for $T^{m-1}(w)$, meaning we still need to verify

$$T \circ j(T^{m-1}(w)) = j \circ T'' \left((T'')^{m-1}(w)\right) = 0.$$

So really, we need to find a $\tilde{w}$ such that $p(\tilde{w}) = w$ and $T^m \tilde{w} = 0$.

Choose $\tilde{\tilde{w}}$ such that only the first condition $p(\tilde{\tilde{w}}) = w$. We want to show that we can set $\tilde{w} = \tilde{\tilde{w}} - u$ for some "correction" term $u \in V'$; note that the projection sends u to zero, so we just need

$$T^m \left(\tilde{\tilde{w}}\right) = T^m(u).$$

Thus we want $T^m \left(\tilde{\tilde{w}}\right) \in \operatorname{im}(T')^m$. But

$$\operatorname{im}(T')^m = \ker(T')^{n-m}$$

because we're in a regular situation. But

$$(T')^{n-m} \left(T^m \tilde{\tilde{w}}\right) = 0$$

follows from $T^n \equiv 0$.

§11 October 7, 2014

Oops we messed up the answer to problem 8 on the last PSet.

I guess T^m vanishes completely too.

§11.1 Order of a Group

Definition 11.1. Let G be a group and g an element of it. We define $\operatorname{ord}_G(g) = \operatorname{ord} g$, the **order** of g, to be the smallest integer $n \geq 1$ with $g^n = 1$, or ∞ if no such element exists.

Lemma 11.2

If $g^m = 1$ then $\operatorname{ord} g \mid m$.

Proof. Clear. □

Note that if $n = \operatorname{ord} g$ then there is a natural injection from $\mathbb{Z}/n\mathbb{Z}$ to G by $1 \mapsto g$.

Definition 11.3. For a finite group G, we say the **order** of G is the number of elements. We write this as $|G|$.

Lemma 11.4

Given a subgroup H of G, the order of H divides the order of G.

Corollary 11.5

Given $g \in G$, $\operatorname{ord} g \mid |G|$.

Proof. The set $\left\{1, g, \dots, g^{n-1}\right\}$ is a subgroup of G, where $n = \operatorname{ord} g$. □

§11.2 Groups of prime powers

Lemma 11.6

$\mathbb{Z}/p\mathbb{Z}$ has no nontrivial subgroups (i.e. other than the trivial group and itself).

Proof. Let H be a subgroup. By our lemma $|H|$ divides p, so either $|H| = 1$ or $|H| = p$. □

Definition 11.7. Let p be a prime. We say that G is a p-group if $|G|$ is a prime power.

Notice that in a p-group the order of g is also a power of p. It turns out that, conversely, if all orders are powers of p then G is a p-group. This will be shown on Thursday.

However, for now we focus on finite abelian groups. Write it additively.

Proposition 11.8

Let A be a finite abelian group.

(1) A is a p-group.

(2) Each element is a power of p.

(3) Multiplication by p (here $p \cdot a = a + \cdots + a$) is nilpotent. (This makes G into a $\mathbb{Z}$-module.)

(4) There exists
$$0 = A_0 \le A_1 \le \cdots \le A_n \le A$$
for which $A_i/A_{i-1} = \mathbb{Z}/p\mathbb{Z}$.

Proof. Go in a line again.

Proof that (i) $\implies$ (ii) The order divides $|A| = p^n$.

Proof that (ii) $\implies$ (iii) Multiplication by p^n is the zero map.

Proof that (iii) $\implies$ (iv) Suppose $|A| = p^n$. We go by induction on n. Take any element g, and suppose it has order $p^m \neq 1$ Then $g' = p^{m-1} \cdot g$ has order p. Now take modulo the subgroup $\langle g' \rangle$. We can decompose $A/\langle g' \rangle$, yay.

Proof that (iv) $\implies$ (i) Check that A_i has order p^i.

Here is a second proof that (iii) $\implies$ (iv). Let $B_i = \ker p^i$. Observe that we have a flag $0 = B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots \subseteq B_n = A$. Now all elements in B_i/B_{i-1} have order p. We can then decompose it into further by $0 = C_0 \subseteq C_1 \subseteq \cdots \subseteq C_t = B_i/B_{i-1}$ for each i; combining these gives a good breakdown of A. □

§11.3 Abelian groups and vector spaces are similar

Let A be a finite abelian group written additively. Let $T : A \to A$ be an endomorphism.

Proposition 11.9

We have $A = \ker(T^\infty) \oplus \operatorname{im}(T^\infty)$.

Proof. We have $A = \ker(T^n) \oplus \operatorname{im}(T^n)$ by first isomorphism theorem. By order, things must stabilize. GG. □

There are actually lots of analogies between abelian groups and vector spaces over algebraically closed fields. Orders correspond to ranks; $\dim \ker T + \dim \operatorname{im} T = \dim V$ corresponds to $|G| = |\ker T| \, |\operatorname{im} T|$.

We exhibit these in the following table.

A	$V \xrightarrow{T} V$
primes	k (algebraically closed)
$\mathbb{Z}$	$k[t]$
$\mathbb{Z}/p\mathbb{Z}$	$\dim V = 1$
p-group	$V^{(\lambda)} = V$
primes dividing $\lvert A \rvert$	$\operatorname{Spec}(T)$
$\sum_p \nu_p(\lvert A \rvert)$	$\dim V$
$\lvert A \rvert \in \mathbb{Z}$	$\mathrm{ch}_T \in k[t]$
maximal order in A	$\min_T \in k[t]$
$\mathbb{Z}/p^n\mathbb{Z}$	$T - \lambda$ is regular and nilpotent
$\mathbb{Z}/n\mathbb{Z}^6$	T is regular
$\bigoplus_p \mathbb{Z}/p\mathbb{Z}$	diagonalizable

In here we used the following.

Lemma 11.10

If $T : V \to V$ and V has no nontrivial T-invariant subspaces, then $\dim V = 1$. Here V is a vector space over an algebraically closed field.

Proof. Take any eigenvector v (possible since k is algebraically closed). □

Why does $\dim V$ correspond to $\sum_p \nu_p(|G|)$? The idea is that "length" corresponds to the length of a flag

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$$

where each quotient is irreducible. Then $n = \dim V$.

§11.4 Chinese Remainder Theorem

Here is the analog of the spectral theorem.

Theorem 11.11

For any finite abelian group A, we have $A = \bigoplus_p A^{(p)}$, where each $A^{(p)}$ is a p-group. Actually, $A^{(p)}$ is the eventual kernel of the map $a \mapsto p \cdot a$.

Definition 11.12. A p-group is **regular** if it is the cyclic group of order p^n.

Proposition 11.13

Let A be a p-group with size p^n. The following are equivalent.

(i) A is regular.

(ii) There exists $a \in A$ which generates A.

(iii) $A^{p^{n-1}} \neq A$.

(iv) $\left| A^{p^i} \right| = p^i$.

(v) $|A^p| = p$.

Here A^n denotes the kernel of the map $a \mapsto n \cdot a$. Finally, we have a "Young diagram".

Theorem 11.14

Let A be a p-group. Then we may write

$$A = \bigoplus_n \mathbb{Z}/p^n\mathbb{Z}.$$

Moreover, the multiset of orders of this group is uniquely determined by A.

§11.5 Not algebraically closed

What occurs if k is not algebraically closed? The point is that the characteristic polynomial of $p \in k[t]$ is not reducible iff the vector space V has no nontrivial T-invariant subspaces. Note that $k[t]/(p)$ is a field.

Hence we've replaced "eigenvalues" with "monic irreducible polynomials". It just so happened that when k is algebraically closed, the "monic irreducible polynomials" were just $t - \lambda$, i.e. the eigenvectors.

A	$V \xrightarrow{T} V$
primes	irreducible polynomials is $k[t]$
$\mathbb{Z}$	$k[t]$
$\mathbb{Z}/p\mathbb{Z}$	$V = k[t]/(p)$; $p = \mathrm{ch}_T$ irreducible
p-group	$p(T)$ is nilpotent
primes dividing $\lvert A\rvert$	factorization of ch_T
$\sum_p \nu_p(\lvert A\rvert)$	$\sum n_i$ where $\mathrm{ch}_T = \prod_i p_i^{n_i}$
$\lvert A\rvert \in \mathbb{Z}$	$\mathrm{ch}_T \in k[t]$
maximal order in A	$\min_T \in k[t]$
$\mathbb{Z}/p^n\mathbb{Z}$	$k[t]/(p^n)$

We actually lied about $V^{\text{non-spec}}$. When we wrote

$$V = \bigoplus_\lambda V^{(\lambda)} \oplus V^{\text{non-spec}}$$

it turns out that $V^{\text{non-spec}}$ breaks down based on irreducible polynomials. All we did was lump together all the guys that were not $t - \lambda$ into one big "junk" pile $V^{\text{non-spec}}$. (And that's why I couldn't find anything on it...)

§12 October 9, 2014

Today's course was taught by the CA's. Gaitsgory was not present.

§12.1 Group Actions

Let G be a group and X a set.

Definition 12.1. A set X is a G-set if there exists $\cdot : G \times X \to X$ such that

1. $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ for all $g_1, g_2, \in G$, $x \in X$.
2. $1 \cdot x = x$ for all $x \in X$.

If X is a G-set then we say that G **acts** on X. We write $G \curvearrowright X$.

Note: the actual lecture used the notation $m(g, x)$ for $g \cdot x$. I think the former notation sucks and will mostly use the latter.

Lemma 12.2

Let X be a G-set. Then the map $X \to X$ by $x \mapsto gx$ is an automorphism.

Proof. Given $g \cdot x_1 = g \cdot x_2$, we find $x_1 = g^{-1} \cdot g \cdot x_1 = g^{-1} \cdot g \cdot x_2 = x_2$. Surjectivity follows from $g \cdot (g^{-1} \cdot x) = x$. □

§12.2 How do G-sets talk to each other?

Definition 12.3. Let X_1 and X_2 be G-sets. A map $\phi : X_1 \to X_2$ is a map of G-sets if $\phi(g \cdot x_1) = g \cdot \phi(x_1)$ holds. We let $\operatorname{Hom}_G(X_1, X_2)$ denote the set of all such maps.

It's equivalent to this digram commuting.

$$\begin{tikzcd} G \times X_1 \arrow[r, "\mathrm{act}_1"] \arrow[d, "\mathrm{id} \times \phi"'] & X_1 \arrow[d, "\phi"] \\ G \times X_2 \arrow[r, "\mathrm{act}_2"'] & X_2 \end{tikzcd}$$

Lemma 12.4

Let ϕ be a G-set map that is also a bijection. Then ϕ^{-1} is also a G-set map.

Proof. Just do it. □

§12.3 Common group actions

Definition 12.5. Suppose $G \curvearrowright X$. Define

$$X^G = \{x \in X : gx = x \forall g \in G\}.$$

If $x \in X^G$ we say X is a G-invariant, and X^G are called the G-invariants.

Example 12.6

Let pt $= \{*\}$ be a set consisting of a single element. Then any group G has exactly one G-action on pt, namely the trivial action.

Let X be a G-set and consider

$$\varphi : \mathrm{Hom}_G(\mathrm{pt}, X) \to X$$

by

$$(f : \mathrm{pt} \to X) \mapsto f(*).$$

Then $\operatorname{im} \phi = X^G$, and this is an bijection.

Example 12.7

Take any group G. Then there exist a canonical action $G \curvearrowright G$ called "left-regular" which is just left-multiplication.

Let X be a G-set. Then we have an isomorphism

$$\phi : \mathrm{Hom}_G(G, X) \to X$$

by

$$(f : G \to X) \mapsto f(1).$$

Example 12.8

Take any group G. There is a canonical action $G \curvearrowright G$ called "right-regular" which by

$$g \cdot x = xg^{-1}.$$

Remark. Note that for G not abelian, the map $g \cdot x \mapsto xg$ is in fact not a group action.

Example 12.9

Combining the previous two examples, we have an action $G \times G \curvearrowright G$ by

$$(g_1, g_2) \cdot g \mapsto g_1 g g_2^{-1}.$$

Example 12.10

Suppose $G \curvearrowright X$ and $\varphi : H \to G$ is a group homomorphism. Then $H \curvearrowright X$ by

$$h \cdot x = \varphi(h) \cdot x.$$

Example 12.11

The **adjoint action** (which I called **conjugation** when I grew up) is the action $G \curvearrowright G$ given by

$$g \cdot x = gxg^{-1}.$$

Example 12.12
Let X be a set, and define $\text{Aut}_{\text{Set}}(X)$ to be the set of automorphisms (for sets, just bijections) on X. You can check this is a group. This gives a tautological action $\text{Aut}_{\text{Set}}(X) \curvearrowright X$. my $\sigma \cdot x = \sigma(x)$.

Definition 12.13. For a positive integer n let

$$S_n \overset{\text{def}}{=} \text{Aut}_{\text{set}}(\{1, 2, \dots, n\}).$$

We call this the **symmetric group** on n letters.

Example 12.14
Let V be a vector space over some field k. Define $GL(V) = \text{Aut}_{k\text{-linear}}(V)$. Then we have an obvious action $GL(V) \curvearrowright V$.

Example 12.15
Let V a be a field. Let $Fl(V)$ denote the set of complete flags in V; i.e. a flag

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V$$

where $n = \dim V$, meaning that $\dim(V_i/V_{i-1}) = 1$ for all i. Observe that if $T \in GL(V)$ then we have a standard action

$$GL(V) \curvearrowright Fl(V).$$

§12.4 More group actions

Let G be a group and H a subgroup. We define $G/H = \{gH \mid g \in G\}$ (cosets, so G/H is not necessarily a group). Let $\pi : G \to G/H$ be the natural projection.

We have a natural group action $G \curvearrowright G/H$ by

$$g \cdot (g'H) = gg'H.$$

Let X be a G-set. We define a map

$$\phi : \text{Hom}_G(G/H, X) \to X$$

by

$$(f : G/H \to X) = f(1H).$$

Lemma 12.16
$\text{im}(\phi) \subset X^H$, and $\phi : \text{Hom}_G(G/H, X) \mapsto \text{im}\,\phi$ is a bijection.

Definition 12.17. Suppose $G \curvearrowright X$. For $x \in X$, we define the stabilizer

$$\text{Stab}_G(x) = \{g \in G : gx = x\}.$$

Example 12.18

Recall the adjoint action $G \curvearrowright G$. Then

$$\operatorname{Stab}_G(g') = Z(g') \stackrel{\text{def}}{=} \left\{z \in G : zg' = g'z\right\}.$$

Example 12.19

Let $S_n \curvearrowright \{1, \dots, n\}$. Prove that

$$\operatorname{Stab}_{S_n}(a) \cong S_{n-1}.$$

Lemma 12.20

Let X be a G-set and let H be a subgroup. If $H \subseteq \operatorname{Stab}_G(x)$, for some $x \in X$ then there exists a unique $\phi : G/H \to X$ such that

$$\phi(1H) = x.$$

Moreover, ϕ is injective if and only if $H = \operatorname{Stab}_G(x)$.

§12.5 Transitive actions

Definition 12.21. Suppose $G \curvearrowright X$. We say the action is **transitive** if

$$\exists g \in G : gx = y \quad \forall x, y \in X.$$

We say it is **simple** if

$$\operatorname{Stab}_G(x) = \{1\} \quad \forall x \in X.$$

We say it is **simply transitive** both conditions hold.

Lemma 12.22

Let $\phi : X_1 \to X_2$ be a map of nonempty G-sets. Suppose $G \curvearrowright X_2$ is transitive. The ϕ is surjective.

Proof. Clear. $\square$

Proposition 12.23

Let G act transitively on X, and take $x \in X$. Then there exists a unique isomorphism of G-sets

$$\phi : G/\operatorname{Stab}_G(X) \to X$$

with $\phi(1) = x$. Here $G/\operatorname{Stab}_G(x)$ is being interpreted as a G-set by left action.

Proof. The only possible map is $g{\cdot}\operatorname{Stab}_G(x) \mapsto gx$. evidently $\phi(g \operatorname{Stab}_G(x)) = \phi(g' \operatorname{Stab}_G(x)) \implies gx = g^{-1}x \implies gg^{-1} \in \operatorname{Stab}_G(X)$. Surjectivity is evident by transitivity. $\square$

§12.6 Orbits

Definition 12.24. Suppose $G \curvearrowright X$. Define an equivalence relation $\sim$ on X by

$$x \sim y \iff \exists g \in G : gx = y.$$

These equivalence classes are called **orbits**.

Note that G acts transitively on orbits by definition.

Proposition 12.25

Suppose G, X are finite. For each orbit $\mathcal{O} \subseteq X$, pick a representative $x_{\mathcal{O}} \in \mathcal{O}$. Then

$$|X| = \sum_{\mathcal{O} \subseteq X} \frac{|G|}{|\operatorname{Stab}_G(X_{\mathcal{O}})|}.$$

Proof. Note that $\mathcal{O} \cong G/\operatorname{Stab}_G(X_{\mathcal{O}})$ by a previous proposition

$$|X| = \sum_{\mathcal{O} \subseteq X} |\mathcal{O}| = \sum_{\mathcal{O} \subset X} \frac{|G|}{|\operatorname{Stab}_G(x_{\mathcal{O}})|}.$$ □

Corollary 12.26

Let G be a p-group and suppose $p \nmid |G|$. Then $X^G \neq \varnothing$.

Proof. If not, then $\operatorname{Stab}_G(x_{\mathcal{O}}) \neq G$ for all $\mathcal{O}$. Take the above proposition modulo p; each term on the RHS is divisible by p but the LHS is not. □

§12.7 Corollaries of Sylow's Theorem

Aaron Landesman takes over at this point.

For the rest of the lecture, we will assuem $|G| = p^n m$, for a prime p, n a positive integer, and m a positive integer not divisible by p.

Definition 12.27. Let $H \subset G$ be a group. It is called a p**-Sylow group** if $|H| = p^n$.

Now here is the nuclear silo.

Theorem 12.28 (Sylow)

Let G be an arbitrary finite group.

(a) A p-Sylow group G_p must exist.

(b) If $H \subset G$ is a p-group then there exists a $g \in G$ such that $gHg^{-1} \subseteq G_p$.

Now for our theorem appreciation session.

Corollary 12.29

Let G be a group, and let G_p be a p-Sylow subgroup.

(i) Any two p-Sylow subgroups is conjugate to G_p.

(ii) In particular, G_p is normal in G if and only if it is the only p-Sylow group.

(iii) In particular, if g has order p^n, then there exists a g' such that $g'g(g')^{-1} \in G_p$.

Proof. (i) Use part (b) of Sylow's Theorem on one of these subgroups.

(ii) Use the previous statement.

(iii) Use the previous statement on $\langle g \rangle$, which is a p-Sylow subgroup.

□

A very obvious corollary is that if a prime q does not divide the order of any element in G, then q does not divide $|G|$ either.

Corollary 12.30

The following are equivalent.

(i) G is a p-group.

(ii) For any g, $\operatorname{ord}(g) = p^k$ for some k (depending on g).

(iii) $p \mid \operatorname{ord} g$ for all $g \neq 1$.

Proof. Trivial. □

§12.8 Proof of (b) of Sylow's Theorem assuming (a)

"We have five minutes so we're going to prove half of Sylow's Theorem".

Let G_p be the p-Sylow subgroup. Look at G/G_p as a set.

Let H be a p-group and consider the action $H \curvearrowright G/G_p$. Now $|G/G_p| = \frac{p^n m}{p^n} = m$. Hence

$$\gcd\left(|H|, |G/G_p| = 1.\right)$$

By our (mod p) prime corollary, that means there exists a $\overline{g} \in G/G_p \cap \operatorname{Stab}(H)$.

We will now prove the following easy lemma; applying it with $G' = G_p$ solves the problem.

Lemma 12.31

Let G be a group, and take $H, G' \subset G$. Let $H \curvearrowright G/G'$. If $\overline{g} \in G/G'$ is H-invariant, then any lifting g has

$$g^{-1}Hg \subseteq G'.$$

Proof. For all h, the action tells us $hg = gg'$ for some $g' \in G_p$. Hence $g^{-1}hg = g' \in G_p$, which is what we wanted. □

§13 October 14, 2014

I was out with the flu this day; by this time I had recovered enough that I could walk out of my room, but not enough to be able to eat meals. Thanks to Wyatt for his notes for this day, which I've adapted below.

§13.1 Proof of the first part of Sylow's Theorem

This is already in the Math 122 notes and is pretty widely known, so I won't bother too much here. I'll give a brief outline though.

The main idea is to consider the subsets of $|G| = p^n m$ with cardinality p^n. There are $\binom{p^n m}{p^n} \not\equiv 0 \pmod p$ of them (Lucas' Theorem). So if G acts on X by left multiplication, there is an orbit $\mathcal{O}$ with $p \nmid |\mathcal{O}|$. Take any representative $x \in \mathcal{O}$. Then show that $|\mathrm{Stab}_G(x)| = p^n$.

§13.2 Abelian group structure on set of modules

Let R be a ring and let M and N be R-modules. Then there's a natural abelian group structure $\mathrm{Hom}_R(M, N)$ by

$$T_1 + T_2 \overset{\text{def}}{=} (m \mapsto T_1(m) + T_2(m)).$$

(But note that the sum of two ring homomorphisms, in contrast, is in general not a ring homomorphism.)

We claim that we can upgrade the structure of $\mathrm{Hom}_R(M, N)$ to a R-module if R happens to be commutative. The multiplication is

$$r \cdot \phi \overset{\text{def}}{=} (m \mapsto r \cdot \phi(m)).$$

Let us explain why we needed R to be commutative. We need for $r \cdot \phi$ to actually be an R-module homomorphism. Any map in $\mathrm{Hom}_R(M, N)$, including $r \cdot \phi$, must obey

$$(r \cdot \phi)(r' \cdot m) = r' \cdot (r \cdot \phi)(m).$$

You can check that's equivalent to $rr' = r'r$.

§13.3 Dual Module

Definition 13.1. For any module M we now define $M^\vee = \mathrm{Hom}_R(M, R)$ as the **dual** of M.

This also has the structure of a right R-module by

$$T \cdot r \overset{\text{def}}{=} (m \mapsto T(m) \cdot r).$$

One way to think of this is as follows: suppose M is the set of column vectors over R of size n. Then $M^\vee$ is the set of row vectors over R of size n, but these aren't points in space, they are $1 \times n$ matrices. In particular they act on M by left multiplication which gives a single element of R.

Lemma 13.2

$(M_1 \oplus M_2)^\vee \simeq M_1^\vee \oplus M_2^\vee$.

Proof. We have the bijection of sets. Check the structure is preserved. ∎

Lemma 13.3

As a right R-module, R is isomorphic to its dual $R^\vee$.

Proof. Same as previous. $\square$

Corollary 13.4

$(R^n)^\vee$ is isomorphic to R^n (again as right modules).

Definition 13.5. Now consider $T : M \to N$. Then we define $T^\vee : N^\vee \to M^\vee$ by

$$T^\vee(\xi) \overset{\text{def}}{=} \xi \circ T.$$

where $\xi \in N^\vee$ and $m \in M$.

The correct way to think of $T^\vee$ is that if ξ is $n \times 1$ matrix, then $T^\vee$ sends ξ to ξT; i.e. $T^\vee$ just applies T on the right. At least in the finite dimensional case.

Exercise 13.6. Show that in the special case $T : R^a \to R^b$, the dual $T^\vee$ is a map $R^b \to R^a$ which is the transpose of T.

§13.4 Double dual

It is natural to construct $(M^\vee)^\vee$. We want a map $M \to (M^\vee)^\vee$. For a given $m \in M$, we can send it to the map

$$\text{can}_M(m) : M^\vee \to R$$

which sends $\xi \in M^\vee$ to map $\xi(m)$. (This is "evaluation at m".) Hence can_M itself is a map from M to $(M^\vee)^\vee$.

Remark 13.7. This map can_M need not be an isomorphism. If $R = \mathbb{Z}$, $M = \mathbb{Z}_2$, we find that $M^\vee$ is trivial.

Lemma 13.8

If $M = R^n$, then can_M is an isomorphism.

Lemma 13.9

If $T : M \to N$ is surjective, then $T^\vee : N \to M$ is injective.

If $T : M \to N$ is injective, then $T^\vee : N \to M$ is surjective, provided that N/M is free or they are finite-dimensional.

§13.5 Real and Complex Vector Spaces

Henceforth, $k \in \{\mathbb{R}, \mathbb{C}\}$ and V is a vector space over k.

Definition 13.10. An **inner form** $(\bullet, \bullet) : V \times V \to k$ is a map with the following properties.

- The form is **linear in the first argument**, so $(a+b,c) = (a,c)+(b,c)$ and $(ca,b) = c(a,b)$.
- If $k = \mathbb{R}$, $(a,b) = (b,a)$ and if $k = \mathbb{C}$, (a,b) is the complex conjugate $\overline{(b,a)}$. For brevity, we'll just write this as
$$(a,b) = \overline{(b,a)}$$
since $\overline{x} = x$ for all $x \in \mathbb{R}$.
- The form is **positive definite**, meaning $(v,v) \geq 0$ is a real number, and equality takes place only if $v = 0$.

Observe that these properties imply $(a,cb) = \overline{c}(a,b)$ but $(a,b+c) = (a,b)+(a,c)$.

Example 13.11

If $V = \mathbb{C}^n$, then an inner form is $(\langle a_n \rangle, \langle b_n \rangle) = \sum_i a_i \overline{b_i}$.

§13.6 Obvious Theorems

Definition 13.12. We define the norm by
$$\|v\| = \sqrt{(v,v)} \geq 0.$$

Theorem 13.13 (Pythagorean Theorem)

If $(v,w) = 0$, then we have $\|v+w\| = \|v\| + \|w\|$.

Theorem 13.14 (Cauchy-Schwarz)

For any $v, w \in V$ we have the inequality
$$|(v,w)| \leq \|v\| \cdot \|w\|.$$
Equality holds if and only if v and w are linearly dependent.

Theorem 13.15 (Triangle Inequality)

We have
$$\|v+w\| \leq \|v\| + \|w\|$$
with equality if and only if v and w are linearly dependent.

§13.7 Inner form induces a map

The inner form yields a map $V \to V^\vee$ by
$$v \mapsto (v, \bullet).$$

Theorem 13.16

If V is finite-dimensional, this map is an isomorphism.

Proof. The map is injective since $(v, w) = 0 \implies w = 0$ (for $v \neq 0$). By dimension arguments it is also an isomorphism. $\square$

§14 October 16, 2014

§14.1 Artificial Construction

Last time we tried to construct

$$V \xrightarrow{B_v} V^\vee = \operatorname{Hom}(V, \mathbb{C})$$

sending $v \to \xi_v$, where ξ_v is the map $w \mapsto (w, v)$. Unfortunately, ξ_v is apparently not $\mathbb{C}$-linear (conjugation problems).

So we do an artificial trick where we biject V to $\overline{V}$, by $v \mapsto \overline{v}$. Call the map φ.

Clearly if $e_1, e_2, \dots, e_n$ is a basis of V then $\varphi(e_1), \dots, \varphi(e_n)$ is an isomorphism.

Let's use the inner form to do the following.

Definition 14.1. For a given v, we can define a map $\xi_v \in V^\vee$ as follows:

$$\xi_v(w) \stackrel{\text{def}}{=} (w, v).$$

Now we can construct a map $B_v : \overline{V} \to V^\vee$ by $v \mapsto \xi_v$. And it's necessary to use $\overline{V}$ instead of V since otherwise you can check that the map is in fact not k-linear.

Lemma 14.2

If V is finite dimensional then B_v is an isomorphism.

Proof. This map is injective since $v \in V$ implies $\xi_v(v) \neq 0$. Then it's bijective for dimension reasons. $\square$

Since B_v is an isomorphism from $\overline{V}$ to $V^\vee$, we obtain the following readily.

Corollary 14.3

For every $\xi \in V^\vee$ there exists a v such that $\xi = \xi_v$.

§14.2 Orthogonal Subspace

Let $i : W \to V$ be an inclusion, where $W \subseteq V$. We now present the two distinct concepts of $W^\perp$ which unfortunately use the same symbol.

First, suppose V doesn't have an inner form. We consider the map $i^\vee$ dual to the inclusion $V^\vee \to W^\vee$. Then we define

$$W^\perp = \ker i^\vee = \left\{ \xi \in V^\vee \mid \xi|_W = 0 \right\}.$$

The second thing we define is $W^\perp$ which lives in V (not $V^\vee$). We'll assume $k = \mathbb{R}$ in what follows. Define

$$\{ v \in V \mid (v, w) = 0 \ \forall w \in W \}.$$

While these live in different spaces, they are isomorphic.

Lemma 14.4

The map $B_v : V \to V^\vee$ is an isomorphism between the $W^\perp$'s.

Corollary 14.5

$\dim W^\perp = \dim V - \dim W$.

For the rest of this lecture, we take the $W^\perp$ inside V.

Proposition 14.6

Let V be a finite-dimensional map. Let W be a subspace of V. Then the map

$$W \oplus W^\perp \to V$$

by $(w, v) \mapsto w + v$ is an isomorphism.

Proof. The dimensions agree. For injectivity, if $w + v = 0$ then $(w', w) + (w', v) = 0$ for any $0 \neq w' \in W$. Now $(w', v) = 0$, so $(w', w) = 0$. □

§14.3 Orthogonal Systems

Definition 14.7. Let $v_1, \dots, v_n \in V$. We say that v_i are **orthogonal** if $(v_i, v_j) = 0$ for all $i \neq j$. Moreover, it is **orthonormal** if $\|v_i\| = 1$ for all i.

Lemma 14.8

Any orthogonal set of vectors is linearly independent.

Proof. Suppose $\sum a_i v_i = 0$. Then $0 = (v_1, \sum a_i v_i) = a_1$. □

Proposition 14.9 (Gram-Schmidt)

Let $v_1, \dots, v_n$ be a linearly independent set. Set $V_i = \operatorname{span}(v_1, \dots, v_i)$. There exists a unique $e_1, \dots, e_n$ with the following properties:

- $e_1, \dots, e_n$ is orthogonal (in particular, also linearly independent)
- For all i, $V_i = \operatorname{span}(e_1, \dots, e_i)$.
- The images of v_{i+1} and e_{i+1} in V/V_i are equal,

Proof. Let $V = V_i \oplus (V_i)^\perp$. Then $v_{i+1} = w + u$ uniquely for some $w \in V_i$, $u \in (V_i)^\perp$. Now it's easy to check that $e_{i+1} = u$ is the only thing that works. □

§14.4 Adjoint operators

In what follows, we'll sometimes clarify which inner form we're using by $(\bullet, \bullet)_V$ denoting the inner form for V. But remember all the inner forms output in the same field k.

Definition 14.10. Suppose that $T : V \to W$, and V and W have inner forms. We define a map $T^* : W \to V$, the **adjoint operator**, as follows. For any w, we let $T^*(w)$ be the unique vector with

$$\xi_{T^*(w)} = (v, T^*(w))_V = (T(v), w)_W$$

for all v.

In other words: given $T : V \to W$ and $w \in W$, we have a map in $V^\vee$ by sending $v \mapsto (T(v), w)_W$. Evidently this equals some ξ_x, and we define $T^*(w)$ to be this x.

Lemma 14.11

$(T^*)^* = T$.

Proof. We wish to check $(T^*)^*(v) = T(v)$. We have

$$(T^*)^*(v) = T(v) \iff ((T^*)^*(v), w) = (T(v), w)$$

but the left is $\overline{(w, (T^*)^*(v))} = \overline{(T^*(v), w)} = (v, T^*(w))$. □

First, let's check the following.

Lemma 14.12

Given any subspace V' of V we have

$$\left((V')^\perp\right)^\perp = V'.$$

Proof. It is not hard to check that $V' \subseteq \left((V')^\perp\right)^\perp$. Then the inclusion is strict for dimension reasons. □

Lemma 14.13

We have

(a) $(\operatorname{im} T)^\perp = \ker T^*$

(b) $(\ker T)^\perp = \operatorname{im} T^*$.

Proof. For the first part, note that

$$\begin{aligned} w \in (\operatorname{im} T)^* &\iff (w, w') = 0 \quad \forall w' \in \operatorname{im} T \\ &\iff (w, T(v)) = 0 \quad \forall v \in V \\ &\iff (T^*(w), v) = 0 \quad \forall v \in V \\ T^*(w) = 0. \end{aligned}$$

□

For the second part, just use that

$$\operatorname{im} T^* = \left((\operatorname{im} T^*)^\perp\right)^\perp = \ker(T^*)^* = (\ker T)^\perp.$$

§14.5 Spectral theory returns

Let $k = \mathbb{C}$.

Definition 14.14. $T : V \to V$ is **diagonalizable** in a way compatible with the inner form if V admits an orthonormal basis of eigenvectors.

Lemma 14.15

The following are equivalent.

(1) T is diagonalizable in a way compatible with the inner form.

(2) We can take $V_\lambda \subseteq V$ such that $T|_{V_\lambda} = \lambda \mathrm{Id}_{V_\lambda}$ such that $\bigoplus_\lambda V_\lambda \sim V$, and $V_\mu \perp V_\lambda$ for any $\mu \neq \lambda$.

(3) T is diagonalizable and $V^\lambda \perp V^\mu$ for all $\lambda \neq \mu$.

Proof. (3) $\implies$ (1) is clear, just use Gram-Schmidt to get what we want.

(1) $\implies$ (2) is easy, just set

$$V_\lambda = \mathrm{Span}\left\{ e_i \mid T(e_i) = \lambda e_i \right\}.$$

(2) $\implies$ (3): We know $V_\lambda \subseteq V^\lambda$, but we have injections

$$\bigoplus_\lambda V_\lambda \to \bigoplus_\lambda V^\lambda \to V.$$ □

Definition 14.16. Given an endomorphism $T : V \to V$, we say T is **normal** if and only if $T^* \circ T = T \circ T^*$.

Theorem 14.17

Let $T : V \to V$ be a map over complex vector spaces. The map T is diagonalizable in a way compatible with the inner form if and only if it is normal.

Proof. First, assume it's diagonalizable in such a way. Let $V = \bigoplus V_\lambda$. The point is that if we have $T_1 : V_1 \to W_1$ and $T_2 : V_2 \to W_2$ with $V_1 \perp V_2$ and $W_1 \perp W_2$, then we want $T^* : W_1 \to V_1$ and $T^* : W_2 \to V_2$. Like we want to check

$$(v_1 + v_2, T_1^*(w_1) + T_2^*(w_2)) = (T_1(v_1) + T_1(v_2), w_1 + w_2).$$

Perpendicularity makes this clear.

Our proof is now by induction on $\dim V$. Let T be normal. Since we are over $\mathbb{C}$ and not $\mathbb{R}$, there exists λ such that $V^\lambda \neq 0$. So

$$(V^\lambda) \oplus (V^\lambda)^\perp \sim V.$$

It is enough to show that $(V^\lambda)^\perp$ is T-invariant, then we can use the inductive hypothesis on $(V^\lambda)^\perp$. Now

$$V^\lambda = \ker(T - \lambda I)$$

so

$$(V^\lambda)^\perp = \mathrm{im}(T^* - \overline{\lambda} I).$$

So we merely need to show T commutes with $T^* = \overline{\lambda} \cdot I$. We know that T commutes with $T^* - \overline{\lambda} \cdot I$. Hence T preserves its image. □

§14.6 Things not mentioned in class that any sensible person should know

There are some really nice facts out there that are simply swept under the rug. For example the following proposition makes it immediately clear what the adjoint T^* looks like.

Theorem 14.18

Let $T : V \to W$, and take an **orthonormal** basis of V and W. Consider the resulting matrix for T as well as $T^* : W \to V$. They are complex transposes.

Proof. Orthonormal basis bashing. □

Note that Gram-Schmidt lets us convert any basis to an orthonormal one. Do you now understand why $(T^*)^* = T$ is completely obvious?

Proposition 14.19

Let T be normal. Then T is self-adjoint if and only if its spectrum lies in $\mathbb{R}$.

Proof. Take an orthonormal basis, now T is a diagonal matrix, and its adjoint just conjugates all the entries on the diagonal. □

§14.7 Useful definitions from the homework

On the homework we also have the following definitions.

Proposition 14.20

Let $T : V \to V$ be normal. The following are equivalent.

- We can take a basis in which the matrix of T is diagonal and has only nonnegative real entries. In particular, T has eigenvalues consisting only of nonnegative reals.
- We have $(T(v), v) \geq 0$ for all v.

Definition 14.21. The map $T : V \to V$ is called **non-negative definite** if it satisfies the above conditions.

Theorem 14.22 (Unitary Maps)
Let $T : V \to W$. The following are equivalent.

- T is invertible and $T^* = T$.
- T respects inner forms, i.e. we have
$$(T(v_1), T(v_2))_W = (v_1, v_2)_V$$
for any $v_1, v_2 \in V$.
- $T : V \to W$ respects just the norms, meaning
$$\|T(v)\|_W = \|v\|_V$$
for any $v \in V$.

Definition 14.23. The map $T : V \to W$ is **unitary** if it satisfies the above conditions.

§15 October 21, 2014

"Today will be a challenging class".

§15.1 Generators

Definition 15.1. Let M be a right R-module and let N be a left R-module. The tensor product $M \otimes_R N$ be the abelian group generated by elements of the form $m \otimes n$, subject to the following relations.

$$(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$$
$$m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$$
$$mr \otimes n = m \otimes rn$$

Not going to write down the definition "generated by elements of λ_i subject to relations". Look up "group presentation" for a human explanation.

Lemma 15.2

Suppose Λ is generated by generators λ_i and some relations. To specify a map $S : \Lambda \to \Omega$ is to specify the image of each generator, preserving all relations.

Proof. Intuitively clear. Too lazy to copy down this proof since it's buried in cokernels and commutative diagrams. □

As usual, abelian groups are being interpreted as $\mathbb{Z}$ modules. Like every element is just a sum of the form $\sum_{(m,n)\in M\times N} c_{m,n}(m \otimes n)$ subject to the above conditions.

OK fine I'll sketch it. Basically, if you have a group G with generators $(g_i)_{i\in I}$ and relations $(R_j)_{j\in J}$ (here I and J are indexing sets), then you can define a map $T : A^{\oplus J} \to A^{\oplus I}$ which sends every element of $A^{\oplus J}$ to the corresponding thing we want to make zero in G. Then $G \cong \operatorname{coker}(T)$.

§15.2 Basic Properties of Tensor Products

Lemma 15.3

To specify a map $M \otimes_R N \xrightarrow{S} \Omega$ is to specify a map $S(m \otimes n)$ where $m \in M$, $n \in N$, satisfying the relations.

Lemma 15.4

$M \otimes_R R \cong R$.

Proof. The point is that $m \otimes 1 \mapsto m$. Explicitly, the bijection is $m \otimes r \mapsto m \cdot r$. Check that it's a map. It is obviously a surjection. □

Lemma 15.5

$(M_1 \oplus M_2) \otimes N \cong (M_1 \otimes N) \oplus (M_2 \otimes N)$.

Proof. Meow. □

Consider an left R-module N. Given a map $T : M_1 \to M_2$ of right R-modules, we can construct a map $T \otimes \mathrm{id}$ by to get a map

$$M_1 \otimes_R N \to M_2 \otimes_R N \text{ by } m_1 \otimes n \mapsto T(M_1) \otimes n.$$

Proposition 15.6

Suppose we have $M_1 \xrightarrow{T} M_2 \xrightarrow{\pi} M \to 0$ are right R-modules, and let N be a right R-module. This gives a map

$$M_1 \otimes_R N \to M_2 \otimes_R N \to M \otimes_R N.$$

Let $M = \operatorname{coker}(T)$. Then $\operatorname{coker}(T \otimes \mathrm{id}) \cong M \otimes_R N$.

Proof. Essentially we want $\overline{(m_2 \otimes n)} \mapsto m \otimes n$ as the bijection. Blah. □

§15.3 Computing tensor products

We begin this with the following quote.

> "I want an honest confession now. Who has underspoken? You will have to construct a map in one direction. Come out of your own free will, or you will have to construct a map in both directions." – Gaitsgory

Example 15.7

We wish to show $\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_3 = 0$. Let $M = \mathbb{Z}_2$, $N = \mathbb{Z}_3$. Let $M_1 = \mathbb{Z}$, $M_2 = \mathbb{Z}$, and consider the map $T = 2$. This gives us the maps

$$\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_3 \xrightarrow{T \otimes \mathrm{id}} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_3.$$

We have an isomorphic map

$$\mathbb{Z}_3 \xrightarrow{2} \mathbb{Z}_3$$

and the cokernel of this map is 0. Hence $\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_3 = 0$.

§15.4 Complexification

Let $T : V_1 \to V_2$. We obtain a natural map.

$$\mathrm{id} \otimes T : \mathbb{C} \otimes_{\mathbb{R}} V_1 \to \mathbb{C} \otimes_{\mathbb{R}} V_2.$$

Let's generalize this. Here's the setup Let $\varphi : R_1 \to R_2$, let M be a right R_1-module, and let N be a left R_2 module.

Definition 15.8. Any left R_2-module N is also a left R_1-module according to the rule

$$r_1 \cdot n \overset{\text{def}}{=} \phi(r_1) \cdot n.$$

We can put an right R_1-module structure on R_2 by saying

$$r_2 \cdot r_1 = r_2\varphi(r_1).$$

Then it makes sense to consider the tensor product

$$X = R_2 \otimes_{R_1} M.$$

One can check that X is a left R_2-module by the structure

$$r_2' \cdot (r_2 \otimes m) \stackrel{\text{def}}{=} (r_2' r_2) \otimes m.$$

Hence by our definition above, X is also an R_1-module.

The result of all this setup is the following.

Theorem 15.9

In the above setup,

$$\operatorname{Hom}_{R_1}(M, N) \cong \operatorname{Hom}_{R_2}(X, N) \text{ by } T \mapsto T \circ T_{\text{univ}}$$

$T_{\text{univ}} : M \to X$ by $m \mapsto 1 \otimes x$.

What does this mean? The motivating example is $M = \mathbb{R}^2$, $N = \mathbb{C}^5$, $R_1 = \mathbb{R}$, $R_2 = \mathbb{C}$. Basically if we specify an $\mathbb{R}$-linear map $\mathbb{R}^2 \to \mathbb{C}^5$, we just have to map each basis element, but this automatically extends to a map $\mathbb{C}^2 \to \mathbb{C}^5$ since we already know how to multiply by i. And of course, the inclusion $\mathbb{R}^2 \hookrightarrow \mathbb{C}^2$ of $\mathbb{R}$-modules means that any map $\mathbb{C}^2 \to \mathbb{C}^5$ gives us a map $\mathbb{R}^2 \to \mathbb{C}^5$. What this means is that

$$\operatorname{Hom}_{\mathbb{R}}\left(\mathbb{R}^2, \mathbb{C}^5\right) \cong \operatorname{Hom}_{\mathbb{C}}\left(\mathbb{C}^2, \mathbb{C}^5\right).$$

You can check that X plays the role of $\mathbb{C}^2$ in our examples, because $X = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^2 \simeq (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R})^2$. But $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}$ is just deciding how to "extend" $\mathbb{R}$ to $\mathbb{C}$.

§16 October 23, 2014

Today we will continue tensor products. For this lecture, assume that R is commutative; hence left and right modules are the same thing. We will abbreviate $\otimes_R$ as $\otimes$ unless specified otherwise.

§16.1 Tensor products gain module structure

As R is commutative, $M \otimes N$ has a structure of an R module by

$$r \cdot (m \otimes n) = rm \otimes n.$$

Indeed, just check this.

We know that to map $M \otimes_R N$ into an arbitrary abelian group Ω, it suffices to map each (m, n) such that $B(rm, n) = B(m, rn)$. Now that our tensor product has an R-module structure, we want to map preserving R-modules.

Lemma 16.1

Let L be an R-module. A map $B : M \otimes N \to L$ is specified by a map from $M \times N$ to L satisfying

$$B(rm, n) = B(m, rn) = rB(m, n).$$

Proof. meow. □

Lemma 16.2

$M \otimes_R N \simeq N \otimes_R M$.

This makes sense since R is commutative, so the concept of left and right module coincide.

Proof. Clear by the preceding lemma. □

§16.2 Universal Property

Now $(M_1 \otimes M_2) \otimes M_3 \cong M_1 \otimes (M_2 \otimes M_3)$. This is really annoying by normal means. So we are going to present something called the "universal property", which lets us not have to deal with this nonsense.

We are going to define a module $M_1 \otimes M_2 \otimes M_3 \overset{\text{def}}{=} M$ in one shot.

Theorem 16.3

We claim there is a unique module M and a map $T_{\text{univ}} : M_1 \times M_2 \times M_3 \to M$ such that for any other map L and $T : M_1 \times M_2 \times M_3 \to L$, there is a unique φ such that

$$\begin{tikzcd} M_1 \times M_2 \times M_3 \arrow[rr, "T_{\text{univ}}"] \arrow[dr, "T"'] & & M_1 \otimes M_2 \otimes M_3 \arrow[dl, "\varphi"] \\ & L & \end{tikzcd}$$

Proof. First, we prove that T_{univ} is unique. Suppose on the contrary that we have the M', M'', T'_{univ}, T''_{univ} with these properties. So consider:

$$\begin{tikzcd}
 & M' \arrow[dd, shift right, "\phi"'] \\
M_1 \times M_2 \times M_3 \arrow[ur, "T'_{\text{univ}}"] \arrow[dr, "T''_{\text{univ}}"'] & \\
 & M'' \arrow[uu, shift right, "\psi"']
\end{tikzcd}$$

We can construct ϕ and ψ based on the universal properties. Now observe

$$\psi \circ \phi \circ T'' = T'' = \text{id} \circ T''$$

and the universal property implies $\psi \circ \phi = \text{id}$. Similarly, $\phi \circ \psi = \text{id}$. Hence M and M' are isomorphic.

One will see this over and over again, in completely different problems with exactly the same argument. This is what motivates category theory.

Now let's prove existence. We will explicitly construct M and T_{univ} by

$$M \stackrel{\text{def}}{=} (M_1 \otimes M_2) \otimes M_3$$

and $T_{\text{univ}}(m_1, m_2, m_3) = (m_1 \otimes m_2) \otimes m_3$. Then we'll be done by our uniqueness.

We have the commutative diagram

$$\begin{tikzcd}
M_1 \times M_2 \times M_3 \arrow[rr, "T_{\text{univ}}"] \arrow[dr, "T"'] & & (M_1 \otimes M_2) \otimes M_3 \arrow[dl, "\varphi"] \\
 & L &
\end{tikzcd}$$

Just do standard blah. $\square$

§16.3 Tensor products of vector spaces

In what follows, we now assume the ground ring is in fact a field. Hence we are considering tensor products of vector spaces.

Let V and W be a finite dimensional vector space over k. We'll identify $V \simeq k^n$. In that case,

$$k^{\oplus n} \otimes W \simeq (k \otimes W)^{\oplus n} \simeq W^{\oplus n}$$

By explicitly writing the bijection, we derive the following.

Corollary 16.4

Let V and W be vector spaces over k with basis $e_1, \dots, e_n$ and $f_i, \dots, f_m$. Then $e_i \otimes f_j$ forms a basis of $V \otimes W$.

Now we will construct a map $V^\vee \otimes W \to \text{Hom}(V, W)$ by $\xi \otimes w \mapsto T_{\xi,w} \in \text{Hom}(V, W)$, where $T_{\xi,w}(v) = \xi(v) \cdot w$.

Lemma 16.5

The above map
$$V^\vee \otimes W \to \mathrm{Hom}(V, W)$$
is an isomorphism provided that V and W are finite-dimensional.

Proof. If $e_1^\vee, \dots, e_n^\vee$ is the basis of $V^\vee$ corresponding to $e_1, \dots, e_n$, we find that the image of the map above is
$$e_i^\vee \otimes f_j \to T_{ij}$$
where T_{ij} is the matrix with 1 in (i, j) and zero otherwise. Hence we send basis to basis, and we're done. $\square$

In the homework one will actually show the map is an isomorphism if and only if at least one of V or W are finite-dimensional.

§16.4 More tensor stuff

Lemma 16.6

The following diagram commutes.

$$\begin{tikzcd}
V^\vee \otimes W \arrow[r] \arrow[d, "\sim"] & \mathrm{Hom}(V, W) \arrow[dd] \\
W \otimes V^\vee \arrow[d] & \\
(W^\vee)^\vee \otimes V^\vee \arrow[r] & \mathrm{Hom}(W^\vee, V^\vee).
\end{tikzcd}$$

This gives us a way to interpret the dual map.

Lemma 16.7

Let $\mathrm{ev} : V \otimes V^\vee \to k$ be a canonical evaluation map. The following diagram commutes.

$$\begin{tikzcd}
(U^\vee \otimes V) \otimes (V^\vee \otimes W) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}(U, V) \otimes \mathrm{Hom}(V, W) \arrow[dd] \\
U^\vee \otimes (V \otimes V^\vee) \otimes W \arrow[d] & \\
U^\vee \otimes k \otimes W \simeq U^\vee \otimes W \arrow[r] & \mathrm{Hom}(U, W).
\end{tikzcd}$$

Definition 16.8. The **trace** $\mathrm{Tr} : \mathrm{End}(V) \to V$ is defined as the unique map which makes the following diagram commute.

$$\begin{tikzcd} V^\vee \otimes V \arrow[r, "\simeq"] \arrow[d, "\mathrm{ev}"] & \mathrm{End}(V) \arrow[dl, "\mathrm{Tr}"] \\ k & \end{tikzcd}$$

One can check that this coincides with the "sum of the diagonals" definition.

§16.5 Q & A

> "The first midterm was a joke. It was completely straightforward. This one will require actual thinking."

First, a note: I kept complaining that $V^\vee$ is just V rotated. This is true, but it is not natural to replace $V^\vee$ with V everywhere. This is because a different choice of basis gives a different isomorphism. So V is not more related to $V^\vee$ than any arbitrary vector space of the same dimension.

Let's do PSet review.

> Prove that any $T : V \to V$ can be uniquely written as $T = T^{\mathrm{diag}} + T^{\mathrm{nilp}}$, with the two components commuting.

Proof. Look at the generalized eigenspaces $V = \bigoplus_\lambda V^{(\lambda)}$ with respect to T. First, we show existence. Let T^{diag} be the map which multiplies on λ on each $V^{(\lambda)}$. Then just set $T^{\mathrm{nilp}} = T - T^{\mathrm{diag}}$. You can check these components commute because we simply need T to commute with T^{diag}, which follows by noting that each $V^{(\lambda)}$ is T-invariant by definition, so we only need to check commuting over each $V^{(\lambda)}$ and this is clear. Also, it is easy to see that $T^{\mathrm{nilp}} = T - T^{\mathrm{diag}}$.

(If T commutes with S, then T preserves the eigenstuff because it preserves the kernels and images.)

Now let's show uniqueness. Consider a decomposition $T = T^{\mathrm{diag}} + T^{\mathrm{nilp}}$. Let $V = \bigoplus V_\lambda$, where V_λ are the λ eigenspaces of T^{diag}. You can check that $T - \lambda$ is nilpotent over each V_λ from the fact that T commutes with T^{diag}. Hence $V_\lambda \subseteq V^{(\lambda)}$ (reminder that V_λ is with respect to T^{diag} and $V^{(\lambda)}$ is with respect to T). And now we have

$$\bigoplus_\lambda V_\lambda \hookrightarrow \bigoplus_\lambda V^{(\lambda)} \simeq V.$$

So $V^{(\lambda)} = V_\lambda$. That's enough to imply that our T^{diag} is the same as the one we explicitly constructed.

Polynomial stuff follows from that you can write T^{diag} using spectral projectors. $\square$

§17 October 28, 2014

§17.1 Midterm Solutions

Darn these solutions are correct why did I lose points qq.

17.1.1 Problem 1

Let V and W be vector spaces (not necessarily finite-dimensional).

(a) Consider the map
$$V^\vee \otimes W^\vee \to (V \otimes W)^\vee$$
that sends $\xi \otimes \eta$ to the functional on $V \otimes W$ that takes $v \otimes w$ to $\xi(v) \cdot \eta(w)$. Show that this map is an isomorphism if one of these vector spaces is finite-dimensional.

Assume by symmetry that V is finite-dimensional. By [Pset 8, Problem 6a], the canonical map $V^\vee \otimes W^\vee \to \operatorname{Hom}(V, W^\vee)$ by $\xi \otimes \eta \mapsto (v \mapsto \xi(v) \cdot \eta)$ is an isomorphism. By [Pset 8, Problem 7a] the map $\operatorname{Hom}(V, W^\vee) \to (V \otimes W)^\vee$ by $\theta \mapsto (v \otimes w \mapsto \theta(v)(w))$ is an isomorphism. Composition shows that the requested map is an isomorphism.

(b) Let V and W be vector spaces with W finite-dimensional. Define a linear map
$$\operatorname{Hom}(V, W) \otimes \operatorname{Hom}(W, V) \to k$$
by
$$T \otimes S \mapsto \operatorname{Tr}_W(T \circ S).$$
Note that by [Pset 8, Problem 7a], the above map defines a map
$$\operatorname{Hom}(V, W) \to (\operatorname{Hom}(W, V))^\vee.$$
Show that this map is an isomorphism.

Let $\Psi : \operatorname{Hom}(V, W) \to (\operatorname{Hom}(W, V))^\vee$ be the map in question, and let W have basis e_1, ..., e_n. Then Ψ is given by $T \mapsto (S \mapsto \operatorname{Tr}_W(T \circ S))$ and we wish to show that this is an isomorphism.

Using [Pset 8, Problem 6a] it is easy to compute
$$\operatorname{Tr}(T \circ S) = \operatorname{ev}\left(\sum_{i=1}^n e_i^\vee \otimes (T \circ S)(e_i)\right) = \sum_{i=1}^n e_i^\vee \left((T \circ S)(e_i)\right).$$
Hence we wish to prove that $\Psi : T \mapsto (S \mapsto \sum_{i=1}^n e_i^\vee (T \circ S)(e_i))$ is an isomorphism. To construct the reverse map, consider the map Θ by
$$\Theta : \xi \mapsto \left[v \mapsto \sum_{i=1}^n \left(\xi\left[e_k \mapsto \begin{cases} v & \text{if } k = i \\ 0 & \text{otherwise.}\end{cases}\right]\right) e_i\right]$$
It is tautological to check that Θ is a two-sided inverse to Ψ. Indeed, verify that
$$\Theta(\Psi(T)) = \left(v \mapsto \sum_{i=1}^n e_i^\vee(T(v)) \cdot e_i\right) = T$$

and

$$\begin{aligned}
\Psi(\Theta(\xi)) &= \left\{ S \mapsto \sum_{i=1}^n e_i^\vee \left(\left[v \mapsto \left\{ \sum_{j=1}^n \left(\xi \left[e_k \mapsto \begin{cases} v & \text{if } k = j \\ 0 & \text{otherwise.} \end{cases} \right] \right) e_j \right\} \right] \Big(S(e_i) \Big) \right) \right\} \\
&= \left\{ S \mapsto \sum_{i=1}^n e_i^\vee \left(\sum_{j=1}^n \left(\xi \left[e_k \mapsto \begin{cases} S(e_i) & \text{if } k = j \\ 0 & \text{otherwise.} \end{cases} \right] \right) e_j \right) \right\} \\
&= \left\{ S \mapsto \sum_{i=1}^n \left(\xi \left[e_k \mapsto \begin{cases} S(e_i) & \text{if } k = i \\ 0 & \text{otherwise.} \end{cases} \right] \right) \right\} \\
&= \left\{ S \mapsto \xi \left[\sum_{i=1}^n \left(e_k \mapsto \begin{cases} S(e_i) & \text{if } k = i \\ 0 & \text{otherwise.} \end{cases} \right) \right] \right\} \\
&= \{ S \mapsto \xi \left[e_k \mapsto S(e_k) \right] \} \\
&= \{ S \mapsto \xi(S) \} \\
&= \xi.
\end{aligned}$$

It follows that Ψ is an isomorphism.

17.1.2 Problem 2

(a) Let V_1, V_2 and W be finite-dimensional vector spaces, and let $T : V_1 \to V_2$ be an injective map. Show that the map $T \otimes \mathrm{id} : V_1 \otimes W \to V_2 \otimes W$ is also injective.

Since T is injective there is a map $S : V_2 \to V_1$ such that $S \circ T = \mathrm{id}$. We claim $S \otimes \mathrm{id}$ is also a left inverse for $T \otimes \mathrm{id}$. To prove that $(S \otimes \mathrm{id}) \circ (T \otimes \mathrm{id})$ is the identity, it suffices to verify this for the spanning set $\{v_1 \otimes w \mid v_1 \in V, w \in V\}$, which is evident:

$$(S \otimes \mathrm{id}) \circ (T \otimes id)(v_1 \otimes w) = S(T(v_1)) \otimes w = v_1 \otimes w.$$

(b) Let $T_{\mathbb{R}}$ be a real $n \times n$ matrix, and let $T_{\mathbb{C}}$ be the same matrix but considered as a complex one. Show that $\min_{T_{\mathbb{C}}} \in \mathbb{C}[t]$ equals the image of $\min_{T_{\mathbb{R}}} \in \mathbb{R}[t]$ under the tautological map $\mathbb{R}[t] \to \mathbb{C}[t]$.

Set $T = T_{\mathbb{C}}$, and let $p = \min_T$. WLOG p is monic. Since $\mathbb{C}$ is algebraically closed, it follows that we can decompose $V = \bigoplus_\lambda V^{(\lambda)}$. Now the minimal polynomial is given by

$$p(t) = \prod_{\lambda \in \operatorname{Spec}(T)} (t - \lambda)^{m_\lambda}$$

where m_λ is the smallest integer such that T^{m_λ} annihilates $V^{(\lambda)}$.

We now prove that p consists of real coefficients. To do so, it suffices to prove that $m_\lambda = m_{\overline{\lambda}}$ for any $\lambda \in \mathbb{C}$. But $\overline{T} = T$ as $T \in \mathbb{R}^{n \times n}$ and the claim follows immediately upon observing that

$$(T - \lambda)^k (v) = 0 \iff \left(T - \overline{\lambda}\right)^k (\overline{v}) = \overline{0} = 0.$$

Hence the minimal k to kill a λ-eigenvector is the same as the minimal k to kill a $\overline{\lambda}$-eignvector.

Alternatively, one can use note that if 1, $T_{\mathbb{R}}$, $\dots$, $T_{\mathbb{R}}^{n-1}$ are linearly independent then so are 1, $T_{\mathbb{C}}$, $\dots$, $T_{\mathbb{C}}^{n-1}$.

17.1.3 Problem 3

(a) Let V be a complex vector space endowed with an inner form, and let $T : V \to V$ be an endomorphism. Show that T is normal if and only if there exists a polynomial $p \in \mathbb{C}[t]$ such that $T^* = p(T)$.

First, suppose $T^* = p(T)$. Then $T^*T = p(T) \cdot T = T \cdot p(T) = TT^*$ and we're done.

Conversely, suppose T is diagonalizable in a way compatible with the inner form (OK since V is finite dimensional). Consider the orthonormal basis. Then T consists of eigenvalues on the main diagonals and zeros elsewhere, say

$$T = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}.$$

In that case, we find that for any polynomial q we have

$$q(T) = \begin{pmatrix} q(\lambda_1) & 0 & \dots & 0 \\ 0 & q(\lambda_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & q(\lambda_n) \end{pmatrix}.$$

and

$$T^* = \begin{pmatrix} \overline{\lambda_1} & 0 & \dots & 0 \\ 0 & \overline{\lambda_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \overline{\lambda_n} \end{pmatrix}.$$

So we simply require a polynomial q such that $q(\lambda_i) = \overline{\lambda_i}$ for every i. Since there are finitely many λ_i, we can construct such a polynomial of degree at most $n - 1$ using Lagrange interpolation. The conclusion follows.

Notice moreover the set of polynomials that work is precisely the set of q such that $q(\lambda) = \overline{\lambda}$ for every $\lambda \in \operatorname{Spec}(T)$ (this becomes important in 3c).

(b) Let V be a real vector space, and let $v_1, ..., v_n$ and w be elements. Let $V_\mathbb{C} := \mathbb{C}_\mathbb{R} \otimes V$ be its complexification, and let $1 \otimes v_1, ..., 1 \otimes v_n$ and $1 \otimes w$ be the corresponding elements of $V_\mathbb{C}$. Show that w belongs to the span of $v_1, ..., v_n$ if and only if $1 \otimes w$ belongs to the span of $1 \otimes v_1, ... 1 \otimes v_n$.

If $w = \sum a_k v_k$ then $1 \otimes w = \sum a_k (1 \otimes w_k)$ is clear (here $a_i \in \mathbb{R}$). Conversely, suppose that $1 \otimes w = \sum (x_k + y_k i) \otimes v_k$, where x_i and y_i are reals. Then $1 \otimes w = 1 \otimes (\sum x_k v_k) + i \otimes (\sum y_k v_k)$, so

$$1 \otimes \left(w - \sum x_k v_k\right) = i \otimes \left(\sum y_k v_k\right).$$

Because of the relations between $v \otimes w$, this can only occur if $w - \sum x_k v_k = \sum y_k v_k = 0$, which is what we wanted to show.

(c) Let V be a real vector space endowed with an inner form, and let $T : V \to V$ be an endomorphism. Show that T is normal if and only if there exists a polynomial $p \in \mathbb{R}[t]$ such that $T^* = p(T)$.

Again if $T^* = p(T)$ then we clearly have $T^*T = p(T)T = Tp(T) = TT^*$.

The hard part is showing that if $T^*T = TT^*$, then $T^* = p(T)$ for some T; we no longer have an orthonormal basis of eigenvectors. However, taking T in $\mathbb{C}$ gives us a polynomial $q \in \mathbb{C}[t]$ for which $q(T) = T^*$.

We observe that since T can be written in a real matrix, it follows that $T(v) = \lambda v \implies T(\overline{v}) = \overline{\lambda}\overline{v}$. So it follows that if $\lambda \in \operatorname{Spec}(T)$ implies $\overline{\lambda} \in \operatorname{Spec}(T)$.

Hence we suppose the distinct eigenvalues of $\operatorname{Spec}(T)$ over $\mathbb{C}$ are $\lambda_i, \overline{\lambda_i}$, for $\lambda_i \in \mathbb{C} - \mathbb{R}$, and $\mu_j \in \mathbb{R}$, and that there are a total of N distinct eigenvalues. Consider the minimal complex polynomial P (by degree) such that

$$P(\lambda_i) = \overline{\lambda_i},\ P(\overline{\lambda_i}) = \lambda_i,\ P(\mu_j) = \mu_j.$$

By Lagrange interpolation we have that $\deg P \le N - 1$, and by Problem 3(a) this polynomial forces $T^* = P(T)$. Let $\overline{P}$ be the polynomial whose coefficients are the conjugate of those in P. We find that

$$\overline{P}(\overline{\lambda_i}) = \lambda_i,\ \overline{P}(\lambda_i) = \overline{\lambda_i},\ \overline{P}(\mu_j) = \mu_j.$$

Hence $\overline{P}$ is also a polynomial of the same degree which also satisfies the conditions. Hence $P - \overline{P}$ is the zero polynomial over N distinct complex numbers, but it has degree at most $N - 1$, so $P - \overline{P} \equiv 0$. Hence $P \equiv \overline{P}$. That means $P(t) \in \mathbb{R}[t]$ and we are done.

§17.2 The space $\Lambda^n_{\text{sub}}(V)$

Oh man this is not in the notes PAY ATTENTION CHILDREN.

Let V be a finite dimensional vector space over a field k, and let n be a positive integer with $n \ge 2$. Consider the space $V^{\otimes n}$.

Definition 17.1. We define

$$\Lambda^2_{\text{sub}}(V) \overset{\text{def}}{=} \operatorname{Span}\left\{v_1 \otimes v_2 - v_2 \otimes v_1 \mid v_1, v_2 \in V\right\}.$$

Then, we define for $n \ge 3$ the space $\Lambda^n_{\text{sub}}(V)$ as

$$\left(\Lambda^2_{\text{sub}}(V) \otimes V^{\otimes(n-2)}\right) \cap \left(V \otimes \Lambda^2_{\text{sub}}(V) \otimes V^{n-3}\right) \cap \left(V^2 \otimes \Lambda^2_{\text{sub}}(V) \otimes V^{n-4}\right) \cap \cdots$$

i.e. we set

$$\Lambda^n_{\text{sub}}(V) = \bigcap_{k=0}^{n-2} \left(V^k \otimes \Lambda^2_{\text{sub}}(V) \otimes V^{n-2-k}\right).$$

Let's focus first on $n = 2$.

Lemma 17.2

If V has a basis $e_1, \dots, e_n$ then the elements $e_i \otimes e_j - e_j \otimes e_i$, $i < j$, are a span for Λ^2_{sub}.

Proof. Clear. In a moment we'll see the elements are linearly independent, so in fact they form a basis. □

On to general n (in the actual lecture, we define Λ^n_{sub} only here). In subsequent weeks we will study actions of group on vector spaces. Actually, we claim that the symmetric group S_n acts on $V^{\otimes n}$. The map is simply

$$v_1 \otimes \cdots \otimes v_n \overset{\sigma \in S_n}{\mapsto} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}.$$

We will now point out that this map is actually very simple, mainly because the case of a transposition is just negation.

Lemma 17.3

Let $\text{sign} : S_n \to \{\pm 1\}$ be the sign of a permutation. The S_n action preserves

$$\Lambda^n_{\text{sub}}(V) \subseteq V^{\otimes n}$$

and actually, for all $w \in \Lambda^n_{\text{sub}}(V)$, we have $\sigma \cdot (w) = \text{sign}(\sigma) \cdot w$.

Proof. Because S_n is generated by transpositions, it's enough to prove the lemma in the special case where $\sigma = (i\ i+1)$. We'll show that for any

$$w \in V^{\otimes(i-1)} \otimes \Lambda^2_{\text{sub}}(v) \otimes V^{\otimes(n-i-1)}$$

we in fact have $\sigma(w) = -w$. The conclusion will follow because this is a subset of the massive intersection which is $\Lambda^n_{\text{sub}}(v)$.

And now we're basically done, because we only look at the middle $\Lambda^2_{\text{sub}}(v)$. In fact we're basically looking at the case $n = 2$, this is clear; note that

$$\sigma(v_i \otimes v_{i+1} - v_{i+1} \otimes v_i) = v_{i+1} \otimes v_i - v_i \otimes v_{i+1}. \qquad \square$$

In the homework, we will show that $\Lambda^n_{\text{sub}}(v)$ is equal to $\text{LH}^n = \{w \mid \sigma(w) = -w\}$ except in the case where the ground field has characteristic 2, since in that case

$$\sigma(v \otimes v) = v \otimes v = -v \otimes v \implies v \otimes v \in \text{LH}^n.$$

§17.3 The space $\Lambda^n_{\text{quot}}(V)$

Definition 17.4. Let us define $V^{\otimes n}_{\text{bad}} \subseteq V^{\otimes n}$ as the span of elements of the form

$$v_1 \otimes \dots \otimes v_{i-1} \otimes v \otimes v \otimes v_{i+2} \otimes \dots \otimes v_n.$$

Then, we define

$$\Lambda^n_{\text{quot}}(V) = V^{\otimes n} / V^{\otimes n}_{\text{bad}}.$$

We will let π denote the projection $V^{\otimes n} \to \Lambda^n_{\text{quot}}(V)$.

Lemma 17.5

$v \otimes w + w \otimes v \in V \otimes V$ is bad. Moreover, $v \otimes w \otimes v \in V^{\otimes 3}$ is bad. More generally, any element of the form

$$v_1 \otimes \dots \otimes v_n$$

is bad if $v_i = v_j$ for some $1 \le i < j \le n$.

Proof. It is equal to $(v + w) \otimes (v + w) - v \otimes v - w \otimes w$. Thus modulo bad guys, we have

$$v \otimes w \otimes v \overset{\text{modulo bad}}{\equiv} -v \otimes v \otimes w \in V_{\text{bad}}.$$

The general claim follows by repeating this swap operation multiple times. $\square$

Hence, like in life, there are more bad guys than one might initially suspect.

§17.4 The Wedge Product

Definition 17.6. We will write $(v_1 \wedge \dots \wedge v_n)_{\text{quot}}$ to denote $\pi(v_1 \otimes \dots \otimes v_n)$.

Note that we can swap two elements for a negation by what we knew above: explicitly

$$(v_1 \wedge \dots \wedge v_i \wedge \dots \wedge v_j \wedge \dots \wedge v_n)_{\text{quot}} = -(v_1 \wedge \dots \wedge v_j \wedge \dots \wedge v_i \wedge \dots \wedge v_n)_{\text{quot}}.$$

Corollary 17.7

Let V be an m-dimensional vector space. Then $\Lambda^n_{\text{quot}}(V)$ is spanned by the elements of the form

$$(e_{i_1} \wedge \dots \wedge e_{i_n})_{\text{quot}}$$

where $1 \le i_1 < \dots < i_n \le m$.

Now let us consider the composite map

$$\Lambda^n_{\text{sub}}(V) \to V^{\otimes n} \to \Lambda^n_{\text{quot}}(V).$$

In the case $n = 2$, we see this map gives

$$v_1 \otimes v_2 - v_2 \otimes v_1 = 2v_1 \wedge v_2.$$

Hence this map is an isomorphism *unless* k has characteristic 2; i.e. this is an isomorphism unless $2 = 0$.

So we need a different approach to an isomorphism...

§17.5 Constructing the Isomorphism

Now let us consider the map

$$T : V^{\otimes n} \to V^{\otimes n}$$

given by

$$v_1 \otimes \dots \otimes v_n \mapsto \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \left(v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}\right).$$

We will prove two facts about T.

Lemma 17.8

T kills the bad guys; i.e. V_{bad} is contained in $\ker T$.

Proof. We wish to show T kills the element $v_1 \otimes \dots \otimes v_i \otimes v_{i+1} \otimes \dots \otimes v_n$ whenever $v_i = v_{i+1}$. Let $s = (i\ i+1)$. Then in the sum over $\sigma \in S_n$, we see that the terms σ and $\sigma \circ s$ cancel each other out. $\square$

Lemma 17.9

$\operatorname{im} T \subseteq \Lambda^n_{\text{sub}}(V)$.

Proof. This is deferred to the homework. $\square$

Now from these two lemmas we can define a map $\tilde{T}$ from the quotient $\Lambda^n_{\text{quot}}(V)$ into $\Lambda^n_{\text{sub}}(V)$. In other words, we have the commutative diagram

$$\begin{tikzcd}
V^{\otimes n} \arrow[r, "T"] \arrow[d, "\text{project}"'] & V^{\otimes n} \\
\Lambda^n_{\text{quot}}(V) \arrow[r, "\tilde{T}"] & \Lambda^n_{\text{sub}}(V) \arrow[u, "\text{inclusion}"']
\end{tikzcd}$$

Choose a basis $e_1, \dots, e_m$ in V.

Proposition 17.10

Consider the spanning set

$$(e_{i_1} \wedge \dots \wedge e_{i_n})_{\text{quot}} \in \Lambda^n_{\text{quot}}(V)$$

for $1 \le i_1 < \dots < i_n \le m$. The images of such elements under $\tilde{T}$ are linearly independent.

Proof. Basically, recall that

$$\tilde{T}\left((e_{i_1} \wedge \dots \wedge e_{i_n})_{\text{quot}}\right. = \sum_{\sigma} \text{sign}(\omega)\left(e_{\sigma(1)} \otimes \dots \otimes e_{\sigma(n)}\right).$$

But the $e_{\sigma(1)} \otimes \dots \otimes e_{\sigma(n)}$ are all linearly independent, and there is no repetition. $\square$

Combining with an earlier observation, we obtain the following.

Corollary 17.11

The $(e_{i_1} \wedge \dots \wedge e_{i_n})$ are actually a basis for $\Lambda^n_{\text{quot}}(V)$.

Proof. We already saw earlier they were a spanning set. Because their images are linearly independent by the proposition, they elements themselves were linearly independent. $\square$

Now we are in a position to prove the following theorem.

Theorem 17.12

The map $\tilde{T}$ defined above is an isomorphism.

Note that since isomorphism take bases to bases, this theorem implies $(e_{i_1} \wedge \dots \wedge e_{i_n})$ is a basis of $\Lambda^n_{\text{sub}}(V)$. We'll prove this next time.

Definition 17.13. Now that we know $\Lambda^n_{\text{sub}}(V) \simeq \Lambda^n_{\text{quot}}(V)$, we will simply denote them both by $\Lambda^n(V)$.

Obviously the theorem gives $\dim(\Lambda^n(V)) = \binom{\dim V}{n}$.

§17.6 Why do we care?

These are notes from a discussion after class with JT (thanks!), and were not part of the lecture.

Let's motivate all of this. It's actually the case that the wedge product (particularly in physics) lets you talk about areas and volumes meaningfully.

Wikipedia motivates this well; see `http://en.wikipedia.org/wiki/Exterior_algebra#Motivating_examples`. The single best line I can quote is

$$(ae_1 + be_2) \wedge (ce_1 + de_2) = (ad - bc)\, e_1 \wedge e_2$$

which gives you the area of a parallelogram. More generally, any $v_1 \wedge \dots \wedge v_n$ reduces to a linear combination of guys like $e_{i_1} \wedge \dots \wedge e_{i_n}$, each corresponding to a hyper-volume that you assign to a certain set of basis elements.

So what does the algebra mean? Well, if we're interested in $v \wedge w$ expressing some notion of "area" we better have

$$v \wedge v = 0.$$

(You probably also want the relation $cv \wedge w = v \wedge cw$, but that follows from the fact that these are tensor products.) This is claim is actually equivalent to the relation $v \wedge w = -w \wedge v$ by writing the $0 = (v + w) \wedge (v + w)$. And the miracle is that the relation $v \wedge v = 0$ is all that you ever really need in order to get the nice geometry analog to work!

If you take this interpretation it's obvious that $v_1 \wedge \dots \wedge v_n$ ought to be zero if $v_i = v_j$ for some $i < j$, because you have zero hypervolume. The "bad" guys that Gaitsgory talks about are just the wedge combinations which have zero volume. Of course you mod out by these. And the homework problem about determinants is natural, because determinants are also ways of expressing hyper-volumes of maps.

So all that Λ_{sub} and Λ_{quot} is doing are expressing two different ways of making $v_1 \otimes \dots \otimes v_n$ formally into $v \wedge \dots \wedge v_n$. And the $\tilde{T}$ just shows you that the these notions are isomorphic.

§18 October 30, 2014

§18.1 Review

Recall that we defined $\Lambda^2_{\text{sub}}(v)$ and

$$\Lambda^n_{\text{sub}}(V) = \left(\Lambda^2_{\text{sub}}(V) \otimes V^{\otimes(n-2)}\right) \cap \dots \cap \left(V^{\otimes(n-2)} \otimes \Lambda^2_{\text{sub}}(V)\right)$$

and

$$V^{\otimes n}_{\text{bad}} \subseteq V^{\otimes n}$$

as the span of elements of the form

$$v_1 \otimes \dots \otimes v_{i-1} \otimes v \otimes v \otimes v_{i+2} \otimes \dots \otimes v_n.$$

Then, we define

$$\Lambda^n_{\text{quot}}(V) = V^{\otimes n}/V^{\otimes n}_{\text{bad}}$$

and let π denote the projection $V^{\otimes n} \to \Lambda^n_{\text{quot}}(V)$.

We see that if $e_1, \dots, e_m$ is a basis of V then

$$e_{i_1} \wedge \dots \wedge e_{i_n} \in \Lambda^n_{\text{quot}}(V)$$

is the basis of $\Lambda^n_{\text{quot}}(V)$ (and henceforth I'm going to drop the $(\cdot)_{\text{quot}}$).

We have an averaging map $\text{Av}^{\text{sign}} : V^{\otimes n} \to V^{\otimes n}$ by

$$v_1 \otimes \dots \otimes v_n = \sum_{\sigma \in S_n} \text{sign}(\sigma) \left(V_{\sigma(1)} \otimes \dots \otimes v_{\sigma(s)}\right).$$

In the homework we see that we have a commutative diagram

$$\begin{tikzcd} V^{\otimes n} \arrow[r, "\text{Av}^{\text{sign}}"] \arrow[d, "\pi"] & V^{\otimes n} \\ \Lambda^n_{\text{quot}}(V) \arrow[r, "\widetilde{\text{Av}}^{\text{sign}}"'] & \Lambda^n_{\text{sub}}(V) \arrow[u, "\subseteq"'] \end{tikzcd}$$

Last time we saw that $\widetilde{\text{Av}}^{\text{sign}}$ sent the basis elements of $\Lambda^n_{\text{quot}}(V)$ to a linearly independent collection. This gave a couple nice corollaries.

Today we will prove that

$$\dim \Lambda^n_{\text{sub}}(V) = \dim \Lambda^n_{\text{quot}}(V).$$

This will complete the proof of the theorem we left out last time.

We first need to recall something though. Suppose we have $W_1 \subseteq W_2$, and we have an inclusion $i : W_1 \to W_2$. Recall that we define $W_1^\perp \subseteq W_2^\vee$ by

$$W_1^\perp = \ker i^\vee = \left\{\xi \in W_2^\vee \mid \xi(w) = 0 \forall w_1 \in W_1\right\}.$$

§18.2 Completing the proof that $\Lambda^n_{\text{sub}}(V) = \Lambda^n_{\text{quot}}(V)$

Notice that we have spaces

$$\Lambda^n_{\text{sub}}(V^\vee) \subseteq (V^\vee)^{\otimes n}$$

and

$$\left(\Lambda^n_{\text{quot}}(V)\right)^\vee \subseteq (V^{\otimes n})^\vee.$$

Of course, $(V^\vee)^{\otimes n} \simeq (V^{\otimes n})^\vee$.

Proposition 18.1

The spaces above coincide, meaning that

$$\Lambda^n_{\text{sub}}(V^\vee) \simeq \left(\Lambda^n_{\text{quot}}(V)\right)^\vee.$$

This will allow us to deduce equality of dimensions, because there is a non-canonical isomorphism between $\Lambda^n_{\text{quot}}(V)^\vee$ and $\Lambda^n_{\text{quot}}(V)$, so the dimensions match up.

Let us illustrate the case for $n = 2$ first. We wish to show that

$$(V^\vee)^{\otimes 2} \supseteq \Lambda^2_{\text{sub}}(V^\vee) = \left(V^{\otimes 2}_{\text{bad}}\right)^\perp \subseteq (V^{\otimes 2})^\vee.$$

We can do this explicitly by writing $e_1, \dots, e_n$ a basis for V and $e_1^\vee, \dots, e_n^\vee$ a basis of $V^\vee$. Then the elements of $(V^\vee)^{\otimes 2}$ are linear combinations of $\sum_{i,j} a_{ij} e_i^\vee \otimes e_j^\vee$. Now

$$V^{\otimes 2}_{\text{bad}} = \operatorname{Span}\left(e_i \otimes e_i, e_i \otimes e_j + e_j \otimes e_i\right).$$

and we're interested in which $A = \sum_{i,j} a_{ij} e_i^\vee \otimes e_j^\vee$ which kill all those guys in $V^{\otimes 2}_{\text{bad}}$; this is equivalent to killing the elements above that span $V^{\otimes 2}_{\text{bad}}$. Compute

$$A(e_i \otimes e_i) = a_{ii}$$

and

$$A\left(e_i \otimes e_j + e_j \otimes e_i\right) = a_{ij} + a_{ji}$$

so the criteria for membership in $(V^{\otimes 2}_{\text{bad}})^\perp$ is simply $a_{ii} = 0$ and $a_{ij} = -a_{ji}$, coinciding with $\Lambda^2_{\text{sub}}(V)$.

Let's prove the proposition now for the general case. First, we need a couple lemmas.

Lemma 18.2

Let W_i be subspaces of W. Consider $\sum_i W_i = \{\sum_i w_i \mid w_i \in W_i\}$. Then

$$\left(\sum W_i\right)^\perp \subseteq W^\vee$$

is equal to $\bigcap W_i^\perp$.

Proof. This is just saying that to annihilate $\sum_i W_i$ is equivalent to annihilating each individual W_i. □

Observe that we used this lemma above already, when we only consider $e_i \otimes e_i$ and $e_i \otimes e_j + e_j \otimes e_i$.

Lemma 18.3

Let $W_1 \subseteq W_2$, and U a finite-dimensional subspace. Then the spaces

$$(W_1 \otimes U)^\perp \subseteq (W_2 \otimes U)^\vee$$
$$W_1^\perp \otimes U^\vee \subseteq W_2^\vee \otimes U^\vee$$

are isomorphic. (Notice that $(W_2 \otimes U)^\vee \simeq W_2^\vee \otimes U^\vee$.)

Proof. Take a basis of U. $\square$

Proof of Proposition. We need only show that

$$\left(V^{\otimes i} \otimes V_{\text{bad}}^{\otimes 2} \otimes V^{n-i-2}\right)^\perp = \left(V^\vee\right)^{\otimes i} \otimes \Lambda^2_{\text{sub}}(V^\vee) \otimes \left(V^\vee\right)^{\otimes(n-2-i)}.$$

but the lemma tells us that the left-hand side is equal to

$$(V^\vee)^{\otimes i} \otimes (V_{\text{bad}}^{\otimes 2})^\perp \otimes (V^\vee)^{\otimes(n-2-i)}.$$

Handling the middle term is just the $n = 2$ case. $\square$

§18.3 Wedging Wedges

What follows is preparation for differential forms, three months from now.

Lemma 18.4

There us a unique map $\wedge : \Lambda^{n_1}(V) \otimes \Lambda^{n-2}(V) \to \Lambda^{n_1+n_2}(V)$ which makes the following diagram commute.

$$\begin{tikzcd}
\Lambda^{n_1}(V) \otimes \Lambda^{n-2}(V) \arrow[r, "\wedge"] & \Lambda^{n_1+n_2}(V) \\
V_1^{\otimes n_1} \otimes V^{\otimes n} \arrow[u] \arrow[r, "\simeq"] & V^{\otimes(n_1+n_2)} \arrow[u]
\end{tikzcd}$$

So we're thinking about the Λ_{quot} interpretation of Λ.

Proof. The proof of uniqueness is immediate from the fact that $V^{\otimes n_1} \otimes V^{\otimes n_2} \to \Lambda^{n_1}(V) \otimes \Lambda^{n_2}(V)$ is surjective.

Now for existence. The obvious proof is to just define $\wedge$ on basis elements by

$$(e_{i_1} \wedge \cdots \wedge e_{i_n}) \wedge (e_{j_1} \wedge \cdots \wedge e_{j_m}) \mapsto (e_{i_1} \wedge \cdots \wedge e_{i_n} \wedge e_{j_1} \wedge \cdots \wedge e_{j_m})$$

and check the diagram commutes on the basis, which is clear. But Gaitsgory says that this proof sucks because it defines things on basis elements.

A second proof of existence is as follows.

Lemma 18.5

Consider short exact sequences of R-modules

$$0 \to I_1 \to M_1 \to N_1 \to 0$$

$$0 \to I_2 \to M_2 \to N_2 \to 0$$

and consider the map

$$\Psi : (I_1 \otimes M_2) \oplus (M_1 \otimes I_2) \to M_1 \otimes M_2$$

using the inclusions $I_1 \otimes M_2 \to M_1 \otimes M_2$ and $M_1 \otimes I_2 \to M_1 \otimes M_2$. Then the kernel of the map $M_1 \otimes M_2 \to N_1 \otimes N_2$ is contained in the image of Ψ.

Proof. Homework. ■

Now we can apply the lemma on short exact sequences

$$0 \to V_{\text{bad}}^{\otimes n_1} \to V^{\otimes n_1} \to \Lambda^n_{\text{quot}}(V) \to 0$$

$$0 \to V_{\text{bad}}^{\otimes n_2} \to V^{\otimes n_2} \to \Lambda^n_{\text{quot}}(V) \to 0$$

By the lemma, we can write

$$\Lambda^{n_1}_{\text{quot}}(V) \otimes \Lambda^{n_2}_{\text{quot}}(V) = \operatorname{coker}\left(V_{\text{bad}}^{n_1} \otimes V^{\otimes n_2} \oplus V_1^{\otimes n_1} \otimes V_{\text{bad}}^{n_2} \to V^{\otimes n_1} \otimes V^{\otimes n_2}\right).$$

□

Lemma 18.6

The diagram

$$\begin{tikzcd}
\Lambda^{n_1}(V) \otimes \Lambda^{n_2}(V) \arrow[r] \arrow[d, "\text{swap} \cdot (-1)^{n_1 n_2}"] & \Lambda^{n_1+n_2}(V) \arrow[d, "\simeq"] \\
\Lambda^{n_2}(V) \otimes \Lambda^{n_1}(V) \arrow[r] & \Lambda^{n_2+n_1}(V)
\end{tikzcd}$$

Here swap is just a standard isomorphism $v \otimes w \mapsto w \otimes v$.

By convention, we now define $\Lambda^0(W) = k$ and $\Lambda^1(W) = W$.

Lemma 18.7

Consider the map

$$\bigoplus_{k=0}^{n} \Lambda^k(V_1) \otimes \Lambda^{n-k}(V_2) \to \Lambda^n(V)$$

by sending each $\Lambda^k(V) \otimes \Lambda^{n-k}(V)$ to $\Lambda^n(V)$ using the wedge. Then this map is an isomorphism.

Proof. Basis bash. □

§19 November 4, 2014

Representation Theory.

In this lecture, G is a fixed group, and k is a fixed field.

§19.1 Representations

Definition 19.1. A **representation** of G is a *pair*

$$\rho = (V, \psi)$$

where V is a finite dimensional vector space over k, and $\psi : G \to \mathrm{Aut}_k(V)$ is an action of G on V.

Hence $\psi(g)$ is an automorphism of V, and $g \cdot v = \psi(g)(v)$ is the associated action.

Definition 19.2. A homomorphism of representations $\rho_i = (V_i, \psi_i)$ for $i = 1, 2$ is a map $T : V_1 \to V_2$ such that

$$\psi_2(g) \circ T = T \circ \psi_1(g)$$

holds for all $g \in G$.

Example 19.3

If $V = W^{\otimes n}$, and $G = S_n$, then S_n acts by permutation on W. So the $(W^{\otimes n}, S_n \curvearrowright W^{\otimes n})$ is a representation.

Example 19.4

If $V = k^n$, then S_n acts on V by permutation the basis elements.

Another simple example is the zero representation.

Example 19.5

The **trivial representation**, denoted triv, is the representation $V = k$ and the trivial action on G (i.e. G takes every automorphism to itself). In other words, $\mathrm{triv} = (k, 1_{\mathrm{Act}})$.

§19.2 Group Actions, and Sub-Representations

Suppose $G \curvearrowright X$.

Definition 19.6. For a set X, we define a vector space

$$\mathrm{Fun}(X) : \{\text{maps } X \to k\}.$$

We let $\mathrm{Fun}_c(X)$ be the subset of $\mathrm{Fun}(X)$ consisting of those maps with finite support, meaning only finitely many inputs give nonzero outputs. It is spanned by basis elements δ_*.

Then we can define an action $G \curvearrowright \mathrm{Fun}(X)$ by

$$(g \cdot f)(x) = f(g^{-1} \cdot x).$$

One can check this works. Thus we can also view $\mathrm{Fun}(X)$ as a representation of G. In what follows we will abuse notation and also let $\mathrm{Fun}(X)$ denote the corresponding representation.

Remark 19.7. All representations with $V = k$ are in bijection with $\mathrm{Hom}_{\mathrm{Grp}}(G, \mathrm{Aut}_k(k)) = \mathrm{Hom}_{\mathrm{Grp}}(G, k^*)$ where k^* is the nonzero elements of k.

Definition 19.8. We say ρ' is a **subrepresentation** of ρ if V' is realized as a subspace of V compatible with the G-action.

Example 19.9

$\mathrm{Fun}_c(X)$ is a subrepresentation of $\mathrm{Fun}(X)$.

Lemma 19.10

If f has finite support, then so does ${}^g f$.

Remark 19.11. Earlier we remarked that S_n acts on $V = k^n$, which gives rise to a representation of G. We can view this as the special case $\mathrm{Fun}(\{1, \dots, n\})$.

§19.3 Invariant Subspaces

Definition 19.12. Let $\rho = (V, G \to \mathrm{Aut}(V))$. We define the **invariant** space

$$\rho^G = \{v \mid g \cdot v = v \; \forall g \in G\}.$$

Lemma 19.13

$\rho^G \simeq \mathrm{Hom}_G(\mathrm{triv}, \rho)$.

Proof. Recall $\mathrm{triv} = (k, 1_{\mathrm{Act}})$. Let $\rho = (V, \psi)$. We identify $\mathrm{Hom}(k, V)$ with V by sending each $v \in V$ to the map $a \mapsto a \cdot v$. Observe that $\mathrm{Hom}_G(\mathrm{triv}, \rho) \subseteq \mathrm{Hom}(k, V) \simeq V$.

For a fixed $a \in k$, the corresponding element of $\mathrm{Hom}(k, V)$ is $a \to a \cdot v$. We want to check $g(av) = av$, which is precisely ρ^G. $\square$

§19.4 Covariant subspace

Definition 19.14. Let $\rho = (G, \psi)$. Define the **covariant subspace** is

$$\rho_G = V / \mathrm{Span}(g \cdot v - v)$$

i.e. it is the quotient of the map $v \to \rho_G$.

Definition 19.15. Define the **dual representation** of a map $\rho = (V, \psi)$ by $\rho^\vee = (V^\vee, \psi')$ where $\psi'(g) \overset{\text{def}}{=} (\psi(g^{-1}))^\vee$.

Lemma 19.16
$(\rho_G)^\vee \simeq (\rho^\vee)^G$.

This is motivated by noting that on the homework, we already had $G = S_n$, $\rho = V^{\otimes n}$, $\rho^G = \mathrm{Sym}^n_{\mathrm{sub}}(V)$ and $\rho_G = \mathrm{Sym}^n_{\mathrm{quot}}(V)$.

Proof. We have $(\rho^\vee)^G \subseteq V^\vee$ and $(\rho_G)^\vee = \mathrm{Hom}(\rho_G, k) \subseteq \mathrm{Hom}(V, k) = V^\vee$. We wish to check that

$$(\psi(g^{-1}))^\vee(\xi) = \xi \ \forall g$$

For all $v \in V$, we have

$$\left\langle (\psi(g^{-1}))^\vee(\xi), v \right\rangle = \langle \xi, v \rangle .$$

By definition, we get $\left\langle \xi, (\psi(g^{-1}))(v) \right\rangle = \langle \xi, v \rangle$. So $\left\langle \xi, \psi(g^{-1}) \cdot v - v \right\rangle = 0$ for all $v \in V$, $g \in G$. ☐

§19.5 Quotient spaces and their representations

Let $H \le G$, $X = G/H$.

Lemma 19.17
$\mathrm{Hom}_G(\mathrm{Fun}_c(G/H), \rho) \simeq \rho^H$.

Proof. Consider a map $T : \mathrm{Fun}_c(G/H) \to V$. We will map $\delta_{\overline{1}} \mapsto T(\delta_{\overline{1}}) \in V$. Here $\delta_{\overline{1}}$ is the function on G/H which sends $\overline{1}$ to $1 \in k$ and everything else to $0 \in k$. Letting $v = T(\delta_{\overline{1}})$, we can check that

$$h \cdot \delta_{\overline{1}} = \delta_{\overline{h}} = \delta_{\overline{1}}.$$

Hence $v \in \rho^H$.

Conversely, suppose $v \in \rho^H$. Then $v \in V$, $h \cdot v = v$. We select the function T sending $T(\delta_{\overline{g}}) = g \cdot v$. You can verify this is well-defined. Proving that it's a two-sided inverse is homework. ☐

Lemma 19.18
$\mathrm{Hom}_G(\rho, \mathrm{Fun}(G/H)) \simeq (\rho_H)^\vee$.

Proof. Homework. ☐

§19.6 Tensor product of representations

Definition 19.19. Let $\rho_1 = (V_1, \psi_1)$ and $\rho_2 = (V_2, \psi_2)$ be representations of G_1 and G_2, respectively. We define $\rho_1 \boxtimes \rho_2$ a representation of $G_1 \times G_2$ by

$$\rho_1 \boxtimes \rho_2 = (V_1 \otimes V_2, \psi_1 \otimes \psi_2) .$$

Here $(\psi_1 \otimes \psi_2)(g_1, g_2)(v_1 \otimes v_2) = \psi(g_1)(v_1) \otimes \psi_2(g_2)(v_2)$. In other words, $(g_1, g_2) \cdot (v_1 \otimes v_2) = (g_1 \cdot v_1) \otimes (g_2 \cdot v_2)$.

Definition 19.20. We define the **tensor representation product** of two representations ρ_1, ρ_2 of G as follows. It is the restriction of $\rho_1 \boxtimes \rho_2$ over $G \times G$ to the diagonal G. We denote this by $\rho_1 \otimes \rho_2$.

Equivalently, $\rho_1 \otimes \rho_2 = (V_1 \otimes V_2, \bullet)$ is a representation of G where $g \cdot (v_1 \otimes v_2) = g \cdot v_1 \otimes g \cdot v_2$.

Now here comes the construction of which the mortality rate is 70%. We use a double underline (!?).

Definition 19.21. We define a representation of $G_1 \times G_2$ by

$$\underline{\underline{\operatorname{Hom}}}(\rho_1, \rho_2) = \left(\operatorname{Hom}(V_1, V_2), (g_1, g_2) \cdot T = g_2 \cdot T \circ g_1^{-1}\right).$$

In other words, the action by (g_1, g_2) sends T to $\psi(g_2) \circ T \circ \psi(g_1)$.

Definition 19.22. If ρ_1 and ρ_2 are representations of G, then we define the **internal Hom** by

$$\underline{\operatorname{Hom}}(\rho_1, \rho_2) = \left(\operatorname{Hom}(V_1, V_2), g \cdot T = g^{-1} \cdot T \circ g\right).$$

In other words, the action sends g to $T \mapsto \psi_2(g^{-1}) \circ T \circ \psi_1(g)$.

The following construction has a 70% mortality rate.

Lemma 19.23

We have

$$\operatorname{Hom}_G(\rho_1, \rho_2) = \left(\underline{\operatorname{Hom}}(\rho_1, \rho_2)\right)^G.$$

Proof. Both sides live inside $\operatorname{Hom}(V_1, V_2)$. A given $T : V_1 \to V_2$ lives on the left-hand side if and only if $\psi_2(g) \circ T = T \circ \psi_1(g)$ for every g. It lives in the right-hand side if and only if $\psi_2(g^{-1}) \circ T \circ \psi_1(g)$ holds for every g. $\square$

§20 November 6, 2014

Protip: a representation is a functor from the category corresponding to a group to vector spaces over k. Then $\underline{\text{Hom}}$ represents natural transformations, and we have an isomorphism

$$\rho_1^\vee \otimes \rho_2 \to \underline{\text{Hom}}(\rho_1, \rho_2).$$

§20.1 Representations become modules

Definition 20.1. Let $k[G]$ be the set $\text{Fun}_c(G)$ endowed with a ring structure as follows. Addition is done canonically, and multiplication is done by noting $\text{Fun}_c(G) = \text{Span}(\delta_g : g \in G)$ and

$$\delta_{g_1} \cdot \delta_{g_2} = \delta_{g_1 g_2}.$$

Lemma 20.2

Representations of G correspond to $k[G]$ modules.

Hence it what follows we will consider modules and representations interchangably.

§20.2 Subrepresentations

Definition 20.3. If $\rho = (V, \psi)$ is a representation, a **subrepresentation** is a representation $\rho' = (V', \psi|_{V'})$, where V' is a subspace of V. Notice that for this definition to make sense we require that V' is invariant under ψ; in other words, V' is G-invariant. We write $\rho' \leq \rho$ to mean ρ' is a subrepresentation of ρ.

Definition 20.4. We say ρ is **irreducible** if it has no nontrivial subrepresentations.

Example 20.5

Let $G = S_3$, and let ρ act on k^3. Then $V' = \text{Span}\left(\langle 1,1,1 \rangle\right)$ gives a canonical subrepresentation.

Lemma 20.6

A representation ρ is irreducible if and only if for all nonzero $v \in V$, $\text{Span}(\{g \cdot v \mid g \in G\}) = V$.

Proof. The set $\text{Span}\left(\{g \cdot v \mid g \in G\}\right)$ is G-invariant. Meow. $\square$

Example 20.7

As above, take ρ with $V = k^3$ and $G = S_3$. Let

$$V' = \{(a, b, c) \mid a + b + c = 0\}.$$

One can check that this gives a nontrivial subrepresentation by showing that $\text{Span}(g \cdot v) = V$, except in characteristic 3.

§20.3 Schur's Lemma

Definition 20.8. A module is **irreducible** if and only if it has no nontrivial submodules.

The following guy is REALLY important.

Theorem 20.9 (Schur's Lemma)

Suppose k is algebraically closed, and let R be a k-algebra. Let M be an irreducible R-module which is finite dimensional as a k-vector space. Let T be an R-endomorphism of M. Then T is multiplication by some $\lambda \in k$.

Proof. There exists an eigenvalue $\lambda \in k$ such that $0 \neq M^\lambda = \{m \in M \mid Tm = \lambda m\}$. Then M^λ is a submodule so $M^\lambda = M$. $\square$

Corollary 20.10

If k is algebraically closed then ρ is irreducible over a finite dimensional vector space, then $\mathrm{End}_G(\rho) = k$.

Example 20.11

This can fail in the case of non-algebraically closed fields. Let $k = \mathbb{R}$, $G = \mathbb{Z}_3$, and let ρ be a representation on the vector space $V = \mathbb{R}^2$ where G acts on $v \in V$ by rotating by $120°$. By interpreting this as complex by multiplication $e^{\frac{2\pi i}{3}}$ we see that the $\mathbb{R}^2$-linear maps which commute with ρ are just multiplication by another complex number. In other words,

$$\mathrm{End}_G(\rho, \rho) \simeq \mathbb{C}.$$

Corollary 20.12

Let R be a commutative k-algebra, where k is algebraically closed. Then any irreducible R-module M is a one-dimensional over k and is given by the set of maps f which obey the following commutative diagram.

$$\begin{tikzcd}
R \arrow[r, "f"] & k \\
k \arrow[u] \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

An example is $R = k[t]$. In this case f is evaluation.

Proof. Because R is commutative, every $r \in R$ leads to an endomorphism of M by the left action of M. Hence by Schur's lemma each r corresponds to some $\chi(r) \in k$ in which this left action is multiplication by $\chi(r)$. In other words, each $m \mapsto r \cdot m$ can be written as $m \mapsto \chi(r)m$ for some $\chi(r) \in k$. $\square$

Corollary 20.13

Let G be commutative. Then any irreducible G-representation which is finite dimensional as a vector space is of the form

$$\rho = (k, \chi : G \to k^*).$$

Proof. Immediate from previous corollary. $\square$

§20.4 Splittings

Theorem 20.14 (Maschke's Theorem)

Let k be a field whose characteristic does not divide $|G|$. Then any finite dimensional representation is a direct sum of irreducibles.

Example 20.15

A counterexample if the characteristic condition is violated is $G = \mathbb{Z}_2$ and $\operatorname{char} k = 2$.

The next example uses the following lemma.

Lemma 20.16

Let ρ be irreducible, and let π be another representation.

(a) If $\pi \to \rho$ is nonzero, then it is surjective.

(b) If $\rho \to \pi$ is nonzero, then it is injective.

Example 20.17

Suppose ρ is a representation of $\mathbb{Z}_2$ over the vector field k^2 and with G being the permutation on two elements. This gives a short exact sequence

$$0 \to \mathrm{triv} \to \rho \to \mathrm{triv} \to 0.$$

But we claim ρ does not decompose. Assume for contradiction that $\rho = \rho_1 \oplus \rho_2$ (sufficient since ρ is two-dimensional).

Assume $\rho_1 \to \mathrm{triv}$ is non-zero. We find that $\rho \to \mathrm{triv}$ is an isomorphism. So the short exact sequence splits, and there exists $v \in k^2$ which is G-invariant which maps non-trivially to triv. But for v to be G-invariant it must be of the form (a, a).

Now we actually write the following.

Theorem 20.18

If $\operatorname{char} k \nmid |G|$, then any short exact sequence of G-representations splits.

Proof. We have a surjection $\rho \to \rho'$ and wish to construct a left inverse. ... $\square$

Proof that Theorem 20.18 implies Theorem 20.14. Assume ρ is not irreducible. Let $0 \neq \rho_1 \subsetneq \rho$. Then we have a short exact sequence

$$0 \to \rho_1 \to \rho \to \rho_2 \to 0.$$

Hence $\rho = \rho_1 \oplus \rho_2$. Now induct downwards on the ρ_1 and ρ_2. $\square$

Now we introduce the following theorem.

Theorem 20.19

If $\operatorname{char} k$ does not divide $|G|$, then for any G-representation π the composed map

$$\pi^G \to V \to \pi_G$$

is an isomorphism.

Proof. We want an inverse map. We do

$$\overline{v} \mapsto \sum_{g \in G} g \cdot v$$

where v is any representative of $\overline{v} \in \pi_G$. You can check using summation formulas that this in facts gives us what we want:

$$\begin{tikzcd} V \arrow[r, "\widetilde{\widetilde{\mathrm{Av}}}_G"] \arrow[d, "\mathrm{proj}"'] & V \\ \pi_G \arrow[r, "\widetilde{\mathrm{Av}}_G"] & \pi^G \arrow[u, hook, "\mathrm{incl}"'] \end{tikzcd}$$

This is not actually an inverse because it turns out to be multiplication by $|G|$, but if $|G| \neq 0$ then we can divide by $|G|$. In other words, the desired map is in fact

$$\overline{v} \mapsto \frac{1}{|G|} \sum_{g \in G} g \cdot v$$

Done. $\square$

Also, nice pizza.

§20.5 Table of Representations

God there is so much notation. Remember that $G \hookrightarrow G \times G$ by $g \mapsto (g, g^{-1})$.

Representation	Group	Space	Action
ρ	V	G	$G \to \mathrm{Aut}(V)$
$\rho^\vee$	$V^\vee$	G	$(g \cdot \xi)(v) = \xi(g^{-1} \cdot_\rho v)$
$\mathrm{Fun}(X)$	G	$\mathrm{Fun}(X)$	$(g \cdot f)(x) = f(g^{-1} \cdot x)$
$\mathrm{Fun}_c(X)$	G	$\mathrm{Fun}_c(X)$	$(g \cdot f)(x) = f(g^{-1} \cdot x)$
$\mathrm{Reg}(G)$	$G \times G$	$\mathrm{Fun}(G, k)$	$((g_1, g_2) \cdot f)(x) = f(g_2 x g_1^{-1})$
${}^f\,\mathrm{Reg}(G)$	$G \times G$	${}^f\,\mathrm{Fun}(G, k)$	$((g_1, g_2) \cdot f)(x) = f(g_2 x g_1^{-1})$
triv_G	G	k	$g \cdot a = a$
$\mathrm{Res}^G_H(\rho)$	H	V	$h \cdot v = h \cdot_\rho v$
$\mathrm{Ind}^G_H(\rho)$	G	$f : G \to V$ with $f(g \cdot h) = h^{-1} \cdot_\rho f(g)$	$(g \cdot f)(x) = f(g^{-1} \cdot x)$
$\rho_1 \boxtimes \rho_2$	$G_1 \times G_2$	$V_1 \otimes V_2$	$(g_1, g_2) \cdot (v_1 \otimes v_2)$ $= (g_1 \cdot_{\rho_1} v_1) \otimes (g_2 \cdot_{\rho_2} v_2)$
$\rho_1 \oplus \rho_2$	G	$V_1 \oplus V_2$	$g \cdot (v_1 + v_2) = (g \cdot_{\rho_1} v_1) + (g \cdot_{\rho_2} v_2)$
$\rho_1 \otimes \rho_2$	G	$V_1 \otimes V_2$	$g \cdot (v_1 \otimes v_2) = (g \cdot_{\rho_1} v_1) \otimes (g \cdot_{\rho_2} v_2)$
$\underline{\underline{\mathrm{Hom}}}(\rho_1, \rho_2)$	$G_1 \times G_2$	$\mathrm{Hom}(V_1, V_2)$	$(g_1, g_2) \cdot T = g_2 \cdot T \circ g_1^{-1}$
$\underline{\mathrm{Hom}}(\rho_1, \rho_2)$	G	$\mathrm{Hom}(V_1, V_2)$	$g \cdot T = g^{-1} \cdot T \circ g$

§20.6 Induced and Restricted Representations

The above table presented $\mathrm{Ind}^G_H(\rho)$ and $\mathrm{Res}^G_H(\rho)$, which I haven't actually defined yet. We didn't cover this during the lecture, so I'll do that here.

Given a representation ρ of H, the representation $\mathrm{Ind}^G_H(\rho)$ is a representation of G. It consists of functions $f : G \to V$ (which should be thought of as "vectors" whose components are indexed by G), with the property that

$$f(gh) = h^{-1} \cdot_\rho f(g) \quad \forall g \in G, h \in H.$$

This condition means that the datum of f is equivalent to specifying f on each coset gH; in other words, as a vector space the set of functions is isomorphic to $V^{\oplus G/H}$. (Again, the "vector" interpretation is the right way to think of this.) The action then translates the components of the vector:

$$(g \cdot f)(x) = f(g^{-1}x).$$

This gives a way of *inducing* any representation of H to one of G, through augmenting the vector space by a factor of G/H.

Conversely, $\mathrm{Res}^G_H(\pi)$ is more tautological: given a G-representation π, we simply forget the action of $G \setminus H$, giving an H-representation. In other words, we *restrict* the action from G to H.

§21 November 11, 2014

I went to sleep instead of attending this lecture. Thanks again to Wyatt for the notes.

§21.1 Review

There were two important theorems:

- Schur's Lemma: If ρ is irreducible, on a finite dimensional vector space, and k is algebraically closed, then $k \xrightarrow{\sim} \operatorname{End}_G(\rho)$.
- Maschke's Theorem: If G is finite and $\operatorname{char} k \nmid |G|$, then any representation may be written as a direct sum of irreducible representations.

Recall also the following two definitions.

Definition 21.1. The **matrix coefficient** map of a representation ρ is denoted $\mathrm{MC}_\rho : V \otimes V^\vee \to \operatorname{Fun}(G, k)$ and is defined by sending $v \otimes \xi$ to $g \mapsto \xi(g \cdot_\rho v)$.

Definition 21.2. The **character** of a representation ρ is the trace of the matrix induced when g acts on V. In symbols, $\operatorname{ch}_\pi(g) = \operatorname{Tr}(T_g)$, where $T_g : V \to V$ by $v \mapsto v \cdot_\rho g$, and hence $\operatorname{ch}_\pi : G \to k$.

The next piece of notation is from a PSet problem, which we will invoke later on.

Definition 21.3. Let V be a finite-dimensional vector space. Then the canonical map $V \otimes V^\vee \to \operatorname{End}(V)$ by $v \otimes \xi \mapsto (w \mapsto \xi(w) \cdot v)$ is an isomorphism. Therefore there exists an element $u_V \in V \otimes V^\vee$ which is the pre-image of $\mathrm{id}_V \in \operatorname{End}(V)$.

§21.2 Homework Solutions

First, we give solutions to Problems 7 and 9 on the previous PSet.

PSet 10, Problem 7

> Let $\psi : k[G] \to \operatorname{Hom}(V, V)$ be the map for which the isomorphism
>
> $$\operatorname{Hom}_{G\times G}({}^f\mathrm{Reg}(G), \underline{\underline{\operatorname{Hom}}}(\pi,\pi)) \simeq \operatorname{Hom}_G(\pi_1, \pi_2)$$
>
> sends ψ to id_V. Show that ψ corresponds to the action of $k[G]$ on V.

Unravelling the proof of Proposition 8.2.3 and Proposition 6.1.4 in the Math 123 Notes, we see that ψ is given by

$$\psi(\delta_g) = T_{\mathrm{id}_V}(\delta_g) = T_{\mathrm{id}_V}(\delta_{\overline{(1,g)}}) = (1, g) \cdot \mathrm{id}_V \in (\underline{\operatorname{Hom}}(\pi,\pi))^G.$$

The endomorphism $(1, g) \cdot \mathrm{id}_V$ sends each v to $\mathrm{id}_V(g \cdot v \cdot 1^{-1}) = g \cdot v$, which is the precisely the action of δ_g on V.

PSet 10, Problem 9

Show that $\mathrm{MC}_\pi(u_V) = \mathrm{ch}_\pi$ for any representation π.

First, we claim that the diagram

$$\begin{tikzcd}
V \otimes V^\vee \arrow[r, leftrightarrow, "\simeq"] & \mathrm{End}(V) \\
V \otimes V^\vee \arrow[u, "T \otimes \mathrm{id}_{V^\vee}"] \arrow[r, leftrightarrow, "\simeq"] & \mathrm{End}(V) \arrow[u, "S \mapsto T \circ S"']
\end{tikzcd}$$

commutes. Indeed, we just compute

$$\begin{tikzcd}
T(w) \otimes \xi \arrow[r, mapsto, "\simeq"] & v \mapsto \xi(v)T(w) \\
w \otimes \xi \arrow[u, mapsto, "T \otimes \mathrm{id}_{V^\vee}"] \arrow[r, mapsto, "\simeq"] & v \mapsto \xi(v) \cdot w \arrow[u, mapsto, "S \mapsto T \circ S"']
\end{tikzcd}$$

Now we consider the commutative diagram

$$\begin{tikzcd}
k & \\
V \otimes V^\vee \arrow[u, "\mathrm{ev}"] \arrow[r, leftrightarrow, "\simeq"] & \mathrm{End}(V) \arrow[ul, "\mathrm{Tr}"'] \\
V \otimes V^\vee \arrow[u, "T \otimes \mathrm{id}_{V^\vee}"] \arrow[r, leftrightarrow, "\simeq"] & \mathrm{End}(V) \arrow[u, "S \mapsto T \circ S"']
\end{tikzcd}$$

and take the special case T_g, which gives

$$\begin{tikzcd}
\mathrm{ev}(T_g \otimes \mathrm{id}_{V^\vee})(u_V) = \mathrm{Tr}(T_g) & \\
(T_g \otimes \mathrm{id}_{V^\vee})u_V \arrow[u, mapsto, "\mathrm{ev}"] \arrow[r, mapsto, "\simeq"] & T_g \arrow[ul, mapsto, "\mathrm{Tr}"'] \\
u_V \arrow[u, mapsto, "T_g \otimes \mathrm{id}_{V^\vee}"] \arrow[r, mapsto, "\simeq"] & \mathrm{id}_V \arrow[u, mapsto, "S \mapsto T_g \circ S"']
\end{tikzcd}$$

But by definition, $\mathrm{ev} \circ (T_g \otimes \mathrm{id}_{V^\vee}) = \mathrm{MC}_\pi$, so we're done.

§21.3 A Theorem on Characters

The main result of today's lecture will be the following theorem.

Theorem 21.4

Let ρ_1 and ρ_2 be irreducible representations of G which have underlying vector spaces V_1 and V_2. Then

$$\sum_{g \in G} \mathrm{ch}_{\rho_1}(g)\mathrm{ch}_{\rho_2}(g^{-1}) = \begin{cases} 0 & \text{if } \rho_1 \not\simeq \rho_2 \\ |G| \dim(\mathrm{End}(\rho)) & \text{if } \rho_1 = \rho_2 = \rho. \end{cases}$$

Here by an integer n we mean $\underbrace{1 + \cdots + 1}_{n \text{ times}}$. In the words of Gaitsgory,

> "We can compare apples to oranges by taking the apples orange times."

The proof proceeds in two phases. First we will explicitly compute the sum $\sum_{g\in G} \mathrm{ch}_\pi(g)$, Secondly, we will show that the sum can be rewritten as $\sum_{g \in G} \mathrm{ch}_{\rho_1 \otimes \rho_2^\vee}(g)$ by passing through the matrix coefficient map; using the sum, this will solve the problem.

§21.4 The Sum of the Characters

It turns out we can explicitly compute the sum of the characters.

Theorem 21.5

Let π be any finite dimensional representation. Then

$$\sum_{g \in G} \mathrm{ch}_\pi(g) = |G| \dim \pi^G.$$

This in fact a special case of a more general theorem. First, recall the averaging function.

Definition 21.6. For a representation ρ of G, we have an associated averaging function $\mathrm{Avg}_G : V \to V$ by

$$\mathrm{Avg}_G^\rho(v) \overset{\text{def}}{=} \sum_g g \cdot_\rho v.$$

Theorem 21.7

Let π be any finite dimensional representation over V. Let $S : V \to V$ be an endomorphism. Then

$$\sum_{g \in G} \mathrm{MC}_\pi(\overline{S})(g) = \mathrm{Tr}\left(S \circ \mathrm{Avg}_G^\pi\right).$$

Here $\overline{S} \in V \otimes V^\vee$ is such that $\overline{S} \mapsto S$ under the isomorphism $V \otimes V^\vee \simeq \mathrm{End}(V)$.

Proof. Tautological since $\mathrm{MC}_\pi(S)(g) = \mathrm{Tr}(S \circ g)$ by definition. $\square$

Now to deduce the original theorem, we now pick $S = \mathrm{id}_V$, so that $\overline{S} = u_V$. Then by PSet 10, Problem 9, we get $\mathrm{ch}_\pi(g) = \mathrm{MC}_\pi(u_V)(g)$. Therefore, the problem is solved once we can prove the following lemma.

Lemma 21.8

We have

$$\operatorname{Tr}\left(\operatorname{Avg}_G^{\pi}\right) = |G| \cdot \dim \pi^G.$$

for every finite-dimensional representation π of G.

Proof. Basis bash. A non-basis proof is on the homework. $\square$

§21.5 Re-Writing the Sum

First, we produce a lemma that allows us to eliminate the inverse signs.

Lemma 21.9

For all representations ρ, we have

$$\operatorname{ch}_\rho(g^{-1}) = \operatorname{ch}_{\rho^\vee}(g).$$

Proof. Consider the following commutative diagram.

$$\begin{tikzcd}
V^\vee \otimes V \arrow[r, leftrightarrow, "\simeq"] \arrow[d, "\text{eval}"'] & V \otimes V^\vee \arrow[r, leftrightarrow, "\simeq"] & (V^\vee)^\vee \otimes V^\vee \arrow[d, "\text{eval}"] \\
\operatorname{End}(V) \arrow[rr, "T \mapsto T^\vee"] \arrow[dr, "\operatorname{Tr}"] & & \operatorname{End}(V^\vee) \arrow[dl, "\operatorname{Tr}"'] \\
& k &
\end{tikzcd}$$

The fact that the outer diagram commutes implies that the inner diagram commutes. Now g acts by g^{-1} in $\operatorname{End}(V^\vee)$, implying the conclusion. $\square$

Next, we show that MC maps can be "combined".

Proposition 21.10

Let π_1 and π_2 be representations with underlying vector spaces V_1 and V_2. The following diagram commutes.

$$\begin{tikzcd}
(V_1 \otimes V_1^\vee) \otimes (V_2 \otimes V_2^\vee) \arrow[r, "\mathrm{MC}_{\pi_1} \otimes \mathrm{MC}_{\pi_2}"] \arrow[d, leftrightarrow, "\simeq"'] & \operatorname{Fun}(G) \otimes \operatorname{Fun}(G) \arrow[d, "\text{mult}"] \\
(V_1 \otimes V_2) \otimes (V_1 \otimes V_2)^\vee \arrow[r, "\mathrm{MC}_{\pi_1 \otimes \pi_2}"] & \operatorname{Fun}(G)
\end{tikzcd}$$

Proof. Tautological. $\square$

Now we can prove the main theorem. We have

$$\begin{aligned}\sum_{g\in G}\operatorname{ch}_{\rho_1}(g)\operatorname{ch}_{\rho_2}(g^{-1}) &= \sum_{g\in G}\operatorname{ch}_{\rho_1}(g)\operatorname{ch}_{\rho_2^\vee}(g)\\ &= \sum_{g\in G}\operatorname{MC}_{\rho_1}(u_{V_1})(g)\operatorname{MC}_{\rho_2^\vee}(u_{V_2^\vee})(g)\\ &= \sum_{g\in G}\operatorname{MC}_{\rho_1\otimes\rho_2^\vee}(u_{V_1}\otimes u_{V_2^\vee})(g)\\ &= \sum_{g\in G}\operatorname{ch}_{\rho_1\otimes\rho_2^\vee}(g)\end{aligned}$$

Now applying our summation earlier,

$$\begin{aligned}&= |G|\dim\left(\rho_1\otimes\rho_2^\vee\right)^G\\ &= |G|\cdot\dim\left(\underline{\operatorname{Hom}}(\rho_2,\rho_1)\right)^G\\ &= |G|\cdot\dim\left(\operatorname{Hom}_G(\rho_2,\rho_1)\right).\end{aligned}$$

Now if $\rho_1 \simeq \rho_2$, then $\operatorname{Hom}_G(\rho_2,\rho_1) = \operatorname{End}(\rho)$. Otherwise, $\operatorname{Hom}_G(\rho_2,\rho_1)$ has dimension zero because it consists of a single element which is the zero map; note that ρ_2 and ρ_1 were assumed to be irreducible.

§21.6 Some things we were asked to read about

Proposition 6.1.7 in the Math 123 Notes reads as follows.

Proposition 21.11

There is a canonical isomorphism

$$\operatorname{Hom}_G(\pi,\operatorname{Fun}(G/H,k)) \simeq \operatorname{Hom}_H(\pi,\operatorname{triv}_H).$$

Proof. The image of $T : \pi \to \operatorname{Fun}(G/H,k)$ is the map $\xi_T : V \to k$ by

$$v \mapsto T(v)(\overline{1}) \in k.$$

□

A special case is the following.

Proposition 21.12

There is an isomorphism

$$\operatorname{Hom}_{G\times G}\left(\rho\boxtimes\pi,\operatorname{Reg}(G)\right) \simeq \operatorname{Hom}_G(\rho\otimes\pi,\operatorname{triv}_G).$$

We can use this in combination with the following proposition.

Proposition 21.13

There is an isomorphism

$$\operatorname{Hom}_G(\rho\otimes\pi,\operatorname{triv}_G) \simeq \operatorname{Hom}_G(\rho,\pi^\vee).$$

Proof. Recall that

$$\operatorname{Hom}(V_1 \otimes V_2, k) \simeq \operatorname{Hom}(V_1, V_2^\vee) \implies \underline{\operatorname{Hom}}(\pi_1 \otimes \pi_2, \operatorname{triv}_G) \simeq \underline{\operatorname{Hom}}(\pi_1, \pi_2^\vee).$$

Taking the G invariants gives

$$\operatorname{Hom}_G(\pi_1 \otimes \pi_2, \operatorname{triv}_G) \simeq \operatorname{Hom}_G(\pi_1, \pi_2^\vee).$$

$\square$

§22 November 13, 2014

In what follows, assume all representations are finite-dimensional. We will let G be a finite group and k an algebraically closed field. Assume $\operatorname{char} k$ does not divide $|G|$.

§22.1 Irreducibles

Definition 22.1. Let $\operatorname{IrredRepr}(G)$ denote the set of isomorphism classes of irreducible representations. For every $\alpha \in \operatorname{IrredRepr}(G)$, we can take a representative ρ_α.

First, we prove the following sub-lemma.

Lemma 22.2

Let N be a vector space. Then

$$\operatorname{Hom}_G(\rho_\alpha, \rho_\beta \otimes N) \simeq \begin{cases} 0 & \text{if } \alpha \neq \beta \\ N & \text{if } \alpha = \beta. \end{cases}$$

Proof. If the assertion holds for N_1 and N_2, then it holds for $N_1 \oplus N_2$. Thus it suffices to show that the result holds for $N = k$. The first assertion is just that there is no nontrivial isomorphisms between distinct irreducibles. The second assertion is Schur's lemma. □

The subsequent lemma basically exactly the multiplicities which appear in Maschke's Theorem.

Lemma 22.3

For any representation π, we have

$$\pi \simeq \bigoplus_\alpha \rho_\alpha \otimes \operatorname{Hom}_G(\rho_\alpha, \pi).$$

Proof. By Maschke, $\pi \simeq \bigoplus_\alpha \rho_\alpha^{n_\alpha}$. Since $\rho_\alpha^{n_\alpha} \simeq \rho_\alpha \otimes k^{n_\alpha}$, we may thus write

$$\pi = \bigoplus_\beta \rho_\beta \otimes M_\beta$$

for some M_β. We wish to show $M_\alpha = \operatorname{Hom}_G(\rho_\alpha, \pi)$. We have

$$\operatorname{Hom}(\rho_\alpha, \pi) \simeq \bigoplus_\beta \operatorname{Hom}_G(\rho_\alpha, \rho_\beta \otimes M_\beta).$$

Applying the sublemma now gives the conclusion. □

For this lecture, we will frequently be using the trick that $\rho^n \simeq \rho \otimes k^n$.

Definition 22.4. The component $\rho_\alpha \otimes \operatorname{Hom}_G(\rho_\alpha, \pi)$ is called the **α-isotypic component** of π.

§22.2 Products of irreducibles

Proposition 22.5

We have the following.

(1) If $\rho_1 \in \text{IrredRepr}(G_1)$ and $\rho_2 \in \text{IrredRepr}(G_2)$, then $\rho_1 \boxtimes \rho_2 \in \text{IrredRepr}(G_1 \times G_2)$.

(2) If $\rho_1 \not\simeq \rho_1'$, then $\rho_1 \boxtimes \rho_2 \not\simeq \rho_1' \boxtimes \rho_2'$.

(3) Any irreducible representation is isomorphic to some element of the form given in (1).

First, we prove (3). Let V be the vector space which underlies ρ. First, we interpret ρ as a G_1 representation ρ^1. Then by Maschke's Theorem, we may write ρ^1 as a direct sum of the irreducibles

$$\rho^1 \simeq \bigoplus_\alpha \rho^1_\alpha \otimes \text{Hom}_{G_1}(\rho^1_\alpha, \rho^1).$$

Now we can put the G_2 representation structure on $\text{Hom}_{G_1}(\rho^1_\alpha, \rho^2)$ by

$$(g_2 \cdot f)(g) = g_2 \cdot_\rho (f(g)).$$

It is easy to check that this is indeed a G_2 representation. Thus it makes sense to talk about the $G_1 \times G_2$ representation

$$\bigoplus_\alpha \rho^1_\alpha \boxtimes \text{Hom}_{G_1}(\rho^1_\alpha, \rho^1).$$

We claim that the isomorphism for ρ^1 as a G_1 representation now lifts to an isomorphism of ρ representation. That is, we claim that

$$\rho \simeq \bigoplus_\alpha \rho^1_\alpha \boxtimes \text{Hom}_{G_1}(\rho^1_\alpha, \rho^1)$$

by the same isomorphism as for ρ^1. To see this, we only have to check that the isomorphism $v \otimes \xi \mapsto \xi(v)$ commutes with the action of $g_2 \in G_2$. But this is obvious, since $g_2 \cdot (v \otimes \xi) = v \otimes (g_2 \cdot \xi) \mapsto g_2 \cdot \xi(v)$.

Thus the isomorphism holds. But now we note that since ρ^1_α is irreducible, exactly one α contributes the direct sum above, since in all other cases we have $\text{Hom}_{G_1}(\rho^1_\alpha, \rho^1) = 0$. Thus we derive the required decomposition of ρ.

We leave (2) as an easy exercise.

Next we establish (1). Suppose $\rho_1 \boxtimes \rho_2$ has a nontrivial subrepresentation of the form $\rho_1' \boxtimes \rho_2'$. Viewing as G_1 representation, we find that ρ_1' is a nontrivial subrepresentation of ρ_1, and similarly for ρ_2. But ρ_1 is irreducible, hence $\rho_1' \simeq \rho_1$. Similarly $\rho_2' \simeq \rho_2$. So in fact $\rho_1' \boxtimes \rho_2' \simeq \rho_1 \boxtimes \rho_2$. Hence we conclude $\rho_1 \boxtimes \rho_2$ is irreducible.

§22.3 Regular representation decomposes

Let us consider $\text{Reg}(G)$ as a representation of $G \times G$.

Theorem 22.6

The map
$$\bigoplus_{\alpha} \rho_\alpha^\vee \boxtimes \rho_\alpha \to \operatorname{Reg}(G)$$
by applying $\mathrm{MC}_{\rho_\alpha}$ to each component is an isomorphism.

This theorem has the following nice consequence.

Corollary 22.7

There are only finitely many non-isomorphic irreducible. In fact,
$$|G| = \sum_{\alpha \in \operatorname{IrredRepr}(G)} |\dim \rho_\alpha|^2 .$$

Proof. Just look at the dimensions in the isomorphism. □

Proof of Theorem. We have that $\operatorname{Reg}(G)$ is the sum of irreducibles
$$\operatorname{Reg}(G) = \bigoplus_{\alpha,\beta} (\rho_\alpha \boxtimes \rho_\beta) \otimes \operatorname{Hom}_{G\times G} (\rho_\alpha \boxtimes \rho_\beta, \operatorname{Reg}(G)) .$$

Now we will compute this homomorphism. But we have
$$\operatorname{Hom}_{G\times G} (\rho_\alpha \boxtimes \rho_\beta, \operatorname{Reg}(G)) \simeq \operatorname{Hom}_G(\rho_\alpha \otimes \rho_\beta, \operatorname{triv}_G) \simeq \operatorname{Hom}_G(\rho_\alpha, \rho_\beta^\vee).$$
by the stuff in the last section of the November 11 lecture. Now by irreducibility, this sum is only interesting when $\operatorname{Hom}_G(\rho_\alpha, \rho_\beta^\vee) \neq 0$, so for every α we are only interested in the unique β such that $\rho_\alpha \simeq \rho_\beta^\vee$. Thus we deduce
$$\operatorname{Reg}(G) = \bigoplus_{\alpha} \left(\rho_\alpha^\vee \boxtimes \rho_\alpha\right) \otimes \operatorname{Hom}_G(\rho_\alpha, \rho_\alpha)$$
and of course $\operatorname{Hom}_G(\rho_\alpha, \rho_\alpha) = k$. All that's left to check is that when we unwind the map
$$\rho_\alpha^\vee \boxtimes \rho_\alpha \to \operatorname{Reg}(G)$$
we get $\mathrm{MC}_{\rho_\alpha}$; this is like homework. □

§22.4 Function invariants

Consider the space
$$\left(\operatorname{Res}_G^{G\times G} \operatorname{Reg}(G)\right)^G = \left\{ f \in \operatorname{Fun}(G) \mid f(g_1 g g_1^{-1}) = f(g) \right\} .$$
where Res is the restriction of the action to G.

Theorem 22.8

The collection $\mathrm{ch}_{\rho_\alpha}$ forms a basis of $\left(\operatorname{Res}_G^{G\times G} \operatorname{Reg}(G)\right)^G$.

Proof. We have the isomorphism

$$\bigoplus_\alpha \mathrm{Res}_G^{G\times G}\left(\rho_\alpha^\vee \otimes \rho_\alpha\right) \xrightarrow{\simeq} \mathrm{Res}_G^{G\times G}\,\mathrm{Reg}(G).$$

Now $(\rho_\alpha^\vee \otimes \rho_\alpha)^G \simeq (\underline{\mathrm{Hom}}(\rho_\alpha, \rho_\alpha))^G \simeq \mathrm{Hom}_G(\rho_\alpha, \rho_\alpha) = k$ (since the ρ_α is irreducible). Note we have a $0 \neq u_\alpha \in \rho_\alpha^\vee \otimes \rho_\alpha$ in this one-dimensional space. Now $\mathrm{ch}_{\rho_\alpha} = \mathrm{MC}_{\rho_\alpha}(u_{\rho_\alpha})$; so our isomorphism with $\mathrm{MC}_{\rho_\alpha}$ sends these u_α to $\mathrm{ch}_{\rho_\alpha}$. □

Corollary 22.9

As a group, the number of conjugacy classes of G is $|\mathrm{IrredRepr}(G)|$.

Proof. The dimension of $\mathrm{Fun}(G)^G$ is equal to the size of the basis $\mathrm{ch}_{\rho_\alpha}$ as α varies. On the other hand by the definition of $\mathrm{Fun}(G)^G$ it is also equal to the number of conjugacy classes. □

§22.5 A Concrete Example

Example 22.10

Let us take the group $G = S_3$, with six elements. Because its conjugacy classes are id, $(\bullet)(\bullet\,\bullet)$ and $(\bullet\;\bullet\;\bullet)$, there should be exactly three irreducible representations. The irreducible representations are as follows.

- triv_G on k, where $\sigma \cdot 1 = 1$.
- sign on k, where $\sigma \cdot 1 = \mathrm{sign}(\sigma)$.
- Take the two-dimensional subspace $\{(a, b, c) \mid a + b + c = 0\}$ of k^3. Then S_3 acts on this.

This gives $3! = 1^2 + 1^2 + 2^2$.

Upon seeing 6, we get the following exchange.

"The largest number we've seen in this class" – James Tao
"I take this as a personal insult" – Gaitsgory

Example 22.11

Let us take the group $G = S_4$ now. There are five conjugacy classes. These are irreducible.

- triv again on k.
- refl_0, which is permutation on $\{(a, b, c, d) \mid a+b+c+d = 0\}$, a three-dimensional space.
- sign, again on k.
- $\mathrm{refl}_0 \otimes \mathrm{sign}$ on k^3. Note you have to show $\mathrm{refl}_0 \neq \mathrm{refl}_0 \otimes \mathrm{sign}$; this occurs in S_3!
- The last 2-dimensional representation is not so easy to describe.

Indeed, $24 = 1^2 + 1^2 + 2^2 + 3^2 + 3^2$.

§23 November 18, 2014

Today we are covering S_5. This will yield the number $5! = 120 \gg 6$, making James happy.

§23.1 Review

In this lecture, k is a field, G is a finite group and $\operatorname{char} k$ does not divide $|G|$. Hence we have both Schur's Lemma and Maschke's Theorem.

Using the irreducible decomposition, we have the following corollaries. We have the following two corollaries.

Corollary 23.1

Let ρ be a representation. If $\dim \operatorname{End}(\rho) = 1$, then ρ is irreducible.

Corollary 23.2

We have $\dim \operatorname{Hom}_G(\rho_1, \rho_2) = \dim \operatorname{Hom}_G(\rho_2, \rho_1)$.

We will focus on the group $\operatorname{Fun}(G/H)$ today. Recall also the following results.

- $\operatorname{Hom}_G(\pi, \operatorname{Fun}(G/H)) \simeq \operatorname{Hom}_H(\operatorname{Res}_H^G(\pi), \operatorname{triv}_H)$.
- $\operatorname{Hom}_G(\operatorname{Fun}(G/H), \pi) \simeq \operatorname{Hom}_H(\operatorname{triv}_H, \operatorname{Res}_H^G(\pi))$.

Here Res_H^G is the *restriction*.

We remind the reader that refl_0 as an S_n representation is the action of S_n on $\{(a_1, a_2, \dots, a_n) \mid a_1 + \dots + a_n = 0\} \in k^n$ by permuting the bases.

We also need the following definition from the PSet.

Definition 23.3. A **character** χ is a homomorphism $G \to k^*$.

§23.2 The symmetric group on five elements

Recall that the conjugacy classes of permutations S_n correspond to "cycle types", i.e. partitions of n. This gives the children's game.

$$\begin{aligned}
5 &= 5 \\
&= 4 + 1 \\
&= 3 + 2 \\
&= 3 + 1 + 1 \\
&= 2 + 2 + 1 \\
&= 2 + 1 + 1 + 1 \\
&= 1 + 1 + 1 + 1 + 1.
\end{aligned}$$

"Whenever I see Young diagrams I always think of New York City" – Gaitsgory
"Yes – on the left is Empire State building, on the right is the Cambridge building code. . . and that's the Green Building at MIT".

In fact, we will see that there is a bijection between the Young diagrams and irreducible representations. This is a very special feature of S_n and does not generalize to other groups.

Definition 23.4. Let χ be a character of G, and ρ be any representation. Then we define

$$\rho^\chi \overset{\text{def}}{=} \rho \otimes k^\chi$$

where k^χ is the representation of χ on k with the following action: g acts via multiplication by $\chi(g)$.

Notice that ρ is irreducible if and only if ρ^χ is irreducible, because the latter is just multiplication by a constant. In fact, we have the following.

Proposition 23.5

For any representations ρ_1, ρ_2 and χ we have $\operatorname{Hom}_G(\rho_1, \rho_2^\chi) \simeq \operatorname{Hom}(\rho_1^{\chi^{-1}}, \rho_2)$.

Proof. Obvious. □

Now we'll be a troll and guess representations.

$$\begin{aligned} \operatorname{triv}_{S_5} &\leftrightarrow 5 \\ \operatorname{refl}_0 &\leftrightarrow 4+1 \\ &\leftrightarrow 3+2 \\ &\leftrightarrow 3+1+1 \\ &\leftrightarrow 2+2+1 \\ \operatorname{refl}_0^{\text{sign}} &\leftrightarrow 2+1+1+1 \\ \operatorname{sign} &\leftrightarrow 1+1+1+1+1. \end{aligned}$$

We'll explain in half an hour what the actual recipe is.

Now there is something to prove: we want to show $\operatorname{refl}_0 \neq \operatorname{refl}_0^{\text{sign}}$.

Proposition 23.6

$\operatorname{Hom}_{S_5}\left(\operatorname{refl}_0, \operatorname{refl}_0^{\text{sign}}\right) = 0$. In particular, $\operatorname{refl}_0 \neq \operatorname{refl}_0^{\text{sign}}$.

Proof. We may interpret refl_0 as $\operatorname{Fun}(\{1,2,3,4,5\}) \simeq \operatorname{Fun}(S_5/S_4)$. Explicitly, we'll identify S_4 as the subgroup of S_5 which fixes the point 5. So

$$\operatorname{Hom}_{S_5}\left(\operatorname{refl}_0, \operatorname{refl}_0^{\text{sign}}\right) \simeq \operatorname{Hom}_{S_4}\left(\operatorname{triv}_{S_4}, \operatorname{refl}^{\text{sign}}\right).$$

Now we claim that there are no nontrivial maps $k \to k^5$. (Here $\operatorname{refl}^{\text{sign}}$ was an S_5 representation, but we've restricted it to an S_4 representation.) We claim there are no nontrivial maps. Consider such a map where the image of $1 \in k$ is (a,b,c,d,e). By taking, say $(2\ 3) \in S_4$ we get $-(a,c,b,d,e) = (a,b,c,d,e) \implies a = 0$, and one can similarly show all the coordinates are zero.

Question: where does this break down if we have S_3 in place of S_5? □

More generally, we establish the following result.

Lemma 23.7

Let X be a set acted on by G. Then for any character χ,

$$\operatorname{Hom}_G\left(\operatorname{triv}_G, \operatorname{Fun}(X)^\chi\right)$$

is isomorphic to

$$\bigoplus_{\text{orbit } \mathcal{O}} \begin{cases} k & \text{if } \chi \text{ restricted to } \operatorname{Stab}_G(x) \text{ is trivial for a representative } x \in \mathcal{O} \\ 0 & \text{otherwise.} \end{cases}$$

Example 23.8

We earlier computed $\operatorname{Hom}_{S_4}(\operatorname{triv}_{S_4}, \operatorname{refl}^{\text{sign}}) = 0$. This follows from the lemma as follows. S_4 acts on $\{1,2,3,4,5\}$ with two orbits.

- In the orbit of $1,2,3,4$ we have $\operatorname{Stab}_{S_4}(1) = S_4$, and $\operatorname{sign}|_{S_4} \neq 1$.
- In the orbit of 5 we have $\operatorname{Stab}_{S_4}(1) = S_3$, and $\operatorname{sign}|_{S_3} \neq 1$.

Now we prove the lemma.

Proof. We have $\operatorname{Hom}_G(\operatorname{triv}_G, \operatorname{Fun}(X)^\chi) = \left(\operatorname{Fun}(X)^\chi\right)^G$.

It suffices to consider the case of a single orbit, since $X = \bigsqcup_{\mathcal{O}} \mathcal{O}$, $\operatorname{Fun}(X) = \bigoplus_{\mathcal{O}} \operatorname{Fun}(\mathcal{O})$ and $\operatorname{Fun}(X)^\chi = \bigoplus_{\mathcal{O}} \operatorname{Fun}(\mathcal{O})^\chi$.

If $f \in \operatorname{Fun}(X)$ is invariant as an element of $\operatorname{Fun}(X)^\chi$, we have then we need

$$f(x) = (g \cdot f)(x) = f(g^{-1} \cdot x)\chi(g)$$

equivalent to

$$f(g^{-1} \cdot x) = \chi(g^{-1}) \cdot f(x).$$

Thus we require $g_1 x = g_2 x \implies \chi(g_1^{-1}) = \chi(g_2^{-1}) \implies \chi(h) = 1$, where $g_1 = g_2 h$. Now $h \in \operatorname{Stab}_G(x)$. $\square$

§23.3 Representations of $S_5/(S_3 \times S_2)$ – finding the irreducible

Note that $S_5/(S_3 \times S_2)$ has order 10. Consider the 10-dimensional space $\operatorname{Fun}(S_5/(S_3 \times S_2))$.

Applying the above lemma, we have

$$\dim \operatorname{Hom}_{S_4}\left(\operatorname{triv}, \operatorname{Fun}(S_5/(S_3 \times S_2))\right) = 1$$

where the character is trivial (and $S_5/(S_3 \times S_2)$ has one orbit).

Now we are going to prove that

$$\dim \operatorname{Hom}_{S_4}\left(\operatorname{refl}, \operatorname{Fun}(S_5/(S_3 \times S_2))\right).$$

Lemma 23.9

Let H_1 and H_2 be orbits of G. Then H_1-orbits on G/H_2 are in bijection with H_2 orbits G/H_1. In fact both are in bijection with $H_1 \times H_2$ orbits on G via the action $(h_1, h_2) \cdot g = h_1 g h_2^{-1}$.

As an example, $S_3 \times S_2$ on orbits S_5 are in bijection with S_3 orbits on $S_5/S_2 \simeq \{1,2,3\}$.

At this point we have refl_0 and triv as irreducible subrepresentations of $\mathrm{Fun}(S_5/(S_2 \times S_3))$. Thus we have

$$\mathrm{Fun}(S_5/(S_2 \times S_3)) = \mathrm{triv} \oplus \mathrm{refl}_0 \oplus \dots$$

We claim that the last guy in ellipses is a single irreducible guy. Since $\dim \mathrm{End}(\rho)$ is equal to the number of irreducible components of ρ, it suffices to show the following.

Claim 23.10. $\dim \mathrm{End}\left(\mathrm{Fun}(S_5/S_3 \times S_2)\right) = 3$.

Proof. We have

$$\mathrm{Hom}_{S_5}\left(\mathrm{Fun}(S_5/S_3 \times S_2), \mathrm{Fun}(S_5/S_3 \times S_2)\right) \simeq \mathrm{Hom}_{S_3 \times S_2}\left(\mathrm{triv}, \mathrm{Fun}(S_5/(S_3 \times S_2))\right).$$

So we want to show the above has dimension 3. Applying the above, it's equivalent to finding the number of orbits of $S_3 \times S_2$ on $S_5/(S_3 \times S_2)$. This is a combinatorics problem. $\square$

§23.4 Secret of the Young Diagrams

So that means we have a five-dimensional irreducible ρ with

$$\mathrm{Fun}(S_5/(S_3 \times S_2)) \simeq \mathrm{triv} \oplus \mathrm{refl}_0 \oplus \rho.$$

We now want to show that $\rho^{\mathrm{sign}} \not\simeq \rho$. We will show that

$$\mathrm{Hom}_{S_5}\left(\mathrm{Fun}(S_5/(S_2 \times S_3)), \mathrm{Fun}(S_5/(S_2 \times S_3))^{\mathrm{sign}}\right) = 0.$$

This will imply that $\mathrm{triv}^{\mathrm{sign}} \neq \mathrm{triv}$, $\mathrm{refl}_0^{\mathrm{sign}} \neq \mathrm{refl}_0$ and also $\rho^{\mathrm{sign}} \neq \rho$. You can check this; it's getting skipped in lecture for time.

Now we have six of the seven representations.

$$\begin{aligned}
\mathrm{triv}_{S_5} &\leftrightarrow 5 \\
\mathrm{refl}_0 &\leftrightarrow 4+1 \\
\rho &\leftrightarrow 3+2 \\
\pi &\leftrightarrow 3+1+1 \\
\rho^{\mathrm{sign}} &\leftrightarrow 2+2+1 \\
\mathrm{refl}_0^{\mathrm{sign}} &\leftrightarrow 2+1+1+1 \\
\mathrm{sign} &\leftrightarrow 1+1+1+1+1.
\end{aligned}$$

We have one last mysterious representation π. Because the sum of the squares of the dimensions is $5! = 120$, so $(\dim \pi)^2 = 120 - 2(1^2+4^2+5^2) = 120 - 2 \cdot 42 = 36$.

Where should we look for π? Answer: $S_5/(S_3 \times S_1 \times S_1)$. Now it should be clear what the Young diagrams are doing.

Now let's begin dissecting $\mathrm{Fun}(S_5/S_3)$. Note that $|S_5/S_3| = 20$. Then

- Because there is only one orbit of S_5/S_3, triv appears only once.
- We will compute $\mathrm{Hom}(\mathrm{refl}, \mathrm{Fun}(S_5/S_3))$ now. We have

$$\begin{aligned}
\mathrm{Hom}_{S_5}\left(\mathrm{refl}, \mathrm{Fun}(S_5/S_3)\right) &= \mathrm{Hom}_{S_5}\left(\mathrm{Fun}(S_5/S_4), \mathrm{Fun}(S_5/S_3)\right) \\
&= \mathrm{Hom}_{S_4}\left(\mathrm{triv}, \mathrm{Fun}(S_5/S_3)\right).
\end{aligned}$$

Check combinatorially there are three orbits. Because $\text{refl} = \text{triv} \oplus \text{refl}_0$, that means there are two orbits.

So we thus far have

$$\text{Fun}(S_5/S_3) = \text{triv} \oplus \text{refl}_0^{\oplus 2} \oplus \dots$$

There is dimension 11 left.

- We'll show ρ appears exactly once now by computing

$$\text{Hom}\left(\text{Fun}(S_5/(S_2 \times S_3)), \text{Fun}(S_5/S_3)\right)$$

and showing it has dimension 4; since we already know $\text{Fun}(S_5/(S_2 \times S_3)) = \text{triv} \oplus \text{refl}_0 \oplus \rho$ this will be enough. We apply the same trick:

$$\text{Hom}_{S_2 \times S_3}\left(\text{triv}, \text{Fun}(S_5/S_3)\right)$$

has four orbits. Hence ρ appears exactly once.

- Hence we have missing dimension 6 is

$$\text{Fun}(S_5/S_3) = \text{triv} \oplus \text{refl}_0^{\oplus 2} \oplus \rho \oplus \dots$$

and we want to show the remaining part is irreducible. We could verify that $\dim \text{Hom}(\text{Fun}(S_5/S_3), \text{Fun}(S_5/S_3)) = 1^2 + 2^2 + 1^2 + 1^2$, but we could also just verify that sign does not appear in here.

§23.5 The General Theorem

In the next lecture we discuss the general theorem.

§24 November 20, 2014

In this lecture, fix $n > 2$ an integer and let k be a field such that $\operatorname{char} k$ does not divide $n! = |S_n|$. The field k need not be algebraically closed.

First, recall some definitions on partitions.

Definition 24.1. For a **partition** $\mathfrak{p} = n_1 \ge \cdots \ge n_k$ of $n = n_1 + \cdots + n_k$ we define

$$S_{\mathfrak{p}} = S_{n_1} \times \cdots \times S_{n_k}$$

and

$$\pi_{\mathfrak{p}} \overset{\text{def}}{=} \operatorname{Fun}(S_n / S_{\mathfrak{p}}).$$

By abuse of notation we will also refer to the corresponding representation as $\pi_{\mathfrak{p}}$.

Definition 24.2. We write $\mathfrak{p} \ge \mathfrak{q}$ for the majorization condition (think Muirhead). Note that this is *not* a total order; not all pairs are comparable.

Definition 24.3. We define the dual partition $\mathfrak{p}^\vee$ by flipping the Young digram (see the October 2 lecture).

The goal of today's lecture will be to classify all partitions of the symmetric group S_n.

Theorem 24.4

There exists a bijection between Young diagrams and irreducible representations of S_n by $\mathfrak{p} \mapsto \rho_{\mathfrak{p}}$ with the following property. For any $\mathfrak{p}$ consider the decomposition

$$\pi_{\mathfrak{p}} = \bigoplus_{\mathfrak{q}} \rho_{\mathfrak{q}}^{\oplus n_{\mathfrak{q}}}.$$

Then $n_{\mathfrak{p}} = 1$, and $n_{\mathfrak{q}} = 0$ for any $\mathfrak{q} \not\ge \mathfrak{p}$.

Note that we aren't claiming anything about the value of $n_{\mathfrak{q}}$ for $\mathfrak{q} > \mathfrak{p}$.

§24.1 Reducing to some Theorem with Hom's

Consider the following theorem.

Theorem 24.5

We have the following two assertions.

(a) If $\mathfrak{q} \not\ge \mathfrak{p}$, then

$$\operatorname{Hom}_{S_n}\left(\pi_{\mathfrak{p}}, \pi_{\mathfrak{q}}^{\text{sign}}\right) = 0.$$

(b) We have

$$\operatorname{Hom}_{S_n}\left(\pi_{\mathfrak{p}}, \pi_{\mathfrak{p}^\vee}^{\text{sign}}\right) \sim k.$$

i.e. the homomorphism space has dimension 1.

We will use Theorem 24.5 to prove Theorem 24.4 To do this we must actually exhibit the $\rho_{\mathfrak{p}}$ first.

Definition 24.6. Up to scaling there is a unique nonzero homomorphism

$$T \in \mathrm{Hom}_{S_n}\left(\pi_{\mathfrak{p}}, \pi_{\mathfrak{p}^\vee}^{\mathrm{sign}}\right).$$

We set $\rho_{\mathfrak{p}} = \mathrm{im}(T) \subseteq \pi_{\mathfrak{p}^\vee}^{\mathrm{sign}}$.

Remark 24.7. Note that $\rho_{\mathfrak{p}}$ is in fact irreducible, since it's the image of a T in a one-dimensional Hom (consider the irreducible decomposition.)

Proof that Theorem 24.5 implies Theorem 24.4. The proof proceeds in several steps.

Claim. We have $\rho_{\mathfrak{p}} \neq \rho_{\mathfrak{q}}$ for any $\mathfrak{p} \neq \mathfrak{q}$.

Proof. Assume $\rho_{\mathfrak{p}} = \rho_{\mathfrak{q}}$; we will prove $\mathfrak{p} = \mathfrak{q}$. It suffices to prove that

$$\mathrm{Hom}_{S_n}\left(\pi_{\mathfrak{p}}, \pi_{\mathfrak{q}^\vee}^{\mathrm{sign}}\right) \neq 0.$$

We have a sequence

$$\pi_{\mathfrak{p}} \twoheadrightarrow \rho_{\mathfrak{p}} \simeq \rho_{\mathfrak{q}} \hookrightarrow \rho_{q^\vee}^{\mathrm{sign}}.$$

Hence $\mathrm{Hom}(\rho_{\mathfrak{p}}, \rho_{\mathfrak{q}^\vee}^{\mathrm{sign}}) \neq 0$. By (a) of Theorem 24.5, we obtain $\mathfrak{p} \geq \mathfrak{q}$. Similarly, $\mathfrak{q} \geq \mathfrak{p}$. So $\mathfrak{p} = \mathfrak{q}$. ■

Claim. All irreducible representations of S_n are of the form $\rho_{\mathfrak{p}}$.

Proof. We have exhibited a distinct irreducible representation for every partition. But the number of partitions of n equals the number of conjugacy classes of S_n. ■

Thus Theorem 24.4 follows: The assertion $n_{\mathfrak{p}} = 1$ follows from (b) of Theorem 24.5, since $\rho_{\mathfrak{q}}$ was defined as the image of the unique (up to scaling) map $\pi_{\mathfrak{q}} \to \pi_{\mathfrak{q}^\vee}$. □

§24.2 Reducing to a Combinatorial Theorem

Now we will deduce Theorem 24.5 from the following theorem.

It may be helpful to view the space $S_{\mathfrak{p}}$ as the Young tableau $\mathfrak{p}$ with its cells labelled by $1, 2, \dots, n$. The equivalence classes then result from shuffling the entries in any row. In particular, the canonical choice is to sort the guy in ascending order.

Theorem 24.8

We have the following two assertions.

(a) If $\mathfrak{q} \not\geq \mathfrak{p}$, then for every $g \in S_n$ there exists a transposition $(i\ j) \in S_{\mathfrak{p}}$ and $h \in S_{q^\vee}$ such that

$$(i\ j) \cdot g = g \cdot h.$$

(b) If $\mathfrak{p} = \mathfrak{q}$ there exists a unique orbit $\mathcal{O} \in S_n / S_{q^\vee}$ (in fact the orbit containing the identity) such that

  (i) $S_{\mathfrak{p}}$ acts simply on $\mathcal{O}$ (meaning that for any $x, y \in \mathcal{O}$ there is in fact exactly one $g \in S_{\mathfrak{p}}$ with $g \cdot x = y$)

  (ii) For all $g \in S_n$ and $\overline{g} \notin \mathcal{O}$, there exists a transposition $(i\ j) \in S_{\mathfrak{p}}$ and $h \in S_{\mathfrak{q}^\vee}$ such that

$$(i\ j) \cdot g = gh.$$

Proof that Theorem 24.8 implies Theorem 24.5. We begin by establishing (a). First, we can compute

$$\begin{aligned}
\operatorname{Hom}_{S_n}(\pi_{\mathfrak{p}}, \pi_{\mathfrak{q}^\vee}^{\text{sign}}) &\simeq \operatorname{Hom}_{S_n}\left(\operatorname{Fun}(S_n/S_{\mathfrak{p}}), \pi_{\mathfrak{q}^\vee}^{\text{sign}}\right) \\
&\simeq \operatorname{Hom}_{S_{\mathfrak{p}}}\left(\text{triv}, \pi_{\mathfrak{q}^\vee}^{\text{sign}}\right) \\
&\simeq \left(\pi_{\mathfrak{q}^\vee}^{\text{sign}}\right)^{S_{\mathfrak{p}}} \\
&= \left(\operatorname{Fun}(S_n/S_{\mathfrak{q}^\vee})^{\text{sign}}\right)^{S_{\mathfrak{p}}}.
\end{aligned}$$

Consider an f in this set. Then for every g,

$$\begin{aligned}
f(g) &= (i\ j)\cdot f(g) \\
&= \operatorname{sign}(i\ j)\cdot f\left((i\ j)^{-1}\cdot g\right) \\
&= -f\left((i\ j)\cdot g\right) \\
&= -f(g\cdot h) \\
&= -f(g)
\end{aligned}$$

so $f \equiv 0$.

For (b) of Theorem 24.5, let $\mathcal{O}$ be the orbit for (b). For g such that $\overline{g} \notin \mathcal{O}$, $f(g) = 0$ by the same argument as in (a). Let $x \in \mathcal{O}$ now, take $f \in \operatorname{Hom}_{S_n}(\pi_{\mathfrak{p}}, \pi_{\mathfrak{p}}^{\text{sign}})$. Then

$$f(x) = g^{-1}\cdot f(x) = \operatorname{sign}(g^{-1})\cdot f(g\cdot x)$$

Thus $f(g\cdot x) = \operatorname{sign}(g)\cdot f(x)$. So f is uniquely determined, which implies that the vector space is one-dimensional. □

§24.3 Doing Combinatorics

Now let's prove the combinatorial result.

Note: the reader is urged to draw Young diagrams for $\mathfrak{p}$ and $\mathfrak{q}$ and figure out what the hell is happening. The rows correspond to the n_i we are modding out by.

Definition 24.9. We say that $g \in S_n$ is "good" if there exists $(i\ j) \in S_{\mathfrak{p}}$ such that

$$(i\ j)\cdot g = g\cdot h$$

for some $h \in S_{\mathfrak{q}^\vee}$.

Set $X = \{1, 2, \dots, n\}$. We denote

$$\mathfrak{q}^\vee : \bigsqcup_i X_i' = X' \sim X$$

and

$$\mathfrak{p} : \bigsqcup_j X_j'' \simeq X.$$

(The X_i and X_j's are rows.) Finally, note that $g \in S_n$ is an automorphism of X. Hence we induce a map for each g:

$$\Phi_g : X' \simeq X \xrightarrow{g} X \simeq X''.$$

We say that Φ_g is "good" if for all i, the elements of X_i' get sent to elements in distinct columns.

Claim 24.10. *g is good if and only if Φ_g is good.*

Proof. Look at the picture. (Draw it now!) $\square$

Definition 24.11. We say that $\Phi : X' \to X''$ is "superb" if Φ is good and we have: for all i (index rows $q^\vee$) and every $x' \in X'$, $\Phi(x')$ is such that all elements underneath $\Phi(x')$ in the Young tableau for p are of the form $\Phi(y')$ where $y' \in X'_j$ for some $j < i$.

Lemma 24.12

For any good map Φ, there exists a unique $h'' \in S_{\mathfrak{p}}$ such that

$$\Phi_s = h'' \circ \Phi$$

is "superb".

Intuition: what this proof is doing is repeatedly pressing left and down in the game 2048. (Thanks James Tao.) So any good map can be made superb, since extra factors of h'' are irrelevant for the existence of guys in $h \in S_{\mathfrak{q}^\vee}$.

Proof. We proceed inductively on i. Suppose Φ_s has been defined on $X'_1, \dots, X'_{i-1}$. Consider $x \in X'_i$ and define the transposition σ_x which puts $\Phi(x)$ in the lowest possible position in the column of $\Phi(x)$ that isn't occupied by $\Phi(y')$ for $y' \in X'_j$ where $j < i$.

Define Φ_s on X'_i by

$$\left(\prod_{x \in X'_i} \sigma_x \right) \circ \Phi.$$

(The order does not matter in the product.) So we have superb-ified Φ on rows $1, \dots, i$. $\square$

Now we can prove (a) of Theorem 24.8.

Claim 24.13. *If there exists a good g, then $\mathfrak{p} \le \mathfrak{q}$. (In fact, by the lemma there is a superb g.)*

Proof. It suffices to show that if there exists a good Φ from X' to X'' then $\mathfrak{p}^\vee \ge \mathfrak{q}^\vee$, which implies $\mathfrak{p} \le \mathfrak{q}$.

WLOG our Φ is superb. Then draw the picture and use Pigeonhole. $\square$

We also wish to prove (b) now. It amounts to the following.

Claim 24.14. *Set $\mathfrak{p} = \mathfrak{q}$. The identity is a good (in fact superb) map, and it is unique in the sense that if Φ is good then*

$$\Phi = h'' \circ \mathrm{id} \circ h'$$

fro some $h'' \in S_{\mathfrak{p}}$ and $h' \in S_{q^\vee}$.

Here $\mathrm{id} \circ h'$ is just hat it means to be in the orbit $S_n / S_{\mathfrak{q}^\vee}$.

Proof. WLOG assume Φ is superb (again by the lemma). Then it suffices to show that $\Phi = \mathrm{id} \circ h'$, but since $\mathfrak{p} = \mathfrak{q}$, we see X' and X'' have the same shape; hence the rows map bijectively and so the only changes from the identity are permutations along each row. So the product of all there permutations (per row) in some element of $S_{\mathfrak{q}^\vee}$, and we just declare it to be h'. $\square$

§25 December 2, 2014

Today is the last official lecture. On Thursday, we will have an unofficial category theory class.

Proposition 25.1

$(\rho_{\mathfrak{p}})^{\text{sign}} \simeq \rho_{\mathfrak{p}^\vee}$.

Proof. We have

$$\text{Fun}(S_n/S_{\mathfrak{p}}) \twoheadrightarrow \rho_{\mathfrak{p}} \hookrightarrow \text{Fun}(S_n/S_{\mathfrak{p}^\vee})^{\text{sign}}$$

so

$$\text{Fun}(S_n/S_{\mathfrak{p}})^{\text{sign}} \twoheadrightarrow \rho_{\mathfrak{p}}^{\text{sign}} \hookrightarrow \text{Fun}(S_n/S_{\mathfrak{p}^\vee}).$$

On the other hand

$$\text{Fun}(S_n/S_{\mathfrak{p}^\vee}) \twoheadrightarrow \rho_{\mathfrak{p}^\vee} \hookrightarrow \text{Fun}(S_n/S_{\mathfrak{p}}).$$

$\square$

§26 December 4, 2014

Bonus lecture on category theory.

I am reading the notes `http://www.maths.ed.ac.uk/~tl/msci/` in my spare time, which I like a lot.

§26.1 Categories

Definition 26.1. A **category** $\mathcal{C}$ consists of the following information.

- A set of objects $\mathrm{Obj}(\mathcal{C})$ and a set of morphisms $\mathrm{Hom}_{\mathcal{C}}(c_1, c_2)$ for every $c_1, c_2 \in \mathrm{Obj}(\mathcal{C})$.
- Moreover, for any c_1, c_2, c_3 we have an *associative* operation $\circ$, called composition, which sends
$$\mathrm{Hom}_{\mathcal{C}}(c_1, c_2) \times \mathrm{Hom}_{\mathcal{C}}(c_2, c_3) \to \mathrm{Hom}_{\mathcal{C}}(c_1, c_3).$$
- For every $c \in \mathrm{Obj}(\mathcal{C})$, there is a map $\mathrm{id}_c \in \mathrm{Obj}(\mathcal{C})$ such that for any $f \in \mathrm{Hom}_{\mathcal{C}}(c, c')$ and $g \in \mathrm{Hom}_C\, a(c', c)$, we have $\mathrm{id}_c \circ f = f$ and $g \circ \mathrm{id}_c = g$.

Examples of categories are the following.

1. A category whose objects are sets and whose morphisms are the maps of sets.
2. Groups, rings, finite abelian groups, etc, with the morphisms between homorphisms.
3. Vector spaces, with morphisms being linear maps.
4. Given a set X, we can construct a category $\mathcal{C}$ whose objects are the elements of X, and the only maps are the identity maps 1_X.

A more important example is the following.

Example 26.2

Consider a category with only one object $*$, and let $M = \mathrm{Hom}_{\mathcal{C}}(*, *)$. Then the axioms on the homomorphisms are equivalent to specifying the structure of a monoid. So the datum of a monoid is equivalent to the datum of a one-object category.

§26.2 Functors

Categories are not useful unless they can talk to each other. This is done by something called a *functor*.

Definition 26.3. A **functor** F is a map between two categories $F : \mathcal{C}_1 \to \mathcal{C}_2$ which induces a map $F : \mathrm{Obj}(\mathcal{C}_1) \to \mathrm{Obj}(\mathcal{C}_2)$ and $\mathrm{Hom}_{\mathcal{C}_1}(c_1', c_1'') \to \mathrm{Hom}_{\mathcal{C}_2}(F(c_2'), F(c_2''))$. It must

(i) send id_c to $\mathrm{id}_{F(c)}$ for each c, and

(ii) preserve composition.

Examples of functors include the following.

1. An identity functor from $\mathcal{C}$ to itself.

2. A map from the category of abelian groups to the category of groups by embedding.
3. A forgetful functor **Gp** $\to$ **Set** sending a group G to its underlying set G, and a homomorphisms to the corresponding map of sets.
4. There is a functor **Gp** $\to$ **Ring** by $G \mapsto k[G]$.
5. For a group H, there is a functor **Gp** $\to$ **Gp** by $G \mapsto G \times H$ and $\phi \mapsto \phi \times \mathrm{id}_H$.
6. There is a functor **Set** $\to$ **Ab** (abelian groups) by
$$S \mapsto \mathbb{Z}^S.$$
7. Given rings R_1 and R_2 and a homomorphism ϕ there is a functor for R_2-modules to R_1-modules by viewing each R_2 module M as an R_1 module.
8. In above, there is also a functor from R_1 modules to R_2 modules by
$$M \mapsto R_2 \otimes_{R_1} M.$$

Now here's another example that came up in class.

Example 26.4 (Yoneda Functor)
Fix a category $\mathcal{C}$ and one of its objects c. A functor $h^c : \mathcal{C} \to \mathbf{Set}$ is defined by
$$c' \mapsto \mathrm{Hom}_{\mathcal{C}}(c, c').$$
Then we also need to add a structure
$$\mathrm{Hom}_{\mathcal{C}}(c', c'') \to \mathrm{Hom}_{\mathbf{Set}}\left(\mathrm{Hom}_{\mathcal{C}}(c, c'), \mathrm{Hom}_{\mathcal{C}}(c, c'')\right)$$
and we can do so by composition.

Exercise. Why does this not work if we use $c' \mapsto \mathrm{Hom}_{\mathcal{C}}(c', c)$?

§26.3 Natural Transformations

Functors are not useful unless they can talk to each other. This is done by something called a **natural transformation**.

Definition 26.5. Let $F, G : \mathcal{C}_1 \to \mathcal{C}_2$. A **natural transformation** T consists of map $T_{c_1} \in \mathrm{Hom}_{\mathcal{C}_2}(F(c_1), G(c_1))$ for each $c_1 \in \mathcal{C}_1$ such that the following diagram commutes for any choice of a morphism $\phi : c_1' \to c_1''$ in $\mathcal{C}$:

$$\begin{tikzcd}
F(c_1') \arrow[r, "F(\varphi)"] \arrow[d, "T_{c_1'}"'] & F(C_1'') \arrow[d, "T_{c_1''}"] \\
G(c_1') \arrow[r, "F(\varphi)"] & G(C_1'').
\end{tikzcd}$$

This gives the following.

Theorem 26.6 (Yoneda Lemma)

Let $\mathcal{C}$ be an arbitrary category, and let $c', c'' \in \text{Obj}(\mathcal{C})$. Consider the set of all natural transformations $h^{c'}$ to $h^{c''}$. Then this set is isomorphic to $\text{Hom}_{\mathcal{C}}(c'', c')$.

Proof. The following diagram commutes for each $c_1, c_2 \in \mathcal{C}$ and $f : c_1 \to c_2$.

$$\begin{tikzcd}
\text{Hom}(c', c_1) \arrow[r, "T_{c_1}"] \arrow[d, "h^{c'}(f)"'] & \text{Hom}(c'', c_1) \arrow[d, "h^{c''}(f)"] \\
\text{Hom}(c', c_2) \arrow[r, "T_{c_2}"] & \text{Hom}(c'', c_2)
\end{tikzcd}$$

Now our map is $T \mapsto T_{c'}(\text{id}_{c'})$ in one direction. For the other direction, $\phi \in \text{Hom}_{\mathcal{C}}(c'', c')$ is sent to the $- \circ \phi$. $\square$