# Math 145b Lecture Notes

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This is Harvard College's *Math 145b*, instructed by Peter Koellner. The formal name for this class is "Set Theory II". This class deals with large cardinals and their inner forms.

The permanent URL is `http://web.evanchen.cc/coursework.html`, along with all my other course notes.

## Contents

# §1 January 29, 2015

Same setup as Math 145: all grading is homework. This is a class on large cardinals and their inner models.

## §1.1 Motivation from Arithmetic

Recall from section one that we had

**Theorem 1.1** (Gödel)
Assume PA is consistent. Then there exists a $\varphi$ in the language of PA such that PA cannot prove either $\varphi$ or $\neg\varphi$.

**Remark 1.2.** This is really Rosser's result; Gödel's original formulation was slightly weaker.

**Theorem 1.3** (Gödel)
Assume PA is consistent. Then PA cannot prove Con(PA).

Fortunately, we can add in "missing sentences".

## §1.2 Motivation from Set Theory

The situation is much more problematic in set theory.

1. (Cardinal arithmetic) The first question we ask is CH, $2^{\aleph_0} = \aleph_1$.
2. (Infinite combinatorics) Suslin's Hypothesis: Trying to weaken Cantor's characterization from $\mathbb{R}$, replacing separability with the Suslin condition.
3. (Descriptive set theory) PM: All projective sets are Lebesgue-measurable.

Here a "projective set" is a set you take from $\mathbb{R}^n$ and taking a closed set, projecting it, complementing it, projecting it, complementing it...

Assuming ZFC is consistent, it cannot prove or refute any of these sentences. (Actually, the last statement requires replacing ZFC with ZFC plus the existence of a strongly inaccessible cardinal.)

We want to find new axioms that settle these independent questions.

## §1.3 Large Cardinals

Recall the von Neuman universe. Everything we do in most math classes lives in at most $V_{\omega+2}$.

In ZFC we can keep going higher than this, all the way up to $V_\kappa$, where $\kappa = \aleph_\kappa$ is the first fixed beth point. But what we *can't* reach is a strongly inaccessible cardinal. Recall the following definition.

**Definition 1.4.** $\kappa$ is strongly inaccessible if $\kappa$ is regular and for all $\overline{\kappa} < \kappa$, then $2^{\overline{\kappa}} < \kappa$.

It's a fact that if $\kappa$ is strongly inaccessible, then $V_\kappa \vDash \text{ZFC}$. Gödel's incompleteness theorem prohibits ZFC from establishing its own consistency, so ZFC cannot actually prove the existence of a strongly inaccessible cardinal.

**Remark 1.5.** In $V = L$, weakly inaccessible and strongly inaccessible are the same thing (because the generalized continuum hypothesis is true). So ZFC actually can't even show the existence of a weakly inaccessible cardinal.

We now give the following informal definition.

**Definition 1.6.** A **small** large cardinal is a large cardinal consistent with $V = L$.

Examples include inaccessible, inaccessible limits of inaccessibles, Mahlos, weakly compact, indescribables, .... We can do this using "reflection principles", deferred to reading.

In this course, we will start with "large" large cardinals – the ones which are *not* compatible with $V \neq L$, but at least still compatible with the Axiom of Choice. In particular, we will start with measurables. So our list will read

- measurables,
- strong
- Woodin
- super-compact
- extendable
- huge
- rank to rank
- $I_0$

But past that there are "very large" cardinals which are not even consistent with AC.

- Reinhardt
- Super-Reindhardt
- Berkeley

There's reasons to think up to Woodin, stuff is consistent. It's a little dicier up to $I_0$, but it's still reasonable. But there are reasons to believe the very large cardinals are not consistent.

## §1.4 Large cardinal axioms "close off independence"

**Theorem 1.7** (Martin, Steel, Woodin)

Assume there are infinitely many Woodin cardinals. Then PM holds.

In fact, forcing turns out to not be strong enough: given a proper class of Woodin cardinals, one cannot use forcing to establish any second-order arithmetic statements.

However,

**Fact 1.8.** The current large cardinal axioms do not resolve CH or SH.

Here CH is third-order arithmetic.

## §1.5 Inner Models of Large Cardinal Axioms

We have a very nice inner model $L$. It satisfies AC, CH, $\neg$ SH, and so on and so forth. It's almost as clear as the natural numbers: you ask a question about it, you can get the answer.

It turns out we can get $L$-like analogs to get measurable, strong, and Woodin cardinals. But we want a "ultimate $L$" which can contain all of these cardinals: this will let us eliminate independence altogether.

## §1.6 One Natural Inner Model: The Constructible Universe

**Theorem 1.9** ($L$-Dichotomoy Theorem)
Exactly one of the two holds:

(1) For all singular cardinals $\gamma$,

  (a) $\gamma$ is singular in $L$

  (b) $(\gamma^+)^V = (\gamma^+)^L$.

  So in this case we say $L$ is close to $V$.

(2) All uncountable cardinals are inaccessible in $L$. In this case way say $L$ is far from $V$.

There is a "switch", namely a real number $0^\sharp$, with the property that

- $0^\sharp$ exists then $L$ is far from $V$
- $0^\sharp$ doesn't exist then $L$ is close to $V$.

It turns out that the existence of measurable cardinals then $0^\sharp$ exists.

## §1.7 Another Natural Inner Model: HOD

**Definition 1.10.** A set $x$ is **ordinal-definable** if $x$ is definable in $V$ from ordinal parameters.

The motivation for this is that most notions of definability let you consider the least ordinal not defined in this way. (For example: "the least natural number not definable in less than 1000 words"). It turns out that ordinal-definability does not suffer from this weakness.

**Fact 1.11.** OD is ordinal-definable.

Now we define a notion of *hereditary ordinal-definability.*

**Definition 1.12.** $x$ is hereditary ordinal-definable if its transitive closure is ordinal definability.

**Theorem 1.13** (Gödel)
HOD proves ZFC.

What are the differences?

1. $L$ is built up from below. HOD is built up from above.

2. $L$ is incompatible with large large cardinals, but HOD is compatible.

We'll have to use a notion we can't fully define in the following theorem.

**Theorem 1.14** (HOD Dichotomy Theorem, Woodin)
Assume that $\delta$ is an extendible cardinals. Then exactly one of the following holds

1. For all singular cardinals $\gamma > \delta$,
   (a) $\gamma$ is singular in HOD.
   (b) $(\gamma^+)^V = (\gamma^+)^{\text{HOD}}$.

2. All cardinals greater than $\delta$ are $\omega$-strongly measurable in HOD.

This leads us to the following hopes.

- (HOD Hypothesis) There is a proper class of cardinals which are not $\omega$-strongly inaccessible in HOD (hence we're in the first case of the HOD dichotomy).

- (HOD Conjecture) ZFC proves the HOD Hypothesis. (Also called the "silly conjecture".)

# §2 February 3, 2015

We're going to try to do measurable cardinals today and Thursday; this may go a little fast.

## §2.1 Filters

**Definition 2.1.** A **filter** $F$ on a set $A$ is a set $F \subseteq \mathcal{P}(A)$ such that

(i) $A \in F$ and $\varnothing \notin F$.

(ii) $\forall X, Y \in F$, $X \cap Y \in F$; meaning we have closure under intersection.

(iii) If $X \subset Y \subset A$, $X \in F \implies Y \in F$. In other words we require that the filter is "upwards closed".

Filters give you sense of "closeness".

**Definition 2.2.** Let $F$ be a filter on $A$. Then

(i) $F$ is an **ultrafilter** if for each $X$, either $X \in F$ or $A \setminus X \in F$.

(ii) $F$ is **principle** if its generated by $Y$; i.e. is of the form $F = \{X \mid X \supseteq Y\}$.

**Remark 2.3.** The "right" way to think of an ultrafilter is that it is *maximal under inclusion.* In particular, we can extend every filter $F$ to an ultrafilter of $A$ by considering the poset of filters of $A$ under inclusion and using Zorn's lemma.

**Example 2.4**

Consider $F$ the subsets of $\omega$ with finite complements (i.e. **cofinite sets**). Then $F$ is a filter, but is not principle nor an ultrafilter.

We'll use $U$ and $V$ for ultrafilters.
Easy lemma:

**Lemma 2.5**

Suppose $U$ is an ultrafilter on $A$. The following are equivalent.

(i) $U$ is principle.

(ii) $\{a\} \in U$ for some $a$.

(iii) $U$ is generated by $\{a\}$.

*Proof.* The content is to show (i) $\implies$ (ii) as the other implications are easy. Suppose $U$ is generated by $Y$ and pick $a \in Y$. Since $U$ is an ultrafilter, either $\{a\} \in U$ or $A \setminus \{a\} \in U$. The second case can't happen since we can just note that

$$Y \cap (A \setminus \{a\})$$

needs to be in $U$, but $U$ was generated by $Y$ and the above set does not contain $a \in Y$. □

**Example 2.6**

Consider $F$ the subsets of $\omega$ generated by $\{17\}$. This is principle (by definition). You can also check that it's an ultrafilter.

**Proposition 2.7**

Any ultrafilter on a finite set is principle.

*Proof.* This turns out to be the same as Arrow's Impossibility Theorem. Let $X$ be a set of voters. Call a subset $S \subseteq X$ *decisive* if only the votes of $S$ matter. This is clearly a filter. The "irrelevance of independent alternatives" is the intersection condition. □

## §2.2 $\kappa$-complete Filters

The following definition is a central notion.

**Definition 2.8.** Suppose $F$ is a filter and $\kappa$ is a cardinal. Then we say $F$ is **$\kappa$-complete** if for all $\gamma < \kappa$ we have

$$(X_\alpha)_{\alpha<\gamma} \in F \implies \bigcap_\alpha X_\alpha \in F.$$

In other words, the intersection of strictly fewer than $\kappa$ sets in $F$ must also lie in $F$.

Then a filter is by definition 2-complete. By induction, we are closed under intersections of $n$ sets; hence every filter is $\aleph_0$ complete.

**Example 2.9**

Let $A = \omega$ and $F$ be the cofinite sets. Like all filters, $F$ is $\aleph_0$ complete. But it is not $\aleph_1$ complete, because we can let

$$X_n = \omega \setminus \{n\}$$

and the countable intersection of the $X_n$'s is $\varnothing \notin F$.

**Definition 2.10.** A filter $F$ is **countably complete** if it is $\aleph_1$-complete: closure under countable intersections.

**Definition 2.11.** A cardinal $\kappa > \omega$ is a **measurable cardinal** if there's a non-principle $\kappa$-complete ultrafilter on $\kappa$.

**Remark 2.12.** This is a $\Sigma_1$ definition.

**Proposition 2.13**

There is a non-principle $\aleph_0$-complete ultrafilter on $\omega$.

*Proof.* Apply Zorn's Lemma to the filter of cofinite sets to get an ultrafilter $U$. Then $U$ contains all cofinite sets. Thus $U$ can't contain any finite sets at all! □

This is highly nonconstrucitve, which is to be expected because we're using the Axiom of Choice via Zorn's lemma!

**Proposition 2.14**

If $\kappa$ is measurable it is strongly inaccessible.

*Proof.* Exercise. (This is why we require $\kappa > \omega$.) □

Later we'll give a new, more natural definition of "measurable" (currently we have a combinatorial definition). Existence of measurable cardinals implies $V \neq L$, and so we will work towards building an $L$-like model containing measurables.

## §2.3 Completeness

**Definition 2.15.** Suppose $F$ is a nonprinciple ultrafilter on $A$. The **completeness of $F$** is the least cardinal $\kappa$ such that there exists $(X_\alpha)_\kappa$ in the filter such that

$$\bigcap X_\alpha \notin F.$$

In other words, it's the least $\kappa$ for which things break down. We write $\kappa = \operatorname{comp}(F)$. (If $F$ is nonprinciple we let $\operatorname{comp}(F) = \infty$.)

**Remark 2.16.** Since $F$ is nonprinciple, we have

$$\bigcap F = \varnothing \notin F.$$

Thus this is a good definition.

**Exercise 2.17.** Show that in fact $\operatorname{comp}(F) \leq |A|$.

*Proof.* I think you just take cofinite sets? TODO check this. □

Standard lemma.

**Lemma 2.18** (Functions Induce Filters)

Suppose $F$ is a filter on $A$ and $f : A \to B$. Let

$$G = \{X \in \mathcal{P}(B) \mid f^{\text{pre}}(X) \in F\}.$$

Then

(i) $G$ is a filter.

(ii) If $F$ is an ultrafilter then so is $G$.

(iii) More strongly, $\operatorname{comp}(F) \leq \operatorname{comp}(G)$.

(iv) If $F$ is an ultrafilter, then $G$ is principal if and only if $f^{\text{pre}}(\{b\}) \in F$ for $b \in B$.

*Proof.* Just do it (unwind definitions). □

The first substantive theorem is:

**Theorem 2.19**

The following are equivalent.

(i) There exists a measurable cardinal.

(ii) There exists a countably complete non-principal ultrafilter.

*Proof.* If $\kappa$ is measurable then there exists a $\kappa$ complete non-principal ultrafilter with $\kappa > \omega$. So (i) $\implies$ (ii) is tautological and we need only prove the converse.

Let $U$ be a $\aleph_1$-complete non-principal ultrafilter on $A$ and set $\kappa = \text{comp}(U)$. By the exercise $\kappa = \text{comp}(U) \le |A|$. Now we wish to specify a function $f : A \to \kappa$.

By definition, we can take

$$\{X_\alpha : 0 < \alpha < \kappa\} \subseteq U$$

such that

$$A' \overset{\text{def}}{=} \bigcap_{0<\alpha<\kappa} X_\alpha \notin U.$$

Then $A - A' \notin U$. We define $f : A \to \kappa$ by

$$a \mapsto \begin{cases} \text{least } \gamma \text{ with } a \notin X_\gamma & a \in A - A' \\ 0 & a \in A'. \end{cases}$$

By the lemma this function induces an ultrafilter $V$ on $\kappa$, with $\text{comp}(V) \ge \text{comp}(U) = \kappa$. But $V$ is a filter on $\kappa$ and $\text{comp}(V) \le |\kappa| = \kappa$ as well. So it suffices to prove $V$ is nonprinciple, i.e. that $f^{\text{pre}}(\gamma) \notin F$ for every $\gamma$. But

- If $\gamma = 0$ we get $f^{\text{pre}}(0) = A' \notin U$.
- If $\gamma \ne 0$ we get $f^{\text{pre}}(\gamma)$, which by definition is by definition disjoint from $X_\gamma$. Hence it can't be in the filter (filters can't contain disjoint sets since their intersection would then be empty).

Here, we used the "countable completeness" to guarantee that $\kappa > \omega$ (since $\kappa = \omega$ would be useless). □

## §2.4 Normal Ultrafilters

**Definition 2.20.** An ultrafilter $U$ on an infinite cardinal $\kappa$ is **normal** if for each $f : \kappa \to \kappa$, if

$$\{\alpha < \kappa \mid f(\alpha) < \alpha\} \in U$$

then

$$\exists \beta \in \kappa : \{\alpha \in \kappa \mid f(\alpha) = \beta\} \in U.$$

**Definition 2.21.** Suppose $\kappa$ is an infinite regular cardinal. Let $\{X_\alpha\}_{\alpha<\kappa}$ be a sequence of subsets of $\kappa$. Then we define the **diagonal intersection** by

$$\triangle_{\alpha<\kappa} X_\alpha = \bigcup_\alpha ([0, \alpha] \cap X_\alpha).$$

**Example 2.22**

Let $X_\alpha = \kappa \setminus \alpha$ for each $\alpha$. Then

$$\bigcap_{\alpha<\kappa} X_\alpha = \kappa$$

yet

$$\triangle_{\alpha<\kappa} X_\alpha = \kappa.$$

Somehow this is less strict than a normal intersection.

**Exercise 2.23.** Let $U$ be an ultrafilter on an infinite $\kappa$. Then $U$ is normal if and only if its closed under diagonal intersection.

*Proof.* Short. Do it yourself. □

**Proposition 2.24**

Let $U$ be a normal ultrafliter on an infinite $\kappa$. The following are equivalent.

(1) $U$ is $\kappa$-complete and non-principle.

(2) $U$ is **uniform**, meaning every $X \in U$ has $|X| = \kappa$.

(3) $U$ is **weakly uniform**, meaning $U$ contains all the tails: for every $\gamma < \kappa$ we have $\gamma \setminus \kappa \in U$.

**Theorem 2.25** (Scott)

If $\kappa$ is a measurable cardinal then there is uniform normal ultrafilter on $\kappa$.

Hence we will generally assume our ultrafilters our normal, since normal ultrafilters are nice.

Later we'll give a geometric characterization of measurability.

# §3 February 5, 2015

Continuing the chapter on measurable cardinals...

The key definition we introduced last time was the notion of a normal ultrafilter: the idea is that if $f : \kappa \to \kappa$ "presses down" on a measure one set (i.e. an element of $U$), then it's actually constant on that measure one set.

## §3.1 Scott's Lemma

**Lemma 3.1** (Scott)

If $\kappa$ is a measurable cardinal, then there exists a *normal* uniform ultralfiter on $\kappa$.

*Proof.* Say a subset of $\kappa$ is *big* if it's a subset of $U$.

Since $\kappa$ is measurable there exists a $\kappa$-complete non-principal ultrafilter $U$ on $\kappa$.

We claim that there exists a function $f : \kappa \to \kappa$ such that

(i) For each $\beta < \kappa$, the set $\{\alpha < \kappa \mid f(\alpha) = \beta\}$ is not in $U$. In other words $f$ is not constant on any big set.

(ii) For each $g : \kappa \to \kappa$, if $\{\alpha < \kappa \mid g(\alpha) < f(\alpha)\} \in U$ then there exists $\beta < \kappa$ such that $\{\alpha < \kappa \mid g(\alpha) = \beta\} \in U$. In other words, if $g < f$ on a big set then $g$ is constant on a big set.

Note that (i) is satisfied by the identity function for $f$. So if $U$ was normal, we'd be done. The claim is essentially saying that we have "normal with respect to $f$".

*Proof of Claim.* Assume for contradiction no such function $f$ exists. Start with $f_0 = \mathrm{id}$. As (2) must fail for $f_0$, we can find $f_1$ such that $f_1 < f$ on a big set but is not constant on any big set. Hence $f_1$ satisfies condition (1), so condition (2) must fail for $f_1$. This lets us construct a sequence $f_0, f_1, f_2, \dots\dots$.

Define

$$X_n = \{\alpha < \kappa \mid f_{n+1}(\alpha) < f_n(\alpha\} \in U.$$

Since $U$ is countably complete (it's $\kappa$-complete!) it follows that

$$\bigcap X_n \in U.$$

But no element can lie in all these $X_n$ since that would give

$$f_0(\alpha^*) > f_1(\alpha^*) > \dots$$

an infinite descending chain of ordinals. This is impossible, and the claim is proved. ∎

So now we have a function $f$ which plays the role of the identity. Set

$$V = \{X \subseteq \kappa \mid f^{\text{pre}}(X) \in U\}.$$

Then $V$ is a $\kappa$-complete non-principle ultrafilter on $\kappa$ (by some exercises), and thus it is uniform. So we only need to show $V$ is normal.

Suppose $g : \kappa \to \kappa$ is such that

$$S \overset{\text{def}}{=} \{\alpha < \kappa \mid g(\alpha) < \alpha\} \in V.$$

We want to show that $g$ is constant on some set of $V$. But $f$ applied to it is in $U$:

$$f``(S) = \{\alpha < \kappa \mid g(f(\alpha)) < f(\alpha)\}.$$

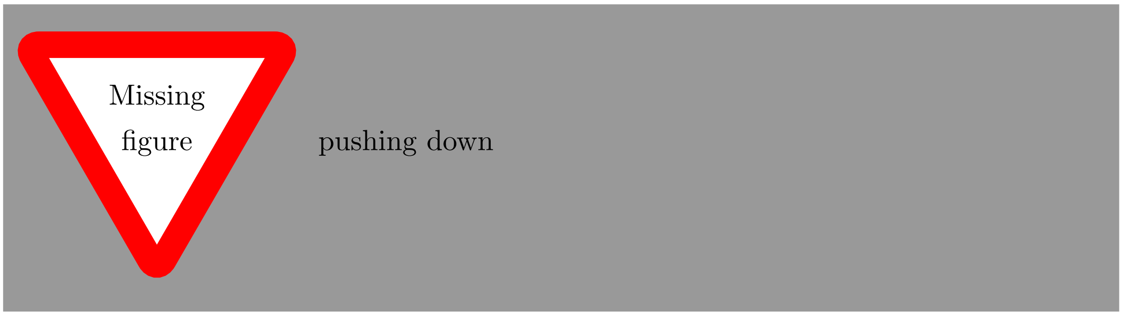
Then $g \circ f$ presses below $f$ on a $U$-big set. By the definition of $f$, we find that $g \circ f$ is constant on a $U$-big set, i.e.

$$\{\alpha < \kappa \mid g(f(\alpha)) = \beta\} \in U$$

It follows that

$$\{\alpha < \kappa \mid g(\alpha) = \beta\} \in V$$

as needed. $\square$




## §3.2 Ultrapowers

Our next goal is to show that $\kappa$ is measurable if there's an elementary embedding of $V$ into an $M$ such that $\kappa$ is the first guy moved. We will have agreement $V_{\kappa+1}^M = V_{\kappa+1}$. To be precise...

## §3.3 Set Models

We focus on models of the form $\mathscr{M} = (M, E)$, where $E \subseteq M \times M$. Suppose we have such a model $\mathscr{M}$ and $U$ is an ultrafliter on some set $A$.

Let's look at the functions $A \to M$. Consider $f, g : A \to M$.

**Definition 3.2.** We say

$$f \sim_{U,M} g$$

if

$$\{a \in A \mid f(a) = g(a)\} \in U.$$

This is the same "big set" intuition: if $f$ and $g$ agree on a $U$-big set, then we consider them the same. Of course, this is an equivalence relation (since $U$ is a filter). So we can consider $(A \to M)/U$ to be the set of these equivalence classes.

Now we can define a membership relation $E_{U,M}$ on equivalence classes $[f]$ and $[g]$. We'll say $[f]\ E_{U,M}\ [g]$ to mean

$$\{a \in A \mid f[a]\ E\ g(a)\} \in U.$$

**Definition 3.3.** Suppose $\mathscr{M} = (M, E)$ is a set model and $U$ is an ultrafilter on $A$. The **ultrapower** of $\mathscr{M}$ with respect to $U$ is the model with points $(A \to M)/U$ and $E_{U,M}$. That is,

$$\Pi_U(\mathscr{M}) = ((A \to M)/U, E_{U,M}).$$

## §3.4 Łoś's Theorem

We'll now drop subscripts.

**Theorem 3.4** (Łoś)

Suppose $\mathscr{M} = (M, E)$ is a set model $U$ is an ultrafilter of $A$. Then for each formula $\phi(x_1, \dots, x_n)$ and each $f_1, \dots, f_n \in A \to M$ we have

$$\Pi_U(\mathscr{M}) \vDash \phi\left[[f_1], \dots, [f_n]\right]$$

if and only if

$$\{a \in A \mid \mathscr{M} \vDash \phi[f_1(a), \dots, f_n(a)]\} \in U.$$

This is the key!

*Proof.* By induction on formula complexity. The atomic base cases $[f_1] = [f_2]$ and $[f_2] \; E_{U,M} \; [f_1]$ are immediate by definition.

Hence it suffices to verify for $\wedge$, $\neg$, $\exists$. For $\wedge$, we essentially use the fact that in any filter $U$, $(X_1 \in U) \wedge (X_2 \in U) \iff (X_1 \cap X_2 \in U)$. (Just chase definitions). $\neg$ is the same, but we use the "ultra" part of ultrafilter: exactly one of $X$ and $U \setminus X$ is in $U$ (for every $X$).

The interesting case is $\exists$, and requires AC. Unwinding,

$$\begin{aligned}
&\exists x_0 \psi : \Pi_U \mathscr{M} \vDash (\exists x_0)\psi[x_0, [f_1], \dots, [f_n]] \\
\iff\; &(\exists f_0 : A \to M) : \Pi_U(\mathscr{M}) \vDash \psi\left[[f_0], \dots, [f_n]\right] \\
\iff\; &(\exists f_0 : A \to M) \{a \in A \mid \mathscr{M} \vDash \psi[f_0(a), \dots, f_n(a)]\} \in U \\
\overset{\text{AC}}{\iff}\; &\{a \in A \mid (\exists b \in M)\mathscr{M} \vDash \psi[b, f_1(a), \dots, f_n(a)]\} \in U \\
\iff\; &\{a \in A \mid \mathscr{M} \vDash \exists x_0 \psi[x_0, f_1(a), \dots, f_n(a)]\} \in U
\end{aligned}$$

Here Axiom of Choice is used to show that if $b \in M$ exists for each $a$, then we can generate the function sending each $a$ to each $b$. □

**Definition 3.5.** Suppose $\mathscr{M} = (M, E)$ and $\mathscr{N} = (N, F)$ are set models. Then $j : M \to N$ is an **elementary embedding** if for all $\phi[x_1, \dots, x_n]$ in LST and $a_1, \dots, a_n \in M$, we have $\mathscr{M} \vDash \phi[a_1, \dots, a_n]$ if and only if $\mathscr{N} \vDash \phi[j(a_1), \dots, j(a_n)]$.

In particular $\mathscr{M}$ and $\mathscr{N}$ have the same sentences.

**Corollary 3.6** (Łoś)

Suppose $\mathscr{M} = (M, E)$ is a set model and $U$ is an ultrafilter on $A$. There is an elementary embedding of $\mathscr{M}$ into $\Pi_U \mathscr{M}$ via the map

$$i_{U,M} : M \to (A \to M)/U$$

given by $m \mapsto [c_m]$, where $c_m : A \to M$ is the constant function sending each $a \in A$ to $m \in M$.

*Proof.* By Łoś, for $\Pi_U \mathscr{M} \vDash \phi[i(a_1), \dots, i(a_n)]$ (where $a_1, \dots, a_n \in M$, sorry) is equivalent to

$$\{x \in A \mid M \vDash \phi[a_1, \dots, a_n]\} \in U.$$

But the left-hand side is either $\varnothing$ or $A$, and it's $A$ exactly when $\mathscr{M} \vDash \phi[a_1, \dots, a_n]$. □

# §4 February 12, 2015

Darn snow.

## §4.1 Mostowski Collapse of $\Pi_U(M)$

Recall last time we had $\Pi_U(M)$ constructed. We want to take the Mostowski collapse of $\Pi_U(M)$ so that it's membership relation is the real $\in$. To do so we need to show that $\Pi_U \mathscr{M}$ is well founded.

**Lemma 4.1**

Suppose $\mathscr{M} = (M, E)$ is a well-founded set model and $U$ is a countably complete ultrafilter. Then $\Pi_U \mathscr{M}$ is well-founded.

*Proof.* Suppose not. Let $\langle [F_n] : n \in \omega \rangle$ be a sequence of functions such that

$$[f_{n+1}] \; E_{U,M} \; [f_n]$$

for all $n$. This means that

$$X_n \overset{\text{def}}{=} \{x \in A : f_{n+1}(x) \; E \; f_n(x)\} \in U$$

is "big" (it's in $U$) for each $n$. But $U$ is countably complete, meaning $\varnothing \neq \bigcap X_n \in U$. But any $x \in \bigcap X_n$, whence $f_{n+1}(x) \; E \; f_n(x)$ which is a contradiction. □

**Lemma 4.2**

Suppose $\mathscr{M} = (M, E)$ is a well-founded set model that satisfies Extensionality. Then there is a unique isomorphism

$$\pi : (M, E) \cong (N, \in).$$

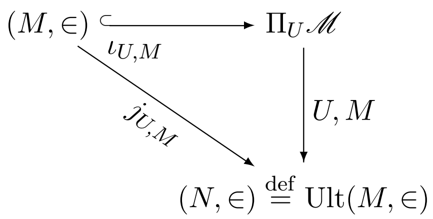
*Proof.* Take the Mostowski collapse. □

## §4.2 Ultrapowers as Transitive Models

In the case of interest, we'll have $E = \in$, giving models $\mathscr{M} = (M, \in)$, where $U$ will be countably complete. In this case we get

$$\pi_{U,M} : \Pi_U \mathscr{M} \cong (N, \in).$$

So we have a commutative diagram as follows.




This means that for any $\mathscr{M}(M, \in)$ which is well-founded we can get an embedding into the ultrapower $\text{Ult}(M, U)$. We'll denote $[f]_{M,U} \in \text{Ult}(M, U)$ the image of $[f]_{M,U} \in \Pi_U \mathscr{M}$.

By Łoś, we have that

$$\text{Ult}(M, U) \vDash \phi[[f_1], \dots, [f_n]] \iff \{x \in A : \mathscr{M} \vDash \phi[f_1(x), \dots, f_n(x)]\} \in U.$$

One can show that if $\kappa$ is measurable, $U$ is an ultrafilter on it, and $\lambda > \kappa$ then the cardinals less than $\kappa$ are preserved under the map $V_\lambda \to \text{Ult}(V_\lambda, U)$, but $\kappa$ goes upwards.

## §4.3 Ultrapowers of Class Models

For us, classes are *definable classes with parameters*. Keep in mind that for a class $\mathcal{C}$, $x \in \mathcal{C}$ really means "$x$ satisfies the definition of $\mathcal{C}$". For example, $x \in \text{On}$ means "$x$ is an ordinal".

Let $\mathscr{M} = (M, E)$ be a **class model**, meaning $M$ is a class given by $\phi_M$ and $E$ is a class given by $\phi_E$. We want to use our ultrafilter $U$ to carve $\mathscr{M}$ into equivalence classes. We cannot just write

$$[f]_{U,M} = \{g : A \to M \mid g \sim_{U,M} f\}$$

because then the $[f]_{U,M}$ would be a class, which would make $\Pi_U \mathscr{M}$ into a class of classes.

The way around this is to use a so-called "Scott's trick" to replace this with a set. So we want to put

$$[f]_{U,M} = \{g : A \to M \mid g \sim_{U,M} f \wedge g \text{ has minimal rank}\}.$$

You can show this is a set. So now we let $\Pi_U \mathscr{M}$ be the class of these sets.

Everything is definable.

Thus we have Łoś in the following form.

**Theorem 4.3** (Łoś)

Suppose $\mathscr{M} = (M, \in)$ is a class model and $U$ is an ultrafilter. Let $\phi$ be a *fixed* formula in the language of set theory. Then for any $f_1, \dots, f_n : A \to M$,

$$\Pi_U(\mathscr{M}) \vDash \phi[[f_1], \dots, [f_n]]$$

if and only if

$$\{a \in A \mid \mathscr{M} \vDash \phi[f_1(a), \dots, f_n(a)]\} \in U.$$

The subtle distinction is classes versus sets. Here there is a different statement of the theorem for every $\phi$.

## §4.4 Set-Like Models

Assume $M$ is well-founded. Hence if $U$ is countably complete then $\Pi_U \mathscr{M}$ is an well-founded. We want to take the Mostowski collapse of this.

**Exercise 4.4.** Define $E$ by $(\alpha, \beta) \in E$ if and only if

- $\alpha < \beta$ and $\alpha$ and $\beta$ are even ordinals,
- $\alpha < \beta$ and $\alpha$ and $\beta$ are odd ordinals,
- $\alpha$ is even and $\beta$ is odd.

Show that the model $(\text{On}, E)$ has no transitive collapse.

The issue is that $(\mathrm{On}, E)$ has order type $\mathrm{On} + \mathrm{On}$. As written above, $(M, \in)$ is not set-like.

**Definition 4.5.** A class-model $\mathscr{M} = (M, E)$ is **set-like** if $\forall x \in M$, the class $y \in M \wedge y \; E \; x$ is a set.

This is the condition required to carry out the Mostowski collapse.

**Lemma 4.6**

Suppose $\mathscr{M} = (M, E)$ is a well-founded class model which is both extensional and set-like. Then there exists a unique isomorphism

$$\Pi : (M, E) \to (N, \in)$$

where $N$ is also a class.

## §4.5 Central Results

**Exercise 4.7.** Suppose $U$ is a principle ultrafilter on a set $A$. Then

$$\mathrm{Ult}(V, U) = V$$

and the map $j_U : V \to V$ is the identity.

*Proof.* Suppose $U$ is generated by $a$. Then all the truth in $\mathrm{Ult}(V, U)$ is determined by $a$: you can "replace" every $f$ as $f(a)$. Even more explicitly

$$[f] \in [g] \iff \{x \in A \mid f(x) \in g(x)\} \iff f(a) \in g(a)$$

and so nothing happens. □

**Lemma 4.8**

Suppose $U$ is a countably complete ultrafilter on a set $A$. Let $\kappa$ denote the completeness of $U$ and

$$j_U : V \to \mathrm{Ult}(V, U)$$

be the associated embedding. Then

(i) $j_U$ is the identity when restricted to $V_\kappa$.

(ii) $j_U(\kappa) > \kappa$.

*Proof.* First, we claim the following.

**Claim 4.9.** $j_U$ restricted to $\kappa$ is the identity.

*Proof of Claim.* Induction on $\alpha < \kappa$. Assume that for all $\gamma < \alpha$ we have $j_U(\gamma) = \gamma$. We want to show that $[c_\alpha]_U = \alpha$, where $c_\alpha$ is the constant function returning $\alpha$.

If $\gamma \in \alpha$, then $\gamma = [c_\gamma] \in [c_\alpha]$ (since we had an elementary embedding). This shows that $\alpha \subseteq [c_\alpha]_U$.

Now suppose $[f] \in [c_\alpha]$. Then by Łoś, this holds if and only if

$$\{x \in A \mid f(x) \in c_\alpha(x) = \alpha\} \in U.$$

By $\kappa$-completeness, of $U$, there must be exist a $\gamma < \alpha$ such that

$$\{x \in A \mid f(x) = \gamma\} \in U.$$

For if not, $\bigcap_\gamma \{x \in A \mid f(x) \neq \gamma\} \in U$ is a measure one set, which is impossible. Hence $[f] = [c_\gamma] = \gamma \in \alpha$. ■

Now we can finish the main proof of the claim by induction on $\alpha < \kappa$. If $\alpha$ is a limit, then it's clear. Now suppose $j_U$ is the identity restricted to $V_\alpha$ and we want to get $V_{\alpha+1}$. Let $x \subseteq V_\alpha$; we wish to check $j_U(x) = x$. For any $y \in x$, our induction hypothesis says $j_U(y) = y \in j_U(x)$, *id est* $x \subseteq j_U(x)$.

So we want to show $j_U(x) \subseteq x$. It's enough to prove $j_U(x) \subseteq V_\alpha$, since then we can repeat the inductive hypothesis.

According to the claim, $j_U(x) \subseteq (V_{j_U(\alpha)})^{\mathrm{Ult}(V,U)} = V_\alpha^{\mathrm{Ult}(V,U)}$, which equals $V_\alpha$ by the induction hypothesis.

Thus we are left to show $[c_\kappa] = j_U(\kappa) > \kappa$. We seek $f$ which will be above all the $[c_\alpha]$ for $\alpha < \kappa$ and yet less than $[c_\kappa]$. We already know there's a sequence $\langle X_\alpha \mid \alpha < \kappa \rangle$ of sets in $U$ whose intersection $\bigcap X_\alpha$ is not in $U$. Hence we let $f : A \to \kappa$ by

$$f(a) = \begin{cases} \text{least } \gamma : a \notin X_\gamma & a \notin \bigcap X_\gamma \\ 0 & \text{otherwise.} \end{cases}$$

Then $f$ is nonzero on a measure one set.

For $\alpha < \kappa$, $\alpha = [c_\alpha] < [f]$. Then $\{a \in A \mid f(\alpha) \leq \alpha)\}$ is certainly in $U$ by $\kappa$-completeness. On the other hand $[f] < [c_\kappa]$ by construction.

Hence there's a strict inequality

$$\sup_{\alpha<\kappa} [c_\alpha] < [f] < [c_\kappa]$$

so $j_U(\kappa) \neq \kappa$. □

Later we are going to show that there exists an elementary embedding with this property if and only if $\kappa$ is measurable, giving us a geometric notion of "measurable cardinal".

# §5 February 17, 2015

Today we'll finish up measurable, showing that the geometric embedding is equivalent to the combinatorial one. We'll then show some properties of measurables; in particular, the existence of a measurable implies $V \neq L$.

Last time we had an embedding

$$j_U : V \to \mathrm{Ult}(V, U)$$

which was the identity restricted to $V_\kappa$ but with $j_U(\kappa) > \kappa$.

## §5.1 Properties of the embedding

**Theorem 5.1**

Suppose $U$ is a countably complete nonprincipal ultrafilter on a set $A$, and let $\kappa$ be its completeness. Let

$$j_U : V \to \mathrm{Ult}(V, U)$$

be the associated embedding. Then

(i) $V_{\kappa+1}^{\mathrm{Ult}(V,U)} = V_{\kappa+1}$

(ii) Each sequence $\mathrm{Ult}(V, U)$ of length $\kappa$ is in the set $\mathrm{Ult}(V, U)$.

(iii) Given $A = \kappa$, the restriction of $j_U$ to $\kappa^+$ is not in $\mathrm{Ult}(V, U)$. Hence we don't preserve $\kappa^+$ sequences.

(iv) $U \notin \mathrm{Ult}(V, U)$. In particular, since $U \subseteq V_{\kappa+2}$, we have $V_{\kappa+2} \subsetneq \mathrm{Ult}(V, U)$.

*Proof.* (i) We already know $V_\kappa^{\mathrm{Ult}(V,U)} = V_\kappa$. Since $\mathrm{Ult}(V, U) \subseteq V$, we know

$$V_{\kappa+1}^{\mathrm{Ult}(V,U)} \subseteq V_{\kappa+1},$$

and so it remains to show the reverse inclusion.

Fix $x \in V_{\kappa+1}$; that is, $x \subseteq V_\kappa$. Since $j_U$ is an elementary embedding Thus $j_U(x) \subseteq V_{j_U(\kappa)}^{\mathrm{Ult}(V,U)}$. Also, for any $y \in x$, we have $j_U(y) \in j_U(x)$. Hence $x = j_U(x) \cap V_\kappa$. So the geometric picture is that we're "inflating" $x$: there are "new elements" being added on.

(ii) Consider a $\kappa$ sequence

$$\langle [f_\alpha] : \alpha < \kappa \rangle .$$

We seek $[g] \in \mathrm{Ult}(V, U)$ such that the first $\kappa$ terms are the sequence above. Let $f$ be such that $[f] = \kappa$.

Suppose (for wishful thinking) that $g$ did exist. Then

$$[g]|_{[f]} = \langle [f_\alpha] : \alpha < [f] \rangle$$

reads (according to Łoś)

$$U \ni \left\{ a \in A \mid V \vDash g(a)|_{f(a)} = \langle f_\alpha(a) \mid \alpha < f(a) \rangle \right\} .$$

So we'll define it this way. We define $g : A \to (\kappa \to V)$ matching each $a \in A$ to the $\kappa$-sequence

$$\langle f_a(\alpha) : \alpha < \kappa \rangle .$$

For $\alpha < \kappa$, the $\alpha$th term of $[g]$ is

$$[g](\alpha) = [g]([c_\alpha]) = [f_\alpha]$$

where the last equivalence holds since

$$\mathrm{Ult}(V,U) \vDash [g_\alpha]([c_\alpha]) = [f_\alpha] \iff U \ni \{a \in A \mid (g(a))(\alpha) = f_\alpha(a)\} = A.$$

(iii) We want to show $j_U$ restricted to $\kappa^+$ is an element of $\mathrm{Ult}(V,U)$, given $A = \kappa$. We claim that

$$j_U(\kappa^+) = \sup j_u``(\kappa^+).$$

Assume not, so we have an inequality of ordinals $\sup j_u``(\kappa^+) < j_U(\kappa^+)$. Let $[f]$ and $[c_{\kappa^+}]$ denote the left and right-hand sides. So $f(\alpha) < \kappa^+$ for most $\alpha$ ("most" in the sense of $U$). Hence by Łoś there exists a $\beta < \kappa^+$ such that $\mathrm{ran}(f) \cap \kappa^+ \subsetneq \beta$. So $[f] < [c_\beta]$ by Łoś which is a contradiction.

(iv) Assume for contradiction that $U \in \mathrm{Ult}(V,U)$. We will show that (iii) fails.

First, consider $\mathcal{P}(\kappa \times \kappa) \in \mathrm{Ult}(V,U)$. By coding, the $\kappa$-sequences of $\kappa^+$ are all in $\mathrm{Ult}(V,U)$.

Now for $\alpha < \kappa^+$, we have

$$j_U(\alpha) = [c_\alpha] = \{[f] \mid f : \kappa \to \alpha\}.$$

since $[f] \in [c_\alpha]$ if and only if $f(\xi) < \alpha$ for most $\xi \in \kappa$. Putting it all together, we can compute $j_U``(\kappa^+)$ (and hence $j_U$ restricted to $\kappa^+$) from the $\kappa$-sequences of $\kappa^+$ and the ultrafilter $U$ (since that's all we need to do this computation). This contradicts (iii).

□

The ultrapower are nice because they "agree with $V$". Here there's two notions of agreement: you can say "agreeing up to rank" (here, $V_\kappa$ is preserved) or you can say they agree on $\kappa$-sequences. (Here $\kappa$ is inaccessible, so there's a bijection $\kappa \leftrightarrow V_\kappa$ and hence preserving $\kappa$ sequences is enough to preserves all of $V_\kappa$.) The point of the theorem above is to show that in both cases, somehow $\kappa$ is the "best possible".

We could demand even more agreement, and the two notions would lead us up different chains of large cardinals.

## §5.2 Geometry and combinatorial views coincide

**Theorem 5.2** (Scott, Keisler)

Suppose $\kappa$ is an ordinal. Then the following are equivalent.

(1) $\kappa$ is measurable.

(2) There is an elementary embedding $j : V \to M$ such that $M$ is transitive and $\kappa$ is the critical point of $j$ (meaning $\kappa$ is the smallest ordinal not preserved by $j$),

(3) There is an elementary embedding $j : V_{\kappa+1} \to N$ such that $N$ is a transitive set with critical point $\kappa$.

*Proof.* (1) $\implies$ (2): done already.

(2) $\implies$ (3): use $j$ restricted to $V_{\kappa+1}$; let $N = M \cap V_{j(\kappa)+1}$.

(3) $\implies$ (1): Let $U = \{x \subseteq \kappa \mid \kappa \in j_U(x)\}$ be a filter on $\kappa$; one can think of $\kappa$ as the "seed" that generates $U$. First, let's check this is a ultrafilter on $\kappa$.

- $\kappa \in j_U(\kappa)$.
- $\varnothing \notin U$ since $\kappa \notin \varnothing$.
- $x, y \in U \implies x \cap y \in U$ because $j(x \cap y) = j_U(x) \cap j_U(y)$.
- $x \in U$ and $x \subseteq y$ implies $y \in U$, since $j_U(y) \supseteq j_U(x) \ni \kappa$.
- For any $\alpha \in \kappa$ we have $\kappa \notin \{\alpha\} = j_U(\{\alpha\})$. So $U$ is nonprincipal.

Next we want to show that for any $\gamma < \kappa$, $U$ is closed under $\gamma$ intersections. Compute

$$\begin{aligned} j_U\left(\bigcap \langle x_\alpha \mid \alpha < \gamma\rangle\right) &= j_U\left(\bigcap \langle x_\alpha \mid \alpha < \gamma\rangle\right) \\ &= \bigcap j_U\left(\langle x_\alpha \mid \alpha < \gamma\rangle\right) \\ &= \bigcap \langle j(x_\alpha) \mid \alpha < \gamma\rangle \\ &\ni \kappa \end{aligned}$$

where we have used the fact that $j_U(\gamma) = \gamma$. □

**Exercise 5.3.** Show that $U$ as above is normal (i.e. closed under diagonal intersections.)

**Theorem 5.4**

Suppose $\kappa$ is a measurable cardinal. Then

(a) $\kappa$ is inaccessible, in fact the $\kappa$th inaccessible in $V$.

(b) $\kappa$ is Mahlo, in fact the $\kappa$th Mahlo cardinal in $V$.

*Proof of Inaccessibility Properties.* Let $j : V \to M$ be such that $M$ is transitive. First, to show $\kappa$ is inaccessible, we need to show it's regular and a strong limit.

Note that if $f : \gamma \to \kappa$ is cofinal (for $\gamma < \kappa$) then $j \circ f : j(\gamma) \to j(\kappa)$ is actually $f$ just because $j$ preserves small numbers:

$$j(f)(\xi) = (j(f))(j(\xi)) = (j(f))(\xi) = j(f(\xi)) = f(\xi)$$

which is impossible since $j(f) = f$ can't be cofinal in the map $\gamma = j(\gamma) \to j(\kappa) > \kappa$.

We also want to show that for $\gamma < \kappa$ we have $2^\gamma < \kappa$. Assume on the contrary $2^\gamma \geq \kappa$, so there is a surjection $f : \mathcal{P}(\gamma) \twoheadrightarrow \kappa$. But $\mathcal{P}(\gamma) \subseteq V_\kappa$. Thus $j(f)$ is now a function $j(\mathcal{P}(\gamma)) = \mathcal{P}(\gamma) \twoheadrightarrow j(\kappa)$. Indeed $j(f) = f$ again.

Hence $\kappa$ is strongly inaccessible.

Now to show $\kappa$ is in fact the $\kappa$th strongly inaccessible. The point is that

$$M \vDash \text{“}\kappa \text{ is strongly inaccessible”}$$

since $V_{\kappa+1} \subseteq M$ and hence $M$ agrees with $V$. (Here $\kappa$ is the real thing, not $j(\kappa)$.)

So $M$ thinks that both $\kappa$ and $j(\kappa)$ are strongly inaccessible, and hence $\kappa < j(\kappa)$. Observe for every $\alpha = j(\alpha) \in M$ with $\alpha < \kappa$, the model $M$ satisfies the sentence "there's a strongly inaccessible between $j(\alpha)$ and $j(\kappa)$" (namely $\kappa$). Hence $V$ satisfies the sentence "there's a strongly inaccessible between $\alpha$ and $\kappa$". And since $\kappa$ is regular we reach the conclusion.

So the point is to use $j$ and diagram chasing. □

Now recall that an inaccessible cardinal $\kappa$ is **Mahlo** if

$$\{\overline{\kappa} < \kappa \mid \overline{\kappa} \text{ is inaccessible}\}$$

is stationary: every club of $\kappa$ intersects it. (Stationary sets are HUGE.)

*Proof of Mahlo Properties.* First we prove $\kappa$ is Mahlo. Let $C$ be any club in $\kappa$. Then $j(C)$ is a club in $j(\kappa)$ (this is not the same as $j``(C)$!) Now $j(C) \cap \kappa$ is cofinal in $\kappa$. Hence $\kappa \in J(C)$!

Hence $M$ satisfies the sentence "$j(C)$ has a strongly inaccessible in it"; hence so does $C$. For the proof that it's the $\kappa$th one, repeat the proof of inaccessible. □

So measurable cardinals transcend Mahlo and inaccessibles not just by being special cases, but having TONS of such cardinals below them. Measurable cardinals "tower over" inaccessibles as such.

# §6 February 19, 2015

**Exercise 6.1.** Suppose $U$ is a $\kappa$-complete non-principal ultrafilter on $\kappa$. Then

$$(2^\kappa)^V \le (2^\kappa)^{\mathrm{Ult}(V,U)} < j_U(\kappa) < \left((2^\kappa)^+\right)^V.$$

**Exercise 6.2.** If for all $\alpha < \lambda$, we have

$$|\kappa \to \alpha|^+ \le \lambda$$

then

$$j_U(\lambda) = \sup_\alpha j_U``\alpha.$$

In addition, if the cofinality of $\lambda$ doesn't equal $\kappa$ then

$$j_U(\lambda) = \lambda.$$

## §6.1 The Constructible Universe

**Theorem 6.3** (Scott)
Suppose $\kappa$ is a measurable cardinal. Then $V \neq L$.

*Proof.* Assume $V = L$. Let $\kappa$ be the least measurable cardinal. Let $U$ be a $\kappa$-complete non-principal ultrafilter on $\kappa$. Let

$$j_U : V \to \mathrm{Ult}(V,U)$$

be the associated embedding (so $j$ has critical point $\kappa$). Then

$$V = L = \mathrm{Ult}(V,U).$$

(Since $L$ is the smallest inner model of ZFC.)

In $L = \mathrm{Ult}(V,U)$, we have

$$\mathrm{Ult}(V,U) \vDash \text{“}j_U(\kappa) \text{ is the least measurable''}$$

which is impossible. □

Assume a measurable cardinal $\kappa$ exists. Since $L$ has the same ordinals as $V$, $\kappa \in L$. This might seem absurd given what we've said above. But the point is that $V \vDash$ “$\kappa$ is measurable'' yet $L \vDash$ “$\kappa$ isn't measurable''.

You can convolute this as the following riddle if you're bored.

> “There is a measurable cardinal in $L$, which is not a measurable cardinal in $L$”.

**Remark 6.4.** $\Sigma_2$ statements are absolute to $L$; that is, for $\phi \in \Sigma_2$ if $V \vDash \phi$ then $L \vDash \phi$. Follows by Löwenheim-Skolem Theorem that “there is a countable transitive model $M$ such that $M$ thinks ZFC and there exists a measurable cardinal”.

This is a $\Sigma_2$ statement, true in $V$. So it's true in $L$.

So $L$ doesn't have large cardinals but it does have countable transitive models which *do* have these large cardinals. You can create these tiny fossils. This leads to a big philosophical debate about whether $V = L$.

## §6.2 Exercises

**Exercise 6.5.** Suppose $U$ is a $\kappa$-complete non-principal ultrafilter on $\kappa$. Then $U$ is normal if and only if $[\mathrm{id}]_U = \kappa$.

The idea is that $[c_\kappa] = j(\kappa) > \kappa$. If $U$ is normal then any function pressed down on by id becomes something less than $\kappa$.

Some other exercises are given. Here's one in particular.

**Exercise 6.6.** Suppose $U$ is a $\kappa$-complete non-principal ultrafilter on $\kappa$. Let $f : \kappa \to V$. Then

$$(j_U(f))(\kappa) = [f].$$

In that sense $\kappa$ is a "seed". It follows that

$$\mathrm{Ult}(V, U) = \mathrm{Hull}^{\mathrm{Ult}(V,U)} \left(\mathrm{im}(j_U) \cup \{\kappa\}\right).$$

## §6.3 Seed Theorem

Hence we've represented $\mathrm{Ult}(V, U)$ by both $[f]$ and $j_U(f)(\kappa)$.

**Theorem 6.7**

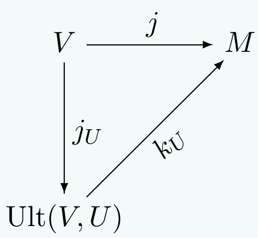
Suppose $j : V \to M$ is a nontrivial elementary embedding with critical point $\kappa$, where $M$ is a transitive model. Define

$$U = \{X \subseteq \kappa \mid \kappa \in j(X)\}.$$

Then the ultrapower $\Pi_U(V)$ is well-founded and has a Mostoswki collapse $\mathrm{Ult}(V, U)$. Moreover, the map

$$k_U : \mathrm{Ult}(V, U) \to M \text{ by } [f]_U \mapsto j(f)(\kappa)$$

gives rise to a commutative diagram of elementary embeddings




*Proof.* Let $\hat{j}_U : V \to \Pi_U(V)$ (well, "$V$" in the latter guy) by

$$x \mapsto [c_x]_x.$$

Consider

$$z = \{j(f)\kappa \mid f : \kappa \to V\}.$$

Let $\hat{k}_U : \Pi_U V \to z$ by

$$[f] \mapsto j(f)(\kappa).$$

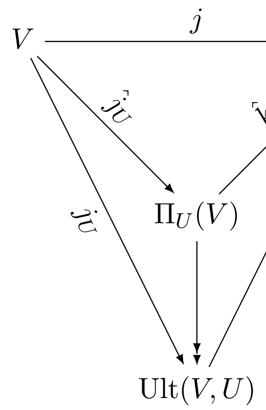
We claim that $\hat{k}_U$ is an isomorphism $\pi_U V \cong z$. For $f, g : \kappa \to V$,

$$\begin{aligned} [f]_U = [g]_U &\iff \{\alpha \in \kappa \mid f(\alpha) = g)\alpha)\} \in U \\ &\iff \kappa \in j(\{\alpha \in \kappa \mid f(\alpha) = g(\alpha)\}) \\ &\iff \kappa \in \{\alpha \in j(\kappa) \mid j(f(\alpha)) = j(g(\alpha))\}) \\ &\iff j(f)(\kappa) = j(g)(\kappa) \end{aligned}$$

Similarly, membership works out, et cetera.

Hence $Z$ sits inside $M$, isomorphic to $\mathrm{Ult}(V, U)$. (It follows that $\mathrm{Ult}(V, U)$ is well-founded, but we already knew that.)

Let $j_U$, $k_U$ be the transitive collapses of $\hat{j}_U$ and $\hat{k}_U$, ie.e. produce a diagram




Next we check that the diagram commutes.

$$\begin{aligned} k_U(j_U(x)) &= k_U([c_x]_U) \\ &= j(c_x)(\kappa) \\ &= c_{j(x)}(\kappa) \\ &= j(x) \end{aligned}$$

So we have a commutative diagram, $j$ and $j_U$ being elementary embedding. Sadly this is not enough to imply $k_U$ is an elementary embedding, so now we check it explicitly.

We first claim that $Z$ is an elementary substructure of $M$. For this we use the Tarski-Vaught test. Assume

$$M \vDash \exists y \phi[y, j(f_1)(\kappa), \dots, j(f_n)(\kappa)],$$

id est

$$\{x \in \kappa \mid V \vDash \exists y \phi[y, f_1(x), \dots, f_n(x)]\} \in U.$$

We seek a $j(g)(\kappa)$ such that

$$M \vDash \phi[j(g)(\kappa), j(f_1)(\kappa), \dots, j(f_n)(\kappa)]..$$

So we just select $g : \kappa \to V$ by a choice function. (This is essentially the same proof as Łoś.)

Thus we have

$$\begin{aligned} &\mathrm{Ult}(V, U) \vDash \phi\,[[f_1], \dots, [f-n]] \\ \iff\ & Z \vDash \phi[j(f_1)(\kappa), \dots, j(f_n)(\kappa)] \\ \iff\ & M \vDash \phi[j(f_1)(\kappa), \dots, j(f_n)(\kappa)] \end{aligned}$$

□

## §6.4 The Inner Model $L[U]$

By Scott's Theorem there are no measurable cardinals in $L$. This is unfortunate, because $L$ has nice structures.

- It has condensation, giving GCH, $\diamond$, ...
- Fine structure.

We call a model "$L$-like" if it has condensation (giving GCH and $\diamond$). Se we want an $L$-like model which does have measurable cardinals. It'd also be cool if it was canonical.

There's a notion of "relative constructibility".

**Definition 6.8.** Suppose $A$ is a set or class. We consider the language of set theory with a distinguished constant $\dot{A}$. We define

$$\begin{aligned} L_0[A] &= \varnothing \\ L_{\alpha+1}[A] &= \operatorname{Def}(L_\alpha[A], \in, \dot{A}) \\ L_\lambda[A] &= \bigcup_{\alpha<\lambda} L_\alpha[A] \\ L[A] &= \bigcup_{\alpha\in\mathrm{On}} L_\alpha[A]. \end{aligned}$$

In other words we allow ourself to use $A$ as a constant; we're allowed to query whether an element is in $A$ in our LST sentences (but we can't actually grab on to it). Definability really is over $L_\alpha[A]$.

**Example 6.9**

Note that if $A \subseteq L$ such that $A \cap x \in L\ \forall x \in L$ (we say "$A$ is amenable to $L$") then $L[A] = L$; the introduction of $A$ doesn't let you define anything new.

**Example 6.10**

Moreover, if $A \subseteq \omega$ is a real number such that $A \notin L$, then $L[A] \supsetneq L$. In this case, $L_n[A] = L_n$ for any natural number $n$ and hence $L_\omega[A] = L_\omega$, but at the $\omega + 1$st level we obtain the real number $A$.

**Example 6.11**

If $A \subseteq \mathrm{On}$ encodes $V$, then $L[A] = V$.

So whether $L[A]$ is desirable depends entirely on the set $A$. Next time, we're going to take $A = U$ a normal uniform ultrafilter on a measurable cardinal $\kappa$. We'll then see that

- $L[U] \vDash$ "$\kappa$ is measurable".
- $L[U] \vDash$ "$\kappa$ is the only measurable".
- $L[U] \vDash \mathrm{GCH}$.
- $L[U]$ is "canonical" in some sense.

$L[U]$ is going to be basically $L$ above $\kappa$, but stranger underneath it.

Later in the class we'll do the same thing with a supercompact cardinal, and then something magical is going to happen.

# §7 February 24, 2015

We're going to deal with $L[A]$ where $U$ is an ultrafilter.

## §7.1 Black boxes

**Theorem 7.1**

There is a $\Pi_2$ sentence "$V = L[A]$" in the Language of Set Theory with distinguished constant $\dot{A}$ such that for any transitive class $M$ and any set or class $A$,

$$\langle M, \in, A \cap M \rangle \vDash \text{“}V = L[A]\text{”}$$

if and only if either $M = L[A]$ or $M = L_\lambda[A]$ for some uncountable limit ordinal $\lambda$.

**Theorem 7.2**

There is a $\Sigma_1$ function

$$\text{“}x <_{L[A]} y''$$

in $\text{LST} \cup \{\dot{A}\}$ such that for any set or class $A$ in $\langle L[A], \in, A \cap L[A] \rangle$ the relation defines a well-ordering of the universe such that for any uncountable $\lambda$,

$$(<_{L_A})^{\langle L_\lambda[A], \in, A \cap L_\lambda[A] \rangle} = (<_{L_A}|_{L_\lambda[A]})^{\langle L_\lambda[A], \in, A \cap L_\lambda[A] \rangle}$$

No idea what this means.

## §7.2 Measurable Cardinals

**Exercise 7.3.** Suppose $U$ is a normal uniform ultrafilter on $\kappa$. Then

$$L[U] \vDash \text{“}U \cap L[U] \text{ is a normal uniform ultrafilter on } \kappa\text{”}.$$

Even if $U \notin L[U]$ we have that

$$U \cap L[U] \in L[U] \text{ and } L[U \cap L[U]] = L[U]$$

so we may as well assume that $U \in L[U]$.

**Definition 7.4.** $\langle L[U], \in, U \rangle$ is a ***$\kappa$*-model** if

$$\langle L[U], \in, U \rangle \vDash \text{“}U \text{ is a normal uniform ultrafilter on } \kappa\text{”}.$$

If $\kappa$ is a measurable cardinal; then there is a $\kappa$-model.

**Exercise 7.5.** Show the converse fails.

## §7.3 Analyzing $L[U]$

To analyze $L[U]$ and show that it is $L$-like we will need two new techniques,

- Iterated ultrapowers, and
- Comparison.

For now, let's see how far we can go with old techniques.

In what follows we abbreviate $\langle L[U], \in, U \rangle$ to just $L[U]$.

**Theorem 7.6** (Solavay)
Suppose $L[U]$ is a $\kappa$-model. Then

(a) $L[U] \vDash (\forall \lambda \geq \kappa)(2^\lambda = \lambda^+)$. In other words we have GCH above $\kappa$.

(b) $L[U] \vDash$ "$\kappa$ is the only measurable".

*Proof.* First we prove GCH holds above $\kappa$. Fix $\lambda \geq \kappa$ and $x \in \mathcal{P}(\lambda) \cap L[U]$. Consider $\xi$ a limit ordinal such that

$$x, U, \lambda \in L_\xi[U]$$

i.e. we have all the ingredients.

Consider

$$\langle H, \in, U \cap H \rangle \prec \langle L_\xi[U], \xi, U \cap L_\xi[U] \rangle$$

such that $\lambda \cup \{x, U\} \subseteq H$, and $|H| = \lambda$. Let

$$\pi : \langle H, \in, U \cap H \rangle \to \left\langle \overline{H}, \in, \overline{U \cap H} \right\rangle$$

be the Mostowski collapse. Since $\forall y \in P(\lambda)(\pi(y) = y)$, $\pi$"$(U \cap H) = U \cap \overline{H}$. Also, since $L_\xi[U] \vDash$ "$V = L[U]''$ (from elementarity), we obtain $\overline{H} = L_\xi[U]$. Moreover, $\left| L_{\overline{\xi}}[U] \right| = \lambda$, so $\mathcal{P}(\lambda) \cap L[U] \subseteq L_{\lambda^+}[U]$.

For the second part, work in $L[U]$. Assume for contradiction that there is another measurable cardinal $\lambda$ and let $U'$ be a normal uniform ultrafilter on $\lambda$. Let

$$j_{U'} : L[U] \to \operatorname{Ult}(L[U], U').$$

Since $U' \notin \operatorname{Ult}(L[U], U')$, it suffices to show that $\operatorname{Ult}(L[U], U') = L[U]$, so this will give the needed contradiction.

We now consider two cases. Suppose $\lambda > \kappa$. Then $\operatorname{Ult}(L[U], U') = L[j_{U'}(U)]$. But $j_{U'}(U) = U$, contradiction.

On the other hand, suppose $\lambda < \kappa$. Let $M$ be the ultrapower $\operatorname{Ult}(L[U], U')$. We claim

$$j_{U'}(U) = U \cap M$$

which suffices since then $M = L[j_{U'}(U)] = L[U \cap M] = L[U]$. (Recall that $l[U]$ is the smallest inner model $M$ such that $U \cap M \in M$.)

We will prove $j(U) \subseteq U$, which implies the conclusion. Suppose $[f]_{U'} \in j(U)$; we wish to show $[f] \in U$. But $j_{U'}(U) = [c_U]_{U'}$. Consider $f : \lambda \to U$. Since $\lambda < \kappa$, we have

$$\bigcap_{\xi < \lambda} f(\xi) \in U.$$

We will show $[f] \in U \cap M$ by exhibiting an object $y \in U \cap M$ such that $y \subseteq [f]$. Actually,

$$j_{U'} \left( \bigcap_{xi < \lambda} f(\xi) \right) = \left[ c_{\bigcap_{\xi < \lambda} f(\xi)} \right] \subseteq [f].$$

so we may set $y = j_{U'} \left( \bigcap_{xi < \lambda} f(\xi) \right)$; hence it remains only to show $y \in U \cap M$.

Let

$$I = \{ \xi < \lambda \mid \xi > \lambda \text{ and } \xi \text{ inaccessible} \}.$$

As $|I| = \kappa$, we have $I \in U$ and $j_U``(I) = I$. Hence

$$j``(I) \cap \left( \bigcap_{\xi<\lambda} f(\xi) \right) = I \cap \left( \bigcap_{\xi<\lambda} f(\xi) \right)$$

which is an intersection of two $U$-big sets, and hence is in $U$. □

Summary: let $U$ be a normal uniform ultrafilter on $\kappa$. Then

$$L[U] \vDash \text{“}\kappa \text{ is measurable with ultrafilter } U\text{”}.$$

Moreover, $L[U]$ satisfies GCH above $\kappa$, and finally thinks that $\kappa$ is the only inaccessible cardinal.

## §7.4 Iterated Ultrapowers

Proceed generally:

(i) Models of ZFC not power set.

(ii) Don't assume $U$ is actually in the model.

**Definition 7.7.** We say $M$ is **fully amenable** to $M$ if for all $x \in M$, $x \cap U \in M$.

**Definition 7.8.** We say $M$ is **$\kappa$-amenable** if for all $x \in M$, if $|x|^M = \kappa$ then $x \cap U \in M$.

**Definition 7.9.** Suppose $M$ is a transitive $\in$-model of ZFC − PowerSet. Then $U$ is **an $M$-ultrafilter on $\kappa$** if

(i) $\langle M, \in, U\rangle \vDash$ “$U$ is a normal uniform ultrafilter on $\kappa$”.

(ii) $\langle M, \in, U\rangle$ is $\kappa$-amenable, *id est*

$$\forall (f : \kappa \to M) \in M : \{\xi < \kappa \mid f(\xi) \in U\} \in U.$$

Another definition:

**Definition 7.10.** Suppose $M$ is a transitive $\in$-model of ZFC − PowerSet. Then $U$ is an $M$-ultrafilter on $\kappa$ if

(i) $U$ is a non-principal ultrafilter and a subset of $\mathcal{P}(\kappa) \cap M$.

(ii) If $\eta < \kappa$, the sequence $\langle x_\xi \mid \xi < \eta\rangle$ is in $M$, and each $x_\eta$ is in $U$, then $\bigcap \langle x_\xi \mid \xi < \eta\rangle$ is in $U$.

(iii) If the sequence $\langle x_\xi \mid \xi < \kappa\rangle$ is in $M$, then so is $\langle \xi \mid x_\xi \in U\rangle$.

For the ultrapower construction, we can proceed as before using $f, g \in (\kappa \to M) \cap M$; the only change is the “$\cap M$” since we have models $M$ rather than the whole universe $M$.

What about $\dot{A}_U$? We have

$$[f] \; E_U \; A_U \iff \{\xi < \kappa \mid f(\xi) \in U\} \in U.$$

Note that $\langle M, \in, U\rangle$ can determine this since $M$ is $\kappa$-amenable.

Hence from Łoś from $\mathrm{LST} \cup \{\dot{A}\}$ using AC and Collection.

$\kappa$-completeness in this context does not guarantee that

$$\langle (\kappa \to M) \cap M/U, E_U, A_U \rangle$$

is well-founded (since $\kappa$ could be countable!), but when it is well-founded we write

$$\mathrm{Ult}(M, U)$$

for the transitive collapse, whose elements are $[f] = \pi([f])$.

**Exercise 7.11.** For any such $M$, prove that

$$j : \langle M, \in, U \rangle \to \mathrm{Ult}(M, U) = \left\langle M', \in, U' \right\rangle$$

is cofinal. Show that

(1) $|M| = |M'|$.

(2) $\forall x \in V_\kappa \cap M$, $j(x) = x$.

(3) $V_\kappa \cap M = V_\kappa \cap M'$.

(4) $V_{\kappa+1} \cap M = V_{\kappa+1} \cap M'$.

(5) $U \notin M'$.

# §8 February 26, 2015

Let $M$ be a transitive $\in$-model of

$$\text{ZFC} - \text{PowerSet}$$

and let $U$ be an $M$-ultrafilter on $\kappa$. We defined the ultrapower of $(M, U)$. If it's well founded, then we can take the Mostowski collapse

$$j : (M, U) \to (M', U')$$

where $M'$ is transitive and $j$ is an elementary embedding.

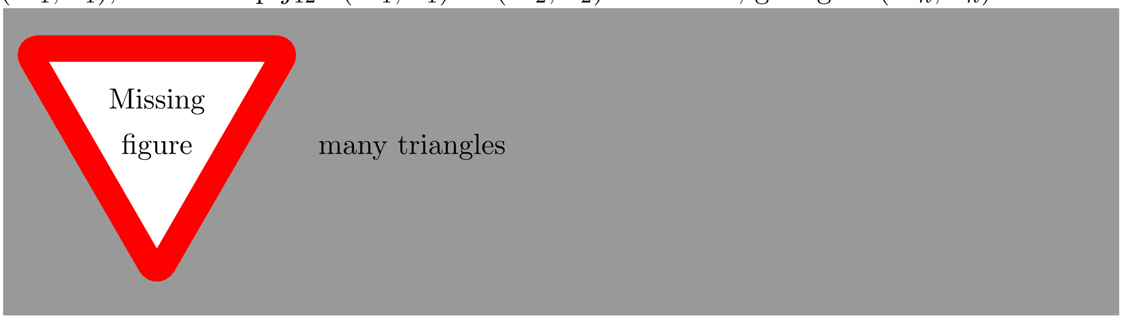
We have that

- $j$ is cofinal,
- $j$ is the identity when restricted to $V_\kappa^M$.
- $(V_{\kappa+1})^M = (V_{\kappa+1})^{M'}$.
- $U \notin M'$.

We also know that $U'$ is an $M'$-ultrafilter (exercise), meaning that it's $\kappa$-amenable to $M'$ and $M'$ thinks $U'$ is a normal uniform ultrafilter over $j(\kappa)$.

## §8.1 Iterated Ultrapowers

Let $M_0 = M$ and $M_1 = M'$ now. We can do the same procedure on the ultrapower $(M_1, U_1)$, then a map $j_{12} : (M_1, U_1) \to (M_2, U_2)$ and so on, giving us $(M_n, U_n)$.




Then we can take the direct limit at limit stages.

**Exercise 8.1.** Specifically, suppose $\lambda$ is a limit ordinal and

$$(\langle (M_\alpha, U_\alpha) : \alpha < \lambda \rangle, \langle j_{\alpha,\beta} : \alpha < \beta < \lambda \rangle)$$

is a directed system of elementary embeddings where each $U_\alpha$ is an $M_\alpha$ ultrafilter of some $\kappa_\alpha$.

Suppose further that the direct limit is well-founded and take the Mostowski collapse $(M_\lambda, U_\lambda)$ and let

$$j_{\alpha,\lambda}(M_\alpha, U_\alpha) \to (M_\lambda, U_\lambda)$$

be the induced embeddings for each $\alpha < \lambda$.

Then $U_\lambda$ is an $M_\lambda$-ultrafilter over $k_\lambda = j_{\alpha\lambda}(\kappa_\alpha)$ for each $\alpha$.

## §8.2 Maximal iterability

So we want some condition which guarantees well-foundedness, because then we can keep iterating.

**Definition 8.2.** Let $\tau$ be the first ordinal stage (if it exists) where well-founedness fails. Otherwise let $\tau = \mathrm{On}$, in which we say $(M_0, U_0)$ is **iterable**. More generally, if $\lambda \in \tau$ we declare that $(M_0, U_0)$ is **$\lambda$-iterable**.

Foreshadowing: We will be interested in $(M_0, U_0)$ which are $(\omega_1 + 1)$ iterable.

**Lemma 8.3**

Suppose $U_0$ is a $M_0$-ultrafilter on $\kappa_0$ and let

$$\langle M_\alpha, U_\alpha, \kappa_\alpha, j_{\alpha,\beta} : \alpha \le \beta < \tau \rangle$$

be the iteration of $(M_0, U_0)$. Suppose $\alpha < \beta < \tau$. Then

(1) The critical point of $j_{\alpha,\beta}$ is $\kappa_\alpha$, and $j_{\alpha,\beta}(\kappa_\alpha) = \kappa_\beta$.

(2) $j_{\alpha,\beta}$ is the identity when restricted to $V_{\kappa_\alpha} \cap M_\alpha$ and

$$V_{\kappa_{\alpha+1}} \cap M_\alpha = V_{\kappa_{\alpha+1}} \cap M_\beta.$$

(3) If $\lambda$ is a limit ordinal then

$$\kappa_\lambda = \sup_{\xi < \lambda} \kappa_\xi.$$

(4) If $M_0$ is a set then in fact

$$|M_\alpha| = |M_0| \cdot |\alpha|.$$

*Proof.* (1) follows by elementarity.

(2) follows by an earlier exercise.

For (3), consider $\xi < \kappa_\lambda$; we seek $\kappa_\beta > \xi$. Then $\overline{\xi} < \kappa_\lambda$, $\eta < \lambda$ be such that

$$\xi = j_{\eta,\lambda}(\overline{\xi}).$$

In $(M_\eta, U_\eta)$ we have $\kappa_\eta > \overline{\xi}$. Thus, since $\kappa_\eta$ is the critical point we obtain $\overline{\xi} = \xi$, hence $\kappa_\eta > \xi$.

The last part follows from the fact that sizes are not increased at any successor stage by a preceding exercise. Thus

$$|M_\alpha| \le |M_0| \cdot |\alpha|.$$

Moreover,

$$|M| \le |M_\alpha|$$

and

$$\{\kappa_{\overline{\alpha}} : \overline{\alpha} < \alpha\} \subseteq M_\alpha$$

which establishes the other inequality. □

**Exercise 8.4.** Show that for $\alpha + 1 \in \tau$ and $x \in \mathcal{P}(\kappa_\alpha) \cap M_\alpha$, we have $x \in U_\alpha$ if and only if $\kappa_\alpha \in j_{\alpha,\alpha+1}(x)$.

## §8.3 Determining $U_\alpha$

**Lemma 8.5**

Let $M$, $\tau$, etc. be as above. Suppose $\lambda \in \tau$ is a limit ordinal. Then for $x \in \mathcal{P}(\kappa_\lambda) \cap M_\lambda$, we have $x \in U_\lambda$ if and only if there exists $\alpha < \lambda$ so that $\langle \kappa_\gamma : \alpha < \gamma < \lambda \rangle$ is a subset of $x$.

*Proof.* Fix any $\gamma < \lambda$ is such that

$$x = j_{\gamma,\lambda}(\overline{x})$$

for some $\overline{x} \in \mathcal{P}(\kappa_\gamma) \cap M_\alpha$.

We have

$$\overline{x} \in U_\gamma \iff \kappa_\gamma \in j_{\gamma,\gamma+1}(\overline{x})$$

according to the exercise. This is true if and only if $\kappa_\gamma \in x$, since the critical point of $j_{\gamma+1,\lambda}$ exceeds $\kappa_\lambda$.

Thus

$$x \in U_\lambda \iff \overline{x} \in U_\gamma \iff \kappa_\gamma \in x. \qquad \square$$

**Lemma 8.6** (Seed Lemma)

Let $M$, $\tau$, etc. be as above. For each $\alpha \in \tau$ and $x \in M_\alpha$ there exists a nonnegative integer $n$ and critical points

$$\kappa_{\xi_1} < \cdots < \kappa_{\xi_n}$$

(where $\xi_1 < \cdots < \xi_n < \alpha$) and $f : [\kappa_0]^n \to M$ a function in $M$ such that

$$x = j_{0,\alpha}(f)\left(\kappa_{\xi_1}, \dots, \kappa_{\xi_n}\right).$$

*Proof.* By induction on $\alpha$. Suppose true for $\alpha$ and $\alpha + 1 \in \tau$. Fix $x \in M_{\alpha+1}$. Suppose $U_\alpha$ is an $M_\alpha$ ultrafilter over $\kappa_\alpha$ we have

$$x = j_{\alpha,\alpha+1}(g)(\kappa_\alpha)$$

for some $g : \kappa_\alpha \to M_\alpha$, $g \in M_\alpha$ (exercise).

Now if $\alpha = 0$ we are done. If $\alpha > 0$ then by the inductive hypothesis we have

$$\overline{g} : [\kappa_0]^n \to M_0$$

in $M_0$ and

$$\xi_1 < \cdots < \xi_n < \alpha$$

such that $g = j_{0,\alpha}(\overline{g})(\kappa_{\xi_1}, \dots, \kappa_{\xi_n})$. So we seek an $f : [\kappa_0]^{n+1} \to M_0$ in $M_0$ such that

$$x = j_{0,\alpha+1}(f)(\kappa_{\xi_1}, \dots, \kappa_{\xi_n}, \kappa_\alpha).$$

Thus we have

$$\begin{aligned}
x &= \left(j_{\alpha,\alpha+1}(g)\right)(\kappa_\alpha) \\
&= \left(j_{\alpha,\alpha+1}(j_{0,\alpha}(\overline{g})(\kappa_{\xi_1}, \dots, \kappa_{\xi_n}))\right)(\kappa_\alpha) \\
&= \left(j_{\alpha,\alpha+1}(j_{0,\alpha}(\overline{g}))\right)\left(j_{\alpha,\alpha_1}(\kappa_{\xi_1}, \dots, \kappa_{\xi_n})\right)(\kappa_\alpha) \\
&= \left(j_{\alpha,\alpha+1}(j_{0,\alpha}(\overline{g}))\right)\left(\kappa_{\xi_1}, \dots, \kappa_{\xi_n}\right)(\kappa_\alpha)
\end{aligned}$$

since the $\kappa_i$ are all less than $\kappa_\alpha$, and hence are preserved. So we want to find $f$ such that this equals

$$(j_{0,\alpha+1}(f))\left(\kappa_{\xi_1},\dots,\kappa_{\xi_n},\kappa_\alpha\right).$$

Define $f$ from $g$ as follows:

$$(\gamma_1,\dots,\gamma_{n+1})\mapsto\left(\overline{g}(\gamma_1,\dots,\gamma_n)\right)(\gamma_{n+1}).$$

This works.

Now for the limit case we fix $x\in M_\lambda$, and $\alpha<\lambda$ and $\overline{x}\in M_\alpha$ such that

$$x=j_{\alpha,\lambda}(\overline{x}).$$

By the inductive hypothesis,

$$\overline{x}=j_{0,\alpha}(f)\left(\kappa_{\xi_1},\dots,\kappa_{\xi_n}\right)$$

but the critical point of $j_{\alpha,\lambda}$ is $\kappa_\alpha>\kappa_{\xi_n}>\dots>\kappa_{\xi_1}$ so $j_{\alpha,\lambda}(\kappa_\xi)=\kappa_\xi$. Thus $x=j_{0,\lambda}(f)(\kappa_{\xi_1},\dots,\kappa_{\xi_n})$. $\square$

This looks hard but it's just like notation.

## §8.4 Comments

Recall that

$$\operatorname{Ult}(V,U)=\operatorname{Hull}^{\operatorname{Ult}(V,U)}\left(\operatorname{ran}(j_u)\cup\{\kappa\}\right)$$

when $U$ is a non-principal ultrafilter on $\kappa$. Contrast this with our new information: (for $\alpha\in\tau$ that)

$$M_\alpha=\operatorname{Hull}^{M_\alpha}\left(\operatorname{ran}(j_{0,\alpha})\cup\{\kappa_\xi:\xi<\alpha\}\right).$$

What we're going to do next is to figure out at which stages we have fixed points of some $j_{0,\alpha}$. We also need to set conditions for iterability. Then we'll go from this local context back to $L[U]$.

# §9 March 3, 2015

## §9.1 A Corollary

**Corollary 9.1**

Let $M = (M_0, U_0), \tau$ et cetera be as in the previous lecture. Then

(1) Suppose $\gamma \in (\mathrm{On})^M$ and $\alpha \in \tau$. Then

$$j_{0,\alpha}(\gamma) < \left(|(\kappa_0 \to \gamma) \cap M| \cdot |\alpha|\right)^+ .$$

(2) Suppose $\nu$ is a cardinal such that

$$|(\kappa \to \kappa) \cap M| < \nu \in \tau.$$

Then $\kappa_\nu \overset{\text{def}}{=} j_{0,\nu}(\kappa_0)$ in fact equals $\nu$.

(3) Suppose $\theta$ is a radical such that $M$ thinks

- ZFC − PowerSet,
- For all $\overline{\theta} < \theta$, there is a set of functions $\kappa_0 \to \overline{\theta}$.
- $\theta$ is a strong limit and $\mathrm{cof}(\theta) > \alpha$.

Assume $\alpha < \min(\theta, \tau)$. Then

$$j_{0,\alpha}(\theta) = \theta.$$

*Proof.* For the first part, let $\gamma > \kappa_0$. Then $j_{0,\alpha}$ takes $\kappa_0$ to $\kappa_\alpha$ and $\gamma \mapsto j_{0,\alpha}(\gamma)$. Consider $\kappa_\alpha < \eta < j_{0,\alpha}(\gamma)$. By the Seed Lemma $\eta$ has the form

$$\eta = j_{0,\alpha}(f)(\kappa_{\xi_1}, \dots, \kappa_{\xi_n}).$$

for some $f : [\kappa_0]^n \to M_0$ in $M_0$. By using $\eta \in j_{0,\alpha}(\gamma)$ and applying Łoś or whatever, you can assume in fact $f : [\kappa_0]^n \to \gamma$. Thus the number of $\eta$ is at most the number of functions times the number of seeds, *id est*

$$|(\kappa_0 \to \gamma) \cap M| \cdot |\alpha^n| = |(\kappa_0 \to \gamma) \cap M| \cdot |\alpha| .$$

This gives the bound on $j_{0,\alpha}(\gamma)$.

In the second part, let

$$\nu \le \kappa_\nu \le \sup\left(\{\kappa_\alpha \mid \alpha < \nu\}\right).$$

Using the above lemma we can bound this by $\nu$, giving $\kappa_\nu = \nu$.

Finally, for the third part, it suffices to show that if $\eta < j_{0,\alpha}(\theta)$ then $\eta < \theta$. Each $\eta < j_{0,\alpha}(\theta)$ has the form

$$j_{0,\alpha}(f)(\kappa_{\xi_1}, \dots, \kappa_{\xi_n})$$

where $f : [\kappa_0]^n \to \theta$ and $f \in M_0$. Since $\mathrm{cof}^{M_0}(\theta) > \alpha$, there exists an ordinal $\overline{\theta} < \theta$ such that in fact $f : [\kappa_0]^n \to \overline{\theta}$. Thus

$$\eta < j_{0,\alpha}(\overline{\theta}) < \left(\left|(\kappa_0 \to \overline{\theta}) \cap M_0\right| |\alpha|\right)^+ \le \theta$$

where we use the fact that $\theta$ is a strong limit for the last inequality, as

$$\kappa_0^{\overline{\theta}} = 2^{\overline{\theta}} < \theta.$$

□

## §9.2 Iterability

Now we want to actually guarantee that these $M_\alpha$ is iterable.

Let $U = U_0$ be an $M = M_0$-ultrafilter on $\kappa = \kappa_0$. Let

$$\langle M_\alpha, U_\alpha, \kappa_\alpha, j_{\alpha,\beta} \mid \alpha \le \beta < \tau \rangle$$

be the iteration of $(M_0, U_0, \kappa_0)$. We want a sufficient condition on $(M, U)$ for iterability; i.e. we want $\tau = \mathrm{On}$.

**Definition 9.2.** We say $U$ is **weakly countably complete** in $V$ if for any sequence $\{X_n : n < \omega\} \subseteq U$, we have

$$\bigcap_{n<\omega} X_n \neq \omega.$$

(Before we wanted the intersection to actually be in $U$.) This is not immediate from $M \vDash$ "$U$ is $\kappa$-complete", but this is not immediate: the $X_n$ sequence comes from $V$, and need not be in $M = M_0$.

We're interested in $\omega_1$ iterability.

**Theorem 9.3**

Let $M = M_0$, $U = U_0$, $\tau$ and so on be as above.

(1) Suppose that $U$ is weakly countably complete the universe of sets.

($\star$) Suppose $(N, V)$ is countable (here $V$ is an ultrafilter, not the constructible universe) such that

(a) $V$ is an $N$-ultrafilter on $\nu$,

(b) There exists an elementary embedding

$$\sigma : (N, V) \to (M, U).$$

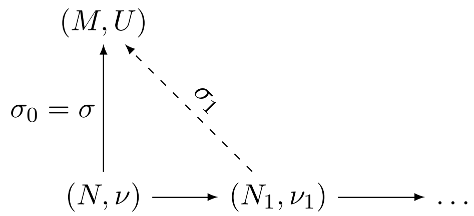
Then $(N, V)$ is $\omega_1$-iterable.

(2) If $(M, U)$ satisfies ($\star$) then $(M, U)$ is iterable.

*Proof.* The proof uses the so-called "realizability". Let

$$\langle N_\alpha, V_\alpha, \nu_\alpha, j_{\alpha,\beta} \mid \alpha \le \beta < \tau \rangle$$

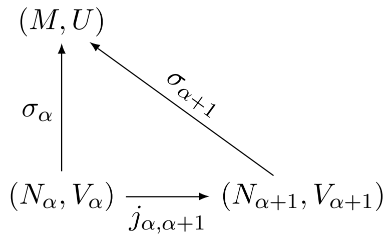
be the iteration of $(N_0, \nu_0) = (N, \nu)$. We want to show $\tau \ge \omega_1$ (and hence $\tau > \omega_1$). The point is to get an elementary embedding of each $(N_\alpha, \nu_\alpha)$ into $(M, U)$.




We proceed by transfinite induction. Consider the *uncollapsed* ultrapower

$$\left\langle (\nu_\alpha \to N_\alpha) \cap N_\alpha / V_\alpha, E_{V_\alpha}, \dot{A}_{V_\alpha} \right\rangle.$$

We seek $\tilde{\sigma}_\alpha$ from this into $\langle M, \in U\rangle$ so that the diagram




commutes. Do some computation with Łoś, using weakly countably complete. The map is

$$[f] \mapsto \sigma_\alpha(f)(\eta_\alpha)$$

for some

$$\eta_\alpha \in \bigcap \{\sigma_\alpha(x) \mid x \in V_\alpha \cap N_\alpha\}$$

which is possible since there are only countably many sets of the form $\sigma_\alpha(x)$ for $x \in N_\alpha$.

The limit case is immediate using properties of direct limits.

For the second part, suppose $(M, U)$ has the property and for contradiction $(M, U)$ is not iterable. Let $\alpha$ be the point so that $(M_\alpha, U_\alpha)$ is ill-founded. There's an infinite descending sequence of $M_\alpha$; cutting off $M$ appropriately, it suffices to consider the case where $M$ is a set.

Let

$$I = \langle M_\alpha, U_\alpha, \kappa_\alpha, j_{\alpha,\beta} \mid \alpha \le \beta < \tau\rangle$$

witness the iterability of $(M, E)$. Hence $\tau < \mathrm{On}$.

Let $\gamma$ be sufficiently large so that $M$, $U$, $I$ are all in $V_\gamma$, and

$$V_\gamma \vDash \text{“}I \text{ is an iteration of } (M, U) \text{ of length } \tau \text{ witnessing the non-iterability of } (M, U).\text{”}$$

Let $H \prec V_\gamma$ be such that $|H| = \omega$ and $M, U, I \in H$ (by reflection).

Take a hull. Blah. □

# §10 March 5, 2015

A small change: we now assume

$$M \vDash \text{ZFC} \setminus \text{PowerSet} + \text{“}V_{\kappa+1} \text{ exists”}.$$

But we actually assume $U \in M$ now. Moreover, $M \vDash$ “$U$ is a normal uniform ultrafilter on $\kappa$”.

## §10.1 Iterability continued

We let $M$, $\tau$, $\kappa$ have the same meanings as last time.

**Corollary 10.1**

The following are equivalent.

(1) $(M, U)$ is iterable.

(2) $(M, U)$ is $\omega_1$-iterable.

(3) There is an $(M_\alpha, U_\alpha)$ such that $U_\alpha$ is weakly countably complete in $V$.

(4) $(M, U)$ has feature $(\star)$

(5) If there is an elementary embedding $\pi : (N, V) \to (M, U)$ then $(N, V)$ is iterable.

*Proof.* (1) $\implies$ (2) is tautological.

(2) $\implies$ (3): We have $(M, U)$ is $\omega_1 + 1$ iterable. Consider $(M_{\omega_1}, U_{\omega_1})$. Suppose $\{X_n : n \in \omega\} \in U_{\omega_1}$. We have that for each $n \in \omega$ there is an $\alpha_n$ such that

$$\{\kappa_\gamma \mid \alpha_n < \gamma < \omega_1\} \subseteq X_n.$$

Let $\alpha = \sup_{n \in \omega} \alpha_n$.

$$\{\kappa_\gamma \mid \alpha < \gamma < \omega_1\} \subseteq \bigcap X_n.$$

Thus $\bigcap_{n \in \omega} X_n$ is nonempty.

(3) $\implies$ (4): By the theorem from last time, we know $(M_\alpha, U_\alpha)$ is iterable. Since $(M_\alpha, U_\alpha)$ is an iterate of $(M, U)$, we get $(M, U)$ is iterable.

(4) $\implies$ (5): $(N, V)$ inherits feature $(\star)$ since it can be embedded in $(M, U)$.

(5) $\implies$ (1): Take the identity embedding $(M, U) \to (M, U)$; hence $(M, U)$ is iterable. □

**Corollary 10.2**

Suppose $N$ is a transitive $\in$-model of ZFC. Assume

(a) $\omega_1 \subseteq N$.

(b) $N$ thinks $U$ is an $M$-ultrafilter on $\kappa$ (where $M$ may be a proper class in $N$).

Then $N$ thinks $(M, U)$ is iterable if and only if $V$ thinks $(M, U)$ is iterable.

*Proof.* Because iterability is equivalent to $\omega_1$-iterability, the condition $\omega_1 \subseteq N$ is sufficient. Indeed, for $\alpha \in \mathrm{On}^N$, we have

$$(M_\alpha, U_\alpha)^N = (M_\alpha, U_\alpha)^V .$$

Hence $V \vDash$ "$(M, U)$ iterable" $\implies N \vDash$ "$(M, U)$ iterable" is immediate.

For the other direction, given the above point and $\omega_1^V \subseteq N$ we have $(M, U)$ is $\omega_1$-iterable. So $V \vDash$ "$(M, U)$ is iterable". □

## §10.2 The Coarse Analysis of $L[U]$

Recall that

$$\langle L[U], \in U \rangle$$

is a **$\kappa$-model** if and only if

$$\langle L[U], \in U \rangle \vDash \text{"}U \text{ is a normal ultrafilter on } \kappa\text{"}.$$

Here we are now assuming $U \in L[U]$.

**Definition 10.3.** We say $\langle M, \in, U \rangle$ is a **premouse at $\kappa$** (plural premice) if

(1) $M$ is a transitive $\in$-model

(2) $M \vDash$ "ZFC \ PowerSet + $\exists V_{\kappa+1}$".

(3) $M$ thinks $U$ is a normal ultrafilter on $\kappa$.

(4) $M$ thinks $V = L[U]$.

It follows that a pre-mouse has form $L[U]$ or $L_\lambda[U]$ for some limit ordinal $\lambda > \kappa \geq \omega$. (A transitive $\in$-model satisfying $V = L[U]$ is of the form $L_\lambda$ as we verified last semester.)

**Definition 10.4.** A **mouse** is an iterable premouse.

**Lemma 10.5**

Suppose $\langle L_\lambda[U], \in, U \rangle$ is a premouse. such that $\omega_1 \subseteq \lambda$. Then it is a mouse.

*Proof.* By previous proof. □

## §10.3 Club filters

**Definition 10.6.** For $\nu > \omega$ a regular cardinal, we define

$$C_\nu = \{x \subseteq \nu \mid \exists \text{club } C \text{ in } \nu, C \subseteq x\} .$$

This is the **club filter** of $\nu$. It's a filter and is $\nu$-complete (check this).

Now for a surprising result.

**Lemma 10.7**

Suppose $\langle M, \in U\rangle$ is a premouse at $\kappa$ which is $(\nu+1)$-iterable, where $\nu$ is a regular uncountable cardinal. Assume $\nu$ exceeds the number of functions $\kappa \to \kappa$ in $M$. Then

$$\langle M_\nu, \in, U_\nu\rangle = \langle L_\alpha[C_\nu], \in, C_\nu \cap L_\alpha[C_\nu]\rangle.$$

In other words, we take this potentially small $\langle M, \in, U\rangle$, and go up to a large $\nu$, the $U_\nu$ are just club filters. Now it's not true that club filters are in general ultrafilters in $V$, but evidently the model thinks $\nu$ is measurable nonetheless.

*Proof.* By an earlier fixed point result, we have

$$j_{0,\nu}(\kappa) = \kappa_\nu = \nu.$$

By an earlier lemma, $U_\nu \subseteq C_\nu \cap M_\nu$. But $U_\nu$ is an ultrafilter on $\mathcal{P}(\nu) \cap M_\nu$, so the inclusion is sharp. □

Even more surprising is the corollary.

**Corollary 10.8**

Suppose there is a $\kappa$-model. Then for all sufficiently large regular cardinals $\nu > \kappa$,

$$\langle L_\nu[C_\nu], \in, C_\nu \cap L[C_\nu]\rangle$$

is a $\nu$-model.

Note that $\kappa$ doesn't even have to be truly measurable. But if it is, then we discover that for sufficiently large $\nu$, building relative to the club filters gives us a ton of such models.

**Definition 10.9.** Suppose $\langle M, \in, U\rangle$ is a premouse at $\kappa$ and $\langle M', \in, U'\rangle$ is a premouse at $\kappa'$. Then we write

$$\langle M, \in, U\rangle \trianglelefteq \langle M', \in, U'\rangle$$

to mean that

(1) $\kappa = \kappa'$

(2) $U = U' \cap M$

(3) $M = L_\alpha[U']$ for some $\alpha \le (\mathrm{On})^M$.

## §10.4 Comparison

**Theorem 10.10** (Comparison)

Suppose $\langle M, \in, U\rangle$ is a mouse at $\kappa$ and $\langle M', \in, U'\rangle$ is a mouse at $\kappa'$. Let $\nu$ be a regular cardinal such that

$$\nu > \max\left\{ \left|(\kappa \to \kappa) \cap M\right|, \left|(\kappa' \to \kappa') \cap M'\right| \right\}.$$

Then either

$$\langle M_\nu, \in, U_\nu\rangle \trianglelefteq \langle M'_\nu, \in, U'_\nu\rangle \quad \text{or} \quad \langle M'_\nu, \in, U'_\nu\rangle \trianglelefteq \langle M_\nu, \in, U_\nu\rangle.$$

*Proof.* Immediate by the lemma. □

This is the pre-cursor of the mice hierarchy: after you let mice run (iterate) for a long time, then you can sort them linearly.

When we get up to Woodin cardinals, mice become less simple: we have a *branching* phenomenon.

## §10.5 Generalized Continuum Hypothesis Holds in $L[U]$

Let's prove GCH ("let's do something").

**Theorem 10.11**

Suppose

$$\langle L[U], \in, U \rangle$$

is a $\kappa$-model. Then

$$\langle L[U], \ni, U \rangle \vDash \text{GCH}.$$

*Proof.* We have by condensation that $\lambda > \kappa$ implied $2^\lambda = \lambda^+$. Transitive collapse of hulls $\subseteq \kappa$ fall back into the $\langle L_\alpha[U] : \alpha \in \text{On} \rangle$ hierarchy.

Now suppose $L_\gamma[U]$ thinks ZFC $\setminus$ PowerSet and $V_{\kappa+1}$ exists. Also, assume $U \in L_\gamma[U]$. Let

$$\langle H, \in, U \cap H \rangle \prec \langle L_\gamma[U], \in, U \rangle$$

be an elementary substructure such that $U \in H$. The model satisfies $V = L[U]$. Hence the transitive collapse of $H$ is of the form $L_{\overline{\gamma}}[\overline{U}]$.

Thus $L_{\overline{\gamma}}[\overline{U}]$ is a mouse (by the corollary).

**Claim 10.12.** Assume $V = L[U]$. For each $\lambda < \kappa$, and $x \subseteq \lambda$, there is a mouse $L_{\overline{\gamma}}[\overline{U}]$ such that

(1) $x \in L_{\overline{\gamma}}[\overline{U}]$

(2) The critical point of $\overline{U}$ exceeds $\gamma$

(3) $\left| L_\gamma[\overline{U}] \right| = \lambda$.

*Proof.* Exercise. ■

Fix a cardinal $\lambda < \kappa$. Our goal is to show that for each subset of $\lambda$ (working in $V = L[U]$)

$$\left| \{ y \subseteq \lambda \mid y <_{L[U]} x \} \right| < \lambda^+.$$

It follows that $L[U]$ satisfies $2^\lambda = \lambda^+$.

This follows from the following claim.

**Claim 10.13.** Assume $V = L[U]$, and let $M^1 = L_{\gamma_1}[U_1]$ and $M^2 = L_{\gamma_2}[U_2]$ be mice. Suppose the critical points of $U_1$ and $U_2$ exceed $\gamma$. Then either $<_{M^1}$ restricted to $\mathcal{P}(\lambda)^{M^1}$ is an initial segment of $<_{M^2}$ restricted to $\mathcal{P}(\lambda)^{M^2}$ or vice-versa.

*Proof.* Since $M^1$ and $M^2$ are iterable, comparison gives a map

$$j^i_{0,\theta} : M^i \to M^i_\theta$$

for $i = 1, 2$. WLOG, $M_\theta^1 \trianglelefteq M_\theta^2$ now. The point is that the critical points exceed $\lambda$: nothing less than $\lambda$ is moved.

That is,

$$\begin{aligned} <_{M_1} \restriction^{M_1}_{\mathcal{P}(\lambda)} &\trianglelefteq <_{M_1} \restriction^{M_\theta^1}_{\mathcal{P}(\lambda)} \\ <_{M_2} \restriction^{M_2}_{\mathcal{P}(\lambda)} &\trianglelefteq <_{M_2} \restriction^{M_\theta^2}_{\mathcal{P}(\lambda)} \\ <_{M_1} \restriction^{M_\theta^1}_{\mathcal{P}(\lambda)} &\trianglelefteq <_{M_2} \restriction^{M_\theta^2}_{\mathcal{P}(\lambda)} \end{aligned}$$

where $\trianglelefteq$ means "initial segment of". ■

Now use the fact that $\left|L_\gamma[\overline{U}]\right| = \lambda$ to bound the size of the ordering. □

# §11 March 12, 2015

## §11.1 Finishing $L[U]$

Recall that in $L$ there is a $\Delta^1_2$ well-ordering of $\mathcal{P}(\omega)$.

**Theorem 11.1**
Suppose $\langle L[U], \in, U \rangle$ is a $\kappa$-model. Then $L[U] \vDash$ "$\mathbb{R}$ has a $\Delta^1_3$ well-ordering".

*Proof.* The main point $\lambda = \omega$ in the previous claim.

We remark that $x <_{L[U]} y$ if there exists a countable premouse $M$ $(\exists)$ such that

(1) $M \vDash$ "$x <_M y$".

(2) For all $\alpha < \omega_1$ $(\forall)$, $M$ is $\alpha$-iterable (a $\Pi_2$ condition I think? Need to check.).

The point is to iterate out the mice until they agree. The complexity is $\exists\forall$. □

**Exercise 11.2.** Suppose $\langle L[U_1], \in, U_i \rangle$ is a $\kappa_1$-model and $\langle L[U_2], \in, U_2 \rangle$ is a $\kappa_2$-model. Show that $\mathcal{P}(\omega) \cap L[U_1] = \mathcal{P}(\omega) \cap L[U_2]$ and the orderings $<_{L[U_1]}$ and $<_L [U_2]$ agree when restricted to $\mathcal{P}(\omega)$.

Summary:

- All $\kappa$-models satisfy GCH.
- All have a $\Delta^1_3$ well-ordering of the reals.
- All have the same reals!

**Lemma 11.3**
Suppose $\langle L[U], \in, U \rangle$ is a $\kappa$-model and let $\Gamma \subseteq \mathrm{On}$. Let

$$\gamma > \sup \Gamma$$

be a limit ordinal. Assume also that $|\Gamma| \geq (\kappa^+)^{L[U]}$. Then

$$\mathrm{Hull}^{L_\gamma[U]}(\kappa \cup \gamma)$$

contains all subsets of $\kappa$ in $L[U]$, *id est* contains $\mathcal{P}(\kappa) \cap L[U]$ as a subset.

*Proof.* Note that the Mostowski collapse of the hull is of the form $L_{\overline{\gamma}}[U]$ for some $\overline{\gamma}$, and we have

$$\overline{\gamma} \geq |\Gamma| \geq (\kappa^+)^{L[U]}.$$

Let $\pi$ be this collapse. By construction, $\pi^{-1}$ is the identity restricted to $\kappa$.

Let $x \subseteq \mathcal{P}(\gamma)$ in the collapse, and let $x'$ be so that $\pi(x') = x$. One can see that $\pi^{-1}(x) \cap \kappa = x$, but the issue is that the hull need not contain $\kappa$ as a point.

So $\forall x \in \mathcal{P}(\kappa) \cap L[U]$ there exists a Skolem term $t$ such that

$$x = t^{\langle L_\gamma[U], \in, U \rangle}(\xi_1, \dots, \xi_n, \nu_1, \dots, \nu_m)$$

where $\xi_1, \dots, \xi_n \in \kappa$ and $\nu_1, \dots, \nu_n \in \gamma$. □

## §11.2 Kunen's Results

**Theorem 11.4** (Kunen)
Suppose $\langle L[U], \in, U\rangle$ and $\langle L[U'], \in, U'\rangle$ is a $\kappa$-model. Then $U = U'$ and $L[U] = L[U']$.

All this is assuming the measures are in the model. (Meaning $U \in L[U]$ and $U' \in L[U']$.)

*Proof.* By Comparison, and the fact that $L[U]$ and $L[U']$ are proper class models, there is a common iterate (because proper class models don't get taller, our theorem that one iterate is an initial segment of the other gives an equality). So we have set-size iterations

$$L[U] \xrightarrow{j} L[W]$$

and

$$L[U'] \xrightarrow{j'} L[W].$$

By the corollary,

$$\left\{\theta > \kappa \mid \theta = |\theta| = j(\theta) = j'(\theta)\right\}$$

is a proper class. Let $\gamma$ be a subset of the above and $\gamma$ is a limit ordinal and $\gamma > \sup\Gamma$. Assume, assume $|\Gamma| = (\kappa^+)^V$.

Given $x \in U$ we wish to show $x \in U'$ (symmetry will do the other direction). By the lemma there is a Skolem term $t$ such that

$$x = t^{\langle L_\gamma[U], \in, U\rangle}\left(\xi_1, \dots, \xi_n, \nu_1, \dots, \nu_m\right).$$

Set

$$x' = t^{\langle L_\gamma[U'], \in, U'\rangle}\left(\xi_1, \dots, \xi_n, \nu_1, \dots, \nu_m\right).$$

Since the elements of $\kappa \cup \gamma$ are fixed by both $j$ and $j'$, we have

$$\begin{aligned} j(x) &= t^{L_{j(\gamma)}[j(U)]}\left(j(\xi_1), \dots, j(\xi_n), j(\nu_1), \dots, j(\nu_m)\right) \\ &= t^{L_\gamma[W]}(\xi_1, \dots, \xi_n, \nu_1, \dots, \nu_m). \end{aligned}$$

Moreover, $j'(x')$ gives the same result. Now, $x = j(x) \cap \kappa$ and $x' = j'(x') \cap \kappa$. Also $j'(x') \in W$, meaning $x' \in U'$. Thus $x = x'$. □

Hence, there is only one $\kappa$ model!

**Theorem 11.5** (Kunen)
Now suppose $M = \langle L[U], \in, U\rangle$ is the $\kappa$-model and $M' = \langle L[U'], \in, U'\rangle$ is the $\kappa'$-model. Given $\kappa < \kappa'$, the model $M'$ is an iterate of $M$.

*Proof.* Let

$$\langle L[U_\alpha], U_\alpha, \kappa_\alpha, j_{\alpha,\beta} \mid \alpha < \beta\rangle$$

be the iteration of $M$. We claim that for some $\beta$ we have $\kappa' = \kappa_\beta$, which is sufficient by the previous theorem.

Suppose not. The sequence of critical points is unbounded, so for some $\beta$ we have

$$\kappa_\beta < \kappa' < \kappa_{\beta+1}.$$

By Comparison, we have maps

$$L[U] \xrightarrow{j=j_{0,\delta}} L[W]$$

and

$$L[U'] \xrightarrow{j'=j'_{0,\delta}} L[W]$$

where $\delta$ is the length of the iteration. Again

$$F = \left\{\theta \mid \theta = |\theta| = j(\theta) = j'(\theta)\right\}$$

is a proper class. Let $\Gamma \cup \{\gamma\} \subseteq F$, $\gamma$ a limit ordinal exceeding $\sup \gamma$ and $|\Gamma| = \kappa_\beta^+$ in the sense of $V$.

We claim $\kappa'$ is in the range of $j'$, which produces the desired contradiction. By the "seed" result from earlier we know

$$\kappa' = (j_{\beta,\beta+1} f)(\kappa_\beta)$$

for some $f : \kappa_\beta \to \kappa_\beta$ with $f \in L[U]$. But $f$ can be written via a Skolem term $t$ with

$$f = t^{\left\langle L_\gamma[U_\beta], \in, U_\beta \right\rangle} (\xi_1, \dots, \xi_n, \nu_1, \dots, \nu_m)$$

where $\xi_i \in \kappa$ and $\nu_i \in \Gamma$. The $\xi_i$ get fixed because they're below the critical point, and the $\nu_i$ get fixed because they are supposed to be fixed points in the entire sequence (by virtue of being in $\Gamma$).

Hence we can get a Skolem term $t'$ which applies $\kappa_\beta$:

$$\begin{aligned} \kappa' &= t'^{\left\langle L_\gamma[U_{\beta+1}], \in, U_{\beta+1} \right\rangle} (\xi_1, \dots, \xi_n, \kappa_\beta, \nu_1, \dots, \nu_m) \\ j_{\beta+1,\delta}(\kappa)' = \kappa' &= t'^{\langle L_\gamma[W], \in, W \rangle} (\xi_1, \dots, \xi_n, \kappa_\beta, \nu_1, \dots, \nu_m) \\ &= j' \left( t'^{\langle L_\gamma[U'], \in, U' \rangle} (\xi_1, \dots, \xi_n, \kappa_\beta, \nu_1, \dots, \nu_m) \right). \end{aligned}$$

Since the critical point of $j'$ is $\kappa' > \kappa_\beta$, we again see that all ordinals $\kappa_\beta$, $\xi_i$, $\nu_i$ are fixed. So $\kappa'$ is in the range of $j'$, which is a contradiction $\square$

# §12 March 24, 2015

Today we will discuss extenders, which are glorified versions of measurables.

## §12.1 Motivating Extenders

Recall that $\kappa$ is measurable if there exists an elementary embedding $j : V \to M$ with critical point $\kappa$. This definition has some nice properties (for example, you can tell that $\kappa$ is strongly inaccessible) but has the following two issues.

1. It's not a formula for LST, since $V$ is big.
2. It doesn't help with combinatorics.

To get around this we obtained an ultrafilter

$$U = \{A \subseteq \kappa \mid \kappa \in j(A)\}.$$

This gives us $j : V \to \mathrm{Ult}(V, U)$.

**Definition 12.1.** A **$\alpha$-strong cardinal** is $\kappa$ for there exists $j : V \to M$ such that $j(\kappa) > \alpha$ and $M$ contains all of $V_\alpha$.

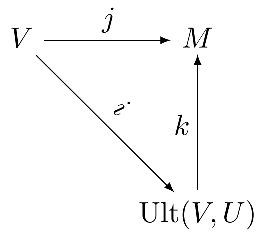
**Definition 12.2.** A **$\lambda$-supercompact cardinal** is $\kappa$ for there exists $j : V \to M$ such that $j(\kappa) > \lambda$ and $M$ is closed under $\lambda$ sequences.

We saw that ultrapowers can't do this. Specifically,

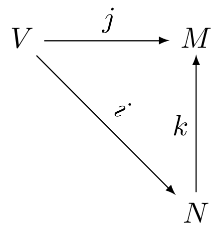
- If $U$ is an ultrafilter on $\kappa$ then $U \notin \mathrm{Ult}(V, U)$.
- If $U$ is an ultrafilter on $\kappa$ and $j_U$ is the canonical map, the restriction $j_U$ to $\kappa^+$ is not in $M$.

So we want to try and get around this problem with extenders.

Recall that if $U$ is derived from $j : V \to M$, we get an embedding $k : \mathrm{Ult}(V, U) \to M$ by $[f] \mapsto j(f)(\kappa)$. Moreover the range of $k$ is $\mathrm{Hull}^M(j``V \cup \{\kappa\})$. The diagram is




Suppose we want $i$ to approximate $j : V \to M$. Take some hull $H$ containing $j``V$ and all of $\lambda$ in $M$. Then we can take its transitive collapse to get $N$; let $k$ be the inverse of this ultrapower. Moreover, $k$ has critical point exceeding $\lambda$ because all of $\lambda$ is in the hull. Now we get a diagram




where we set $i = j \circ k^{-1}$. Thus $i$ approximates $j$ well.

It remain to show how to view $N$ as an ultrapower.

## §12.2 Derived Extenders

Fix some $\lambda$ whose agreement we want to force. Let $j : V \to M$ be an elementary embedding, and let $N$ be the (transitive collapse) of some hull in $M$ which contains all of $\lambda$.

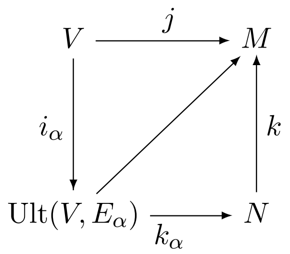
For any $\alpha$ we define

$$E_\alpha = \{A \subseteq \kappa_\alpha : \alpha \in j(A)\}.$$

Here

$$\kappa_\alpha \overset{\text{def}}{=} \min\{\beta : j(\beta) \geq \alpha\}.$$

This is a $\kappa$-complete ultrafilter and so we can construct $\mathrm{Ult}(V, E_\alpha)$. That gives us



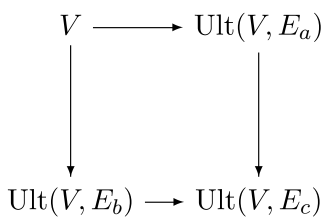

The fact that we can take $k_\alpha$ is a consequence of $\alpha < \lambda$, so the image of the diagonal embedding $\mathrm{Ult}(V, E_\alpha) \to M$ is included in the stuff before we take the transitive collapse $k^{-1}$.

We're going to try to realize $N$ as the direct limit of all the $\mathrm{Ult}(V, E_\alpha)$. But to do this we need that $\mathrm{Ult}(V, E_\alpha)$ and $\mathrm{Ult}(V, E_\beta)$ to embed into a larger system $\mathrm{Ult}(V, E_{\{\alpha,\beta\}})$. So we can define this via

$$E_{\{\alpha,\beta\}} = \left\{A \in [\kappa_{\alpha,\beta}]^2 : \{\alpha, \beta\} \in j(A)\right\}.$$

Then there's a natural embedding from $\mathrm{Ult}(V, E_\beta)$ by $[f]_{E_\beta} \mapsto [f_*]$, where $f_*(\{x, y\}) = f(y)$. Similarly, for $E_\alpha$.

Thus we keep going and we want to define $\mathrm{Ult}(V, E_a)$ for any finite subset $a$ of our $\lambda$. It's a directed system viz




Here's how we define it. IF $a \subseteq c$ and let $x$ be a $|c|$-sized subset of the ordinals. Let $x_a^c$ be so that

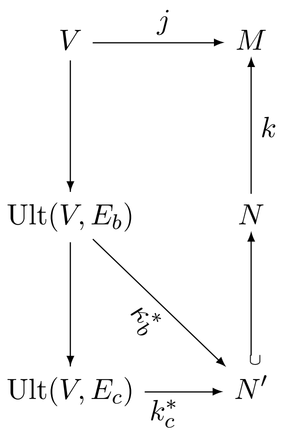
$$(x, x_a^c, \in) \cong (a, c, \in)$$

meaning *$x$ sits inside $x_a^c$ the same way $c$ sits inside $a$.* The ultrapower $\mathrm{Ult}(V, E_a)$ consists of functions $\binom{\kappa_\alpha}{|a|} \to V$, so we define

$$f_{a,c} : \binom{\kappa_c}{|c|} \to V$$

by $f_{a,c}(x) = f(x_a^c)$, and hence we get a map $\mathrm{Ult}(V, E_a) \to \mathrm{Ult}(V, E_c)$ by $f \mapsto f_{a,c}$.

Thus we have a directed system of $\text{Ult}(V, E_a)$, and we can take the direct limit to get $N'$ so that we have something like




Since these all map into $N$ our map factors through the direct limit $N'$ Since $N$ was supposed to be a hull, we get $N = N'$, as we wanted.

## §12.3 Explicitly Presentation of Derived Extenders

If $j : V \to M$, the $(\kappa, \lambda)$-extender derived from $j$ is the sequence

$$\langle E_a : a \text{ finite subset of } \lambda \rangle .$$

Then the direct limit is

$$\text{Ult}(V, E) = \left\{ [f, a] \mid f : \binom{\kappa_a}{|a|} \to V \text{ and } a \subseteq \lambda \text{ finite} \right\}.$$

Moreover, two $[f, a]$ and $[g, b]$ are equal if their image in some higher $\text{Ult}(V, E_c)$ coincide.

Finally, there's a canonical embedding $j_E : V \to \text{Ult}(V, E)$ into the ultrapower by simply

$$j_E(x) = [c_x, \varnothing]$$

where $c_x$ is the constant function which returns $x$ everywhere.

Observe that if we take $j = j_E$ now, we get $E$ back.

## §12.4 Defining Extenders Without Referencing Embeddings

We now seek to give a definition of the extender which doesn't refer to the embedding $j : V \to M$, but such that $\langle E_a \rangle$ is an extender under the new definition if and only if it's the derived extender for some $j : V \to M$.

Here's the definition.

**Definition 12.3.** A sequence

$$E = \langle E_a : a \subseteq \lambda \text{ finite} \rangle$$

is a $(\kappa, \lambda)$-**extender** if the following holds.

(1) ($E_a$ are ultrafilters) For each $a \subseteq \lambda$ finite, let $\kappa_a$ be the smallest cardinal such that $\binom{\kappa_a}{|a|}$ is in $E_a$.[1] Then $E_a$ is a $\kappa$-complete ultrafilter on $\binom{\kappa_a}{|a|}$.

[1] Not critical, but we need to cut off $E_a$ somewhere, so we just cut it off by $\kappa_a$.

(2) (Compatibility; the $E_a$ form a directed system) If $a \subseteq b$ and $X \in E_a$, then the lifted set $X_a^b$ is in $E_b$ This makes the embedding $\text{Ult}(V, E_a) \to \text{Ult}(V, E_b)$ an elementary embedding, since then by Łoś we have

$$\begin{aligned} \text{Ult}(V, E_a) \vDash \phi([f]) &\iff \{x : \phi(f(x))\} \in E_a \\ &\iff \{x^* : \phi(f_{a,b}(x^*))\} \in E_b \\ &\iff \text{Ult}(V, E_b) \vDash \phi([f_{a,b}]). \end{aligned}$$

(3) (Normality, guarantees that $E$ is the extender of $j_E$) We now know we can take a direct limit since we've got a directed system. The next condition is that if for some $a$, $f$ and an index $i$ we have

$$\left\{ x \in \binom{\kappa_a}{|a|} : f(x) < x_i \right\} \in E_a$$

then there is some $\beta < a_i$ such that if $a' = a \cup \{\beta\}$, then

$$\left\{ x \in \binom{k_{a'}}{|a'|} : f_{a,a'}(x) = x_\beta^{a'} \right\} \in E_{a'}.$$

(4) We need one more condition to guarantee that $\text{Ult}(V, E)$ is well-founded.

**Remark 12.4.** We can see how (3) is true for a derived extender as follows. The condition holds, by definition, exactly when

$$\begin{aligned} a \in j\left(\left\{ x \in \binom{\kappa_a}{|a|} : f(x) < x_i \right\}\right) \\ = \left\{ x \in \binom{j(\kappa_a)}{|a|} : j(f)(x) < x_i \right\} \end{aligned}$$

and thus $j(f)(a) < a_i$. Letting $\beta = j(f)(a)$ does the trick.

You might notice that this looks a lot like the "pressing down" condition.

Now we check that these definitions are equivalent.

**Theorem 12.5**

Given the conditions above, we may form $\text{Ult}(V, E)$, and moreover the sequence $E$ is the extender of the canonical map $j_E : V \to \text{Ult}(V, E)$.

*Proof.* We can form $\text{Ult}(V, E)$ because of conditions (1) and (2). We want to see that

$$x \in E_a \iff a \in j_E(X).$$

As in the case of ultrapowers,[2] it suffices to show that $[\text{id}, a] \in \text{Ult}(V, E)$ is actually equal to $a$. Indeed,

$$\begin{aligned} a \in j_E(x) &\iff [\text{id}, a] \in j_E(x) \\ \iff E_{a \cup \{\varnothing\}} &\ni \left\{ y \in \binom{\kappa_a}{|a|} : \text{id}(y) \in c_x(\varnothing) \right\} \\ &= x. \end{aligned}$$

So let's now show $[\text{id}, a] = a$.

[2] This is analogous to showing that $x \in U \iff \kappa \in j(x)$, where we checked that $[\text{id}] \in j(x)$, and then observing this means $U_j \ni \{y : \text{id}(y) \in x\} = y$.

**Claim 12.6.** Let $\cup$ be the union function. Then $[\cup, \{\alpha\}] = \alpha$ for $\alpha < \lambda$.

The $\cup$ is just to combat type mismatch; $\cup\{x\} = x$. That's just because $E_\alpha$, in all strictness, should be $E_{\{\alpha\}}$.

*Proof.* We use normality here. Proceed by transfinite induction; assume it holds for $\beta < \alpha$. If $[f, \alpha] \in [\cup, \{\alpha\}]$, then by definition with $a' = a \cup \{\alpha\}$ this is

$$E_{a'} \ni \left\{ x \in \binom{\kappa_{a'}}{|a'|} \mid f_{a,a'}(x) \in \cup x^{a'}_{\{a\}} = x^{a'}_a \right\}.$$

Now we apply the normality condition to get that for some $\beta < (a')^{a'}_a = \alpha$, if we let $b = a \cup \{\alpha, \beta\}$ we have

$$E_b \ni \left\{ x \in \binom{\kappa_b}{|b|} \mid f_{a,b}(x) = x^b_\beta \right\}.$$

If we set $a'' = a \cup \{\beta\}$ then by compatibility this is equivalent to

$$E_{a''} \ni \left\{ x \in \binom{\kappa_{a''}}{|a''|} \mid f_{a,a''}(x) = x^{a''}_\beta \right\}.$$

which amounts to saying $[f, a] = [\cup, \{\beta\}]$. By the inductive hypothesis, $[f, a] = \beta < \alpha$. So all elements of $[\cup, \{\alpha\}]$ are ordinals less than $\alpha$. ■

Now we show that $[\mathrm{id}, a] = a$ for any finite subset $a \subseteq \lambda$. Suppose that $[f, b] \in [\mathrm{id}, a]$ meaning

$$E_{a \cup b} \ni \left\{ x \in \binom{\kappa_{a \cup b}}{|a \cup b|} \mid f_{b, a\cup b}(x) \in \mathrm{id}_{a, a\cup b} x = x^{a \cup b}_a \right\}.$$

As $x^{a\cup b}_a$ is a finite set, by Infinite Pigeonhole there's some fixed "index" $\alpha \in a$ such that we may replace $x^{a\cup b}_a$ with $x^{a \cup b}_{\{\alpha\}}$ and thus obtain

$$E_{a \cup b} \ni \left\{ x \in \binom{\kappa_{a \cup b}}{|a \cup b|} \mid f_{b, a\cup b}(x) \in \cup x^{a \cup b}_{\{\alpha\}} \right\}.$$

(The "$\cup$" is the stupid type mismatch again.) This is equivalent to

$$[f, b] = [\cup, \{\alpha\}] = \alpha \in a.$$

as desired. □

# §13 March 26, 2015

## §13.1 Notions of Strength

There are two ways of securing strength: agreement by *rank* (meaning $V_\kappa$ matches), and *closure*.

To repeat what was said earlier...

**Definition 13.1.** We say $\kappa$ is **$\gamma$-strong** if there exists a $j : V \to M$ with critical point $\kappa$ (where $M$ is a transitive class), such that $j(\kappa) > \gamma$ and $V_\gamma^M = V_\gamma$.

Thus if $\kappa$ is measurable then it's $(\kappa+1)$-strong. Note that this only concerns the behavior of sets with lower rank.

**Definition 13.2.** We say $\kappa$ is **$\lambda$-supercompact** if there exists $j : V \to M$ with critical point $\gamma$ such that $j(\kappa) > \lambda$ and such that $M$ is closed under $\gamma$-sequences (meaning any function $\gamma \to M$ is also in $M$).

Note that this implies $\lambda$-supercompact implies $\lambda$-strong for large enough $\lambda$. This is a *global* property not restricted to the lower ranks of $M$.

**Definition 13.3.** We say $\kappa$ is **strong** if its $\gamma$-strong for all $\gamma$. We say $\kappa$ is **supercompact** if its $\lambda$-strong for all $\lambda$.

In all of this, we're trying to get $M$ to approximate $V$ well.

## §13.2 Extenders and Reflection

The point of extenders is to get more agreement than just a single ultrapower $\mathrm{Ult}(V,U)$; if $\kappa$ is a measurable cardinal then we have rank agreement up to $\kappa+1$ but not $\kappa+2$, and closure up to $\kappa$ but not $\kappa^+$. (So a measurable cardinal is $(\kappa+1)$-strong but not $(\kappa+2)$-strong, and $\kappa$-supercompact but not $\kappa^+$-supercompact.)

Last time we started with *any* $j : V \to M$ (say $\gamma$-strong or $\lambda$-supercompact) and we derived a $(\kappa,\lambda)$-extender $E$. Conversely, given such an extender we can form $j_E : V \to \mathrm{Ult}(V,E)$.

Now, we'd like to put conditions on $E$ which give us strong levels of agreement. This has the nice property that it's first order (one doesn't need to worry about "proper classes", since extenders are a set-sized object). It also shows various reflection properties. For example, if $\kappa$ is $(\kappa+2)$-strong then there's an ultrafilter $U$ living in $V_{\kappa+2}$; hence $M$ can see it too and $M \vDash$ "$\kappa$ is measurable". Thus $M \vDash$ "$\exists \alpha < j(\kappa)$ measurable"; then $V$ thinks this as well and we can keep reflecting like we did before. The key is that our set-sized object $U$ witnessing $\kappa$ is measurable actually lives inside the model $M$ because of $(\kappa+2)$-strength.

## §13.3 Review of Last Time

Suppose $E$ is a $(\kappa,\lambda)$-extender.

**Definition 13.4.** For $\alpha < \lambda$ define

$$\mathrm{pr}_\alpha : \binom{\kappa_{\{\alpha\}}}{1} \to \kappa_{\{\alpha\}}$$

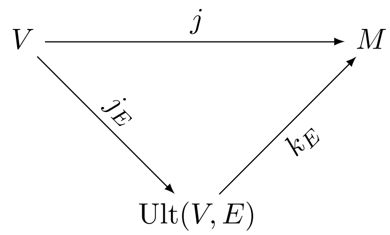
by $\{\xi\} \mapsto \xi$.

So this is the same "$\cup$" typehack as earlier. Recall that $\kappa_{\{alpha\}}$ is the first cardinal getting shot above $\alpha$.

Last time we showed that for all $\alpha < \lambda$, we have $\alpha = [\{\alpha\}, \mathrm{pr}_\alpha]$ and for all finite subsets $a \subseteq \lambda$, we have $a = [a, \mathrm{id}]$. Thus we showed that

> If $E$ is a $(\kappa, \lambda)$-extender then $E$ is the $(\kappa, \lambda)$-extender derived from $j_E : V \to \mathrm{Ult}(V, E)$.

This gives us a commutative diagram




where $k_E : [a, f] \mapsto j(f)(a)$.

**Fact 13.5.** $j_E(f)(a) = [a, f]$.

Let's compile these facts.

**Lemma 13.6**

Let $j$, $M$, etc. be as above. Then

(1) $k_E : \mathrm{Ult}(V, E) \to M$ is an elementary embedding.

(2) $k_E \circ j_E = j$.

(3) $k_E$ is the identity on any finite subset of $\lambda$.

(4) If $\gamma$ is such that $j(\gamma) \le \lambda$ then $j_E$ and $j$ agree on all ordinals not exceeding $\gamma$.

Thus the critical point of $\kappa_E$ is somewhere above $\lambda$.

**Theorem 13.7** (Agreement)

Let $j$, $M$, $E$, $\kappa$, $\lambda$, etc. be as above. Let $\gamma \le \lambda$ (here $\lambda$ is a limit ordinal) be such that

$$\left| V_\gamma^M \right|^M \le \lambda.$$

Then

(1) $V_\gamma^{\mathrm{Ult}(V,E)} = V_\gamma$. This is $\gamma$-strength.

(2) $k_E$ is the identity when restricted to $V_\gamma^{\mathrm{Ult}(V,E)}$.

So we just need to pick $\lambda$ large enough to capture large amounts of agreement. For example, if we wish to capture $(\gamma + 2)$-strength we just pick $\lambda$ so that $\lambda > \left| V_{\kappa+2}^M \right|^M$.

*Proof.* Let $\nu = \left| V_\gamma^{\mathrm{Ult}(V,E)} \right|^{\mathrm{Ult}(V,E)}$, so by hypothesis,

$$\nu \le k_E(\nu) = \left| V_\gamma^M \right|^M \le \lambda.$$

We claim that $\nu = k_E(\nu)$ now. If not, then $\nu < k_E(\nu)$, so $\nu < \lambda$. But by the lemma, $k_E$ ought to be the identity on the singleton $\{\nu\}$, which is a contradiction.

Let

$$\langle X_\xi : \xi < \nu \rangle$$

be an enumeration of $V_\gamma^{\mathrm{Ult}(V,E)}$ in $\mathrm{Ult}(V,E)$. Then

$$k_E\left(\langle X_\xi : \xi < \nu \rangle\right) = \langle k_E(X_\xi) : \xi < \nu \rangle$$

is an enumeration of $V_\gamma^M$ in $M$, *id est* the map $k_E$ is surjective. Since $k_E$ is elementary, this implies that $k_E$, when restricted to $V_\gamma^{\mathrm{Ult}(V,E)}$, induces an isomorphism

$$\left\langle V_\gamma^{\mathrm{Ult}(V,E)}, \in \right\rangle \cong \langle V_\gamma^M, \in \rangle$$

in the sense that, say $\xi_1 \in \xi_2 \iff k_E(\xi_1) \in k_E(\xi_2)$.

Thus this map must be the identity as both these guys are transitive. (In general, there are no nontrivial automorphisms of transitive well-founded models.) □

**Theorem 13.8** (Closure)
Let $E$ be a $(\kappa, \lambda)$-extender and let $j_E : V \to \mathrm{Ult}(V,E)$ be the associated embedding. Assume that $\gamma$ is such that

(1) The $\gamma$-sequences of $\lambda$ are contained in $\mathrm{Ult}(V,E)$; *id est* $E$ is $\gamma$-complete, and

(2) $j_E``\gamma \in \mathrm{Ult}(V,E)$ holds.

Then $\mathrm{Ult}(V,E)$ is $\gamma$-supercompact: all $\gamma$-sequences of $\mathrm{Ult}(V,E)$ are in $\mathrm{Ult}(V,E)$.

*Proof.* Omitted. (It's not especially nice.) □

## §13.4 Extender Formulations

**Definition 13.9.** Let $E = \langle E_a : a \subseteq \lambda \text{ finite} \rangle$ be a $(\kappa, \lambda)$-extender.

- The **critical point** of $E$ is defined as $\kappa$.
- The **length** of $E$ is defined as $\lambda$.
- The **support** of $E$ is $\sup \kappa_a$ across all such $a$.
- The **strength** of $E$ is the largest $\gamma$ such that $V_\gamma$ is contained inside $\mathrm{Ult}(V,E)$
- The **closure** of $E$ is the smallest $|\gamma|$ such that $\mathrm{Ult}(V,E)$ is *not* closed under $\gamma$-sequences.

**Theorem 13.10** (Extenders Witness $\gamma$-Strength)
Let $\gamma > \kappa$. The following are equivalent.

(1) $\kappa$ is a $\gamma$-strong cardinal.

(2) There exists an extender $E$ such that
   (a) The critical point of $E$ is $\kappa$.
   (b) $j_E(\kappa) > \gamma$.
   (c) The strength of $E$ exceeds $\gamma$.

*Proof.* (2) $\implies$ (1) is immediate.

Conversely, suppose $j : V \to M$ witnesses that $\kappa$ is $\gamma$-strong, meaning that

(a) the critical point of $j$ is $\kappa$,

(b) $j(\kappa) > \gamma$, and

(c) $V_\gamma^M = V_\gamma$.

We need to exhibit a $(\kappa, \lambda)$-extender $E$ derived from $j$. The question is: which $\lambda$ should we pick? We just take $\lambda > \max\left\{\gamma, |V_\gamma|^M\right\}$. □

The conditions (a), (b), (c) are $\Sigma_2$. Hence asserting strength (requiring another $\forall$ to express $\alpha$-strength for all $\alpha$) is $\Pi_3$.

**Theorem 13.11** (Extenders Witness $\lambda$-Closure)
Let $\lambda > \kappa$. The following are equivalent.

(1) $\kappa$ is a $\lambda$-supercompact cardinal.

(2) There exists an extender $E$ such that
   (a) The critical point of $E$ is $\kappa$.
   (b) $j_E(\kappa) > \lambda$.
   (c) The closure of $E$ exceeds $\lambda$.

*Proof.* Note that we unfortunately will be talking about $(\kappa, \lambda')$-extenders since we've re-used the symbol $\lambda$.

We take $\lambda' > \lambda$ such that $\lambda' = |V_{\lambda'}|$ (arbitrarily large $\lambda'$ with this property exist) and so that the cofinality of $\lambda'$ exceeds $\lambda$. Then we let $E$ be the $(\kappa, j(\lambda'))$-extender derived from $j$.

By Agreement, we have
$$V_{j(\lambda')}^{\mathrm{Ult}(V,E)} = V_{j(\lambda')}^M.$$
Thus
$$j_E``(\lambda) \in V_{j(\lambda)+1}^M \subseteq \mathrm{Ult}(V, E).$$

To see that the $\lambda$-sequences of $j(\lambda')$ are all in $\mathrm{Ult}(V, E)$, we use the condition on cofinality. Note that if $f : \lambda \to j(\gamma')$ is such a sequence, then $f \in M$ since $M$ is closed under $\lambda$-sequences. We need to get these into $\mathrm{Ult}(V, E)$.

**Claim 13.12.** The cofinality of $j(\lambda')$ in the sense of $M$ exceeds $\lambda$.

*Proof.* $\text{cof}(\lambda') > \lambda$, so $\text{cof}(j(\lambda'))^M = j(\text{cof}(\lambda')) > j(\lambda) \geq \lambda$. ■

Thus $f$ is bounded, implying $f \in V_\gamma^M$ for some $\gamma < j(\lambda')$. Hence $f \in V_\gamma^M$, which agrees with $\text{Ult}(V, E)$ by Agreement. □

Again, supercompacts are $\Pi_3$.

## §13.5 A Word on Superstrong Cardinals

Strong cardinals seem to be as powerful as you can get: given any $\gamma$, there is an embedding $j_\gamma : V \to M$ with critical point $\kappa$ which agrees up to $\gamma$. It turns out you can't have full agreement; it's not possible that $V = M$.

But in all of these cases, the agreement is less than $j_\gamma(\kappa)$. The image of the critical point is always hovering above the level of agreement. What if we made these coincide?

**Definition 13.13.** We say $\kappa$ is **superstrong** if there exists $j : V \to M$ with critical point $\kappa$ such that

$$V_{j(\kappa)}^M = V_{j(\kappa)}.$$

(Note that superstrong need not be strong. Oops.)

# §14 March 31, 2015

Today we discuss large cardinal beyond Choice.

## §14.1 Motivation

Recall that if $\kappa$ is measurable as witnessed by $U$, then we have rank agreement up to $\kappa+1$ but not $\kappa+2$ (since $U \notin \mathrm{Ult}(V,U)$), and closure up to $\kappa$ but not $\kappa^+$ (since $j$ restricted to $\kappa^+$ is not in $\mathrm{Ult}(V,U)$). Thus we introduced the notion of $\gamma$-strong and $\lambda$-supercompact.

What happens if we demand full agreement (and thus full closure, since full agreement is equivalent to full closure?). In his 1967 dissertation Reinhardt proposed the axiom

$$\exists j : V \to V \quad j \neq \mathrm{id}.$$

**Definition 14.1.** Work in ZF with an additional predicate $j$ (called $\mathrm{ZF}_j$). Then $\kappa$ is a **Reinhardt** cardinal if $\exists j : V \to V$ such that $j \neq \mathrm{id}$ and the critical point of $j$ is $\kappa$.

**Remark 14.2.** Although I didn't say it earlier: it's true in general that non-trivial embeddings have a critical point. See e.g. `http://en.wikipedia.org/wiki/Critical_point_(set_theory)`.

With reflection properties we see that $j$ cannot be captured by set-size properties, because then you could generate a contradiction. Finally, to talk about $j : V \to V$ being an "elementary embedding" we should instead say it's a $\Sigma_1$ embedding.

## §14.2 Kunen's Inconsistency Theorem

Now we have the following theorem.

**Theorem 14.3** (Kunen's Inconsistency Theorem)
Assume Choice. Then there are no Reinhardt cardinals.

*Proof (Woodin).* Assume for contradiction that $j : V \to V$,then $j \neq \mathrm{id}$, and let $\kappa$ be its critical point. We will use the following (black-box) lemma.

**Lemma 14.4** (Solovay)
Suppose $\kappa > \omega$ is a regular cardinal and $S \subseteq \kappa$ is stationary (meaning it intersects all clubs of $\kappa$). Then there exists a partition $\langle S_\alpha : \alpha < \kappa \rangle$ of $S$ into stationary sets.

We will break the lemma assuming a Reinhardt cardinal exists. (Needless to say, the lemma uses AC; in fact Woodin has said this lemma is the "purest manifestation of AC".)

Let $\kappa_0$ be the critical point of $j$ and inductively define $\kappa_{n+1} = j(\kappa_n)$ for each $n \in \omega$. Set

$$\lambda = \sup_{n \in \omega} \kappa_n.$$

By construction,

$$j(\lambda) = j(\sup \{\kappa_n\}) = \sup \{j(\kappa_n)\} = \lambda.$$

Moreover, $j(\lambda^+) = j(\lambda)^+ = \lambda^+$.

Now define

$$S_\omega^{\lambda^+} = \left\{ \alpha < \lambda^+ \mid \operatorname{cof}(\alpha) = \omega \right\}.$$

Observe that it's fixed by $j$, since

$$\begin{aligned} j\left(S_\omega^{\lambda^+}\right) &= \left\{ \alpha < j(\lambda^+) \mid \operatorname{cof}(\alpha) = j(\omega) \right\} \\ &= \left\{ \alpha < \lambda^+ \mid \operatorname{cof}(\alpha) = \omega \right\} \\ &= S_\omega^{\lambda^+}. \end{aligned}$$

By the lemma, we can partition the above set into $\langle S_\alpha \rangle$. Let

$$\left\langle T_\alpha : \alpha < \lambda^+ \right\rangle = j\left( \left\langle S_\alpha : \alpha < \lambda^+ \right\rangle \right).$$

So the $T_\alpha$ is also a partition of $S_\omega^{\lambda^+}$ into stationary sets. We see that we have

$$j(S_\alpha) = T_{j(\alpha)}.$$

In particular, for $\alpha < \kappa$ (or just any fixed point) we get $T_\alpha = j(S_\alpha)$. Thus $T_\lambda = j(S_\lambda)$. However, $T_\kappa$ is *not* of the form $j(S_\alpha)$.

Now define an **$\omega$-club** in a regular $\kappa > \omega$ to be a subset which is unbounded and contains all $\omega$-limits (this is weaker than a being club). Then set

$$F_j^{\lambda^+} = \left\{ \alpha < \lambda^+ \mid j(\alpha) = \alpha \right\}.$$

We claim this is a $\omega$-club. It's unbounded because sup's are fixed points (the same way we started). It's trivially closed under $\omega$ limits.

Observe that $T_{\kappa_0}$ is a stationary subset. It hits every club. Thus it hits the closure of $F_j^{\lambda^+}$ at some point $\eta$, which has cofinality $\omega$. Then $\eta$ belongs to $F_j^{\lambda^+}$ itself (and not just its closure), since $F_j^{\lambda^+}$ is an $\omega$-club. Now

$$\eta = j(\eta) \in j(S_{\alpha_0}) = T_{j(\alpha_0)}.$$

But $\eta \in T_{\kappa_0}$, meaning $\kappa_0 = j_{\alpha(0)}$, which is impossible. □

Hence the ultimate axiom of full agreement is way too strong.

Stronger forms of the theorem are as follows.

**Theorem 14.5** (Kunen's Inconsistency Theorem)
Assume Choice. Let $j : V \to M$ be a nontrivial elementary embedding. Let $\lambda$ be the supremum of the critical sequence (as defined in the proof above). Then

(a) If $M = V$, then $j$ does not preserve $V_{\lambda+2}$.

(b) $\mathcal{P}(\lambda)$ is not contained in $M$.

In the notion of super-strength defined at the end of the last lecture, (a) says we can't have $(\omega + 1)$-superstrength.

## §14.3 Very Large Cardinals

It is an open question whether $\mathrm{ZF}_j + \exists$Reinhardt is consistent. Perhaps it is and it is the start of a hierarchy of large cardinals *beyond Choice*. Putting our doubts aside...

Assume we have a Reinhardt cardinal...

**Definition 14.6.** A cardinal $\kappa$ is **super-Reinhardt** if for every ordinal $\gamma$, there exists $j : V \to V$ with critical point $\kappa$ such that $j(\kappa) > \gamma$.

Let's try to rank reflect a Reinhadt cardinal, that is, to get $\gamma < \kappa$ such that

$$(V_\gamma, V_{\gamma+1}) \vDash \mathrm{ZF}_2 + \exists\text{Reinhardt}$$

then we can pick such $\gamma < \kappa$, as $\kappa$ is super Reinhardt, and we can reflect that property.

Thus suffices to find *any* $\gamma$ such that

(a) $\gamma$ is inaccessible (implying the $\mathrm{ZF}_2$), and

(b) $j(\gamma) = \gamma$ (thus $j|_{V_\gamma}$ witnesses the existence of a Reinhardt cardinal).

Observe that in general, $\kappa_1$ is inaccessible. Since we can make $\kappa_1$ as high as we like; hence there are arbitrarily high accessible above $\lambda$. So, we let $\gamma$ be the first inaccessible above $\gamma$. As $\lambda$ is fixed, it follows that

$$j(\gamma) = \gamma$$

since $\gamma$ can be defined as "largest inaccessible above $\lambda$". This $\gamma$ works, with $j$ as the original $j$ restricted to $V_\gamma$.

Let's explicitly do the reflection back down. Pick a $\hat{j}$ with critical point $\kappa$ such that $\hat{j}(\kappa) > \gamma$. Then $V_{\hat{j}(\kappa)}$ thinks that there exists a good $\gamma$ as above thus so does $V_\kappa$.

Hence super-Reinhardt rank-reflects Reinhardt cardinals. (Trivially, every super-Reinhardt is Reinhardt.)

**Question 14.7.** Is it true that every super-Reinhardt cardinal has a Reinhardt cardinal below it?

## §14.4 Berkeley cardinals

**Definition 14.8.** A **proto-Berkeley cardinal** is a cardinal $\delta$ such that if $M \ni \delta$ is a transitive set, then there is an elementary embedding $j : M \to M$ with critical point less than $\delta$.

A Berkeley cardinal implies the consistency of a Reinhardt cardinal.

# §15 April 2, 2015

We'll now discuss cardinals between the supercompact and the cardinal beyond Choice. This includes the huge cardinals and $I_0$ cardinals.

In particular, we're going to use extenders to show that larger large cardinals reflect smaller large cardinals. This means that the large cardinals actually line up into a well-ordering.

## §15.1 Relativization

First, we're going to relativize the ultrapower construction.

Suppose $M$ is a transitive class and $E \in M$ an extender such that $M$ thinks $E$ is an extender. Then we can form

$$\mathrm{Ult}(M, E)$$

in exactly the same way we formed

$$\mathrm{Ult}(V, E)$$

except this time we use $f : \binom{\kappa_a}{|a|} \to M$ such that $f \in M$.

Now we want to ask: how do the above two ultrapowers compare?

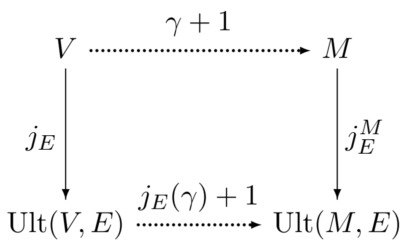
**Theorem 15.1**

Suppose $E$ is a $(\kappa, \lambda)$-extender and $M$ is a transitive class of ZFC, with $E \in M$. Then

(1) $M$ thinks $E$ is an extender.

(2) If $\gamma \geq \kappa$ is such that $V^V_{\gamma+1} = V^M_{\gamma+1}$, then $j^M_E$ and $j^V_E$ agree up to $\gamma^+$ and

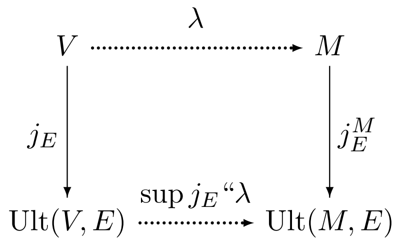
$$V^{\mathrm{Ult}(M,E)}_{j_E(\gamma+1)} = V^{\mathrm{Ult}(V,E)}_{j_E(\gamma+1)}.$$




*Proof.* For (1), the point is that for any $\kappa_a$ in the support of$E$, we have $V_{\kappa_a+1} \subseteq M$ and hence $E_a \subseteq M$. Since the finite subsets of $\lambda$ are in $M$, we find that $M$ has all the information of $E$ inside it. (Here we use absoluteness of well-foundedness for the well-founded condition of extenders.)

For (2), this follows since $M$ and $V$ have the same functions $f : \binom{\kappa_a}{|a|} \to \gamma^+$ and $g : \binom{\kappa_a}{|a|} \to V_{\gamma+1}$; indeed such functions can be coded by elements of $V_{\gamma+1} = (V_{\gamma+1})^M$. (Since $\kappa_a$ is regular, its functions are bounded, so we can replace $f$ with $\gamma^+$, yada yada...). So the coding in $V_{\gamma+1}$ gives us (2). $\square$

So we actually get




but not necessarily up to $j_E(\lambda)$.

## §15.2 Reflection

**Theorem 15.2**

Suppose $\kappa$ is $(\kappa+2)$-strong. Then we may obtain a nonprincipal uniform ultrafilter on $\kappa$ with

$$\{\overline{\kappa} < \kappa \mid \overline{\kappa} \text{ measurable}\} \in U.$$

This is kind of absurd (there are LOTS of measurables, in fact there are measure one many).

Indeed, consider an embedding $j : V \to M$ with critical point $\kappa$ with agreement up to $V_{\kappa+2}$. Hence $\kappa$ is measurable and there's a measure $U$ on it, derived from $j$. So $M$ thinks $\kappa$ is measurable too; but it also thinks $j(\kappa)$ is measurable. Since $M$ thinks there's a measurable below $j(\kappa)$, so does $V$; and keep reflecting...

*Proof.* Let $U$ be the derived ultrafilter; we claim it works. Set $A = \{\overline{\kappa} < \kappa \mid \overline{\kappa} \text{ measurable}\}$. Thus $A \in U$ if and only if $\kappa \in j(A)$. But

$$A = \{\overline{\kappa} < j(\kappa) \mid \overline{\kappa} \text{ measurable}\}.$$

Hence clearly $\kappa \in A$. QED. ☐

OK, so suppose $\kappa$ is $(\kappa+3)$-strong. Is $\kappa$ a limit of $\overline{\kappa}$'s which are $(\overline{\kappa}+2)$-strong? The issue is whether $V_{\kappa+3} \subseteq M$ is enough to ensure that $M$ thinks that $\kappa$ is $(\kappa+2)$-strong.

The extender

$$E = \langle E_a \mid A \subseteq \lambda \text{ finite}\rangle$$

will work as long as it has length at least $|V_{\kappa+2}|^M$. We'd have to squeeze this entire $E$ into the ultrapower. Since $|V_{\kappa+2}|^M < |V_{\kappa+2}|^+$, we would need that instead.

To summarize

- If $\kappa$ is $(\kappa+2)$-strongs it is a limit of measurables.
- If $\kappa$ is $|V_{\kappa+2}|^+$-strong it is a limit of $\overline{\kappa}$'s which are $\overline{\kappa}+2$ strong.

Gabe seems to think that by encoding a total order of $|V_{\kappa+2}|^+$ can actually get replaced by $\kappa+3$... Let's get back on that.

## §15.3 More Reflections

Let's black box the following.

**Lemma 15.3**

The following are equivalent.

(a) $\gamma$ is a strong limit.

(b) $\gamma = |V_\gamma|$

(c) $V_\gamma = \mathrm{Hull}(\gamma)$

(d) $V_\gamma \prec_{\Sigma_1}$, $V_\gamma$ agrees with $V$ on $\Sigma_1$ sentences.

*Proof.* Omitted. □

**Lemma 15.4**

Suppose $\kappa$ is strong. Then

(1) Assuming GCH below $\kappa$, GCH holds everywhere.

(2) $V_\kappa$ is a $\Sigma_2$ substructure of $V$

*Proof.* (1) is immediate as follows: to get GCH up to $\gamma$ embed $\kappa$ up to $j(\kappa) > \gamma$ via $V \to V$.

For (2), using the preceding lemma and GCH we have $V_\kappa \prec_{\Sigma_1} V$. $\Sigma_2$ statements have the form $\exists \lambda\, (V_\lambda \vDash \phi[\vec{a}])$. Suppose such a statement is true in $V$. Since $\kappa$ is strong, we can get an elementary embedding we can throw $V \to V$ and then $\kappa$ above $V_\lambda^M$ and then reflect back to $V$. □

**Theorem 15.5**

There need not be an inaccessible above a strong cardinal.

*Proof.* Assume $\kappa$ is strong and $\kappa' > \kappa$ is the least inaccessible. Take $V_{\kappa'}$. We claim that $V_{\kappa'} \vDash$ "$\kappa$ strong". To show this we need to check that for all $\gamma$ there exists an extender $E \in V_{\kappa'}$ with critical point $\kappa$ such that $j_E(\kappa) > \gamma$ with strength at least $\gamma$.

Fix $\gamma$; let $j : V \to M$ give an extender $E$ of length $\left|V_\gamma^M\right|^M$. Inaccessibility shows $E \in V_{\kappa'}$. □

Indeed, "there are arbitrarily large measurables" is $\Pi_3$.

What if it's super-strong?

**Theorem 15.6**

Suppose $\kappa$ is superstrong. Then there are inaccessible above $\kappa$.

**Lemma 15.7**

The following are equivalent.

(1) $\kappa$ is superstrong.

(2) There is an extender $E$ such that the critical point of $E$ is $\kappa$ and the strength of $E$ is at least $j_E(\kappa)$.

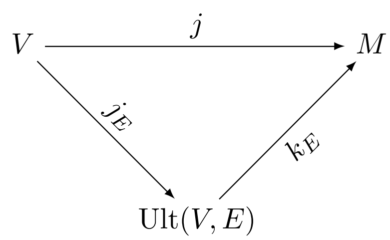
*Proof.* (2) $\implies$ (1) is immediate by definition.

For (1) $\implies$ (2), suppose $j : V \to M$ witnesses superstrength. Let $E$ be the $(\kappa, j(\kappa))$-extender derived from $j$.

By Agreement,

$$\forall \eta < j(\kappa) : V_\eta^{\mathrm{Ult}(V,E)} = V_\eta^M.$$

Thus $V_{j(\kappa)}^{\mathrm{Ult}(V,E)} = V_{j(\kappa)}^M$.




Upon showing $j_E(\kappa) = j(\kappa)$ we are done. Since $j(\kappa) = k_E(j_E(\kappa)) \ge j_E(\kappa)$. We cannot have $j_E(\kappa) < j(\kappa)$ since otherwise $k_E$ would be the identity on it ($k_E$ is the identity up to $j(\kappa)$) and chasing the diagram gives a contradiction. □

**Lemma 15.8**

The statement "$\kappa$ is superstrong" is $\Sigma_2$.

*Proof.* It expands as

$$\exists \gamma \exists \lambda \exists E \left( (\kappa < \lambda < \gamma) \wedge (E \text{ is a } (\kappa, \lambda)\text{-extender}) \wedge (\gamma = |V_\gamma|) \wedge (E \in V_\gamma) \wedge \dots \right).$$

□

**Lemma 15.9**

Suppose $\kappa$ is superstrong. Then for some $\gamma < \kappa$ such that

$$V_\gamma \vDash \mathrm{ZFC} + \exists \text{ strong}.$$

*Proof.* Exercise. □

But you can't guarantee that there's a real strong cardinal above a superstrong.

- Since superstrong is $\Sigma_2$ if there exists a strong cardinal above then there exists one below.
- If $\kappa$ is the least superstrong cardinal then there are no strong cardinals below $\kappa$.

# §16 April 7, 2015

Supercompact cardinals today.

## §16.1 Motivation

So far we have two formulations,

- elementary embeddings, and
- extenders.

Today we give more useful formulations for inner model theory:

- Normal fine ultrafilters on $\mathcal{P}_\kappa\lambda$ (we'll explain in a moment).
- Magidor extenders.

Here is some motivation. Recall that the existence of measurable implies $V \neq L$, and we have an "$L$-like model $L[U]$". If there exist two measurables, then $V \neq L[U]$. The pattern continues in inner model theory.

More generally, suppose $\Phi$ is a large cardinal axiom. One builds an "$L$-like" model $L_\Phi$ which can satisfy $\Phi$. One can consider the axiom

$$V = L_\Phi.$$

Examples of success in inner model theory include constructions for the following $\Phi$'s:

- There is a proper class of measurables.
- There is a strong cardinal.
- There is a Woodin cardinal.
- Current record: There is a Woodin cardinal which is a limit of Woodin cardinals.

The issue is that $\Phi_2$ is a stronger large cardinal axiom than $\Phi_1$, then

$$\Phi_2 \implies V \neq L_{\Phi_1}.$$

In fact $\Phi_2$ "humiliates" $L_{\Phi_1}$ in much the same sense that measurables "humiliate" $L$. In some sense, if there is a measuarble, then $L$ is some *tiny* fraction of $V$. So we build $L[U]$, but if there's another measurable then $L[U]$ is tiny.

So the phenomenon we observe is that every candidate axiom "$V = L_{\Phi_1}$" gets shot down by a stronger large cardinal axiom $V = L_{\Phi_2}$. It made it seem like large cardinals would be a long march; while we could understand large cardinals, we weren't making progress on $V$.

Remarkably , recent results of Woodin suggest that if we can get an "$L$-like model" for *one supercomact* then there is an "overflow" and all the various types of large cardinals manifest.

## §16.2 Ultrafilters on Supercompacts

We will look at models that are analogous to $L[U]$, only now $U$ is a "supercompactness measure", on $\mathcal{P}_\kappa(\lambda)$ (for $\kappa$ a $\lambda$-supercompact).

Recall that $\kappa$ is $\lambda$-supercompact if there is an elementary embedding $j : V \to M$ with critical point $\kappa$ such that $j(\kappa) > \lambda$ and $M$ is closed under $\lambda$-sequences.

**Definition 16.1.** Let $\kappa \le \lambda$ be cardinals. Then

$$\mathcal{P}_\kappa(\lambda) \overset{\text{def}}{=} \{X \subseteq \lambda \mid |X| < \kappa\}.$$

Before we just considered measures on $\kappa$; now we're bumping up to consider "small subsets of $\lambda$".

**Definition 16.2.** Suppose $U$ is an ultrafilter on $\mathcal{P}_\kappa(\lambda)$ (hence $U \subseteq \mathcal{P}_\kappa(\lambda)$ and has the usual properties). We say $U$ is **fine** if $\forall \alpha < \lambda$,

$$\{x \in \mathcal{P}_\kappa(\lambda) \mid \alpha \in x\} \in U.$$

We say $U$ is **normal** if for all $f : \mathcal{P}_\kappa(\lambda) \to \lambda$ such that

$$\{x \in \mathcal{P}_\kappa(\lambda) \mid f(x) \in x\} \in U$$

(meaning for most $x$, $f$ sends $x$ into itself; $f$ "presses down"; hence $f$ is *not* like sup) then $f$ is *constant* on a measure one set, meaning there is an $\alpha < \lambda$ such that

$$\{x \in \mathcal{P}_\kappa(\lambda) f(x) = \alpha\} \in U.$$

We are going to see that the ultrafilters derived from supercompacts satisfy these key properties.

**Theorem 16.3**

Suppose $\kappa \le \lambda$ are cardinals. Then $\kappa$ is $\lambda$-supercompact if an only if there's a $\kappa$-complete normal fine ultrafilter on $\mathcal{P}_\kappa\lambda$.

The problem with our old ultrafilters is that we can't get $\kappa^+$ closure. Supercompacts let us do this by bumping up the space.

**Remark 16.4.** "The right way to think about large cardinals is not by the measures and other concrete objects, but in terms of the closure the embeddings have. We're picking up first-order "shrapnel" from the embedding, and seeing if we can reconstruct the embedding from the data."

*Proof of* ($\implies$). Assume $\kappa$ is $\lambda$-supercompact and $j : V \to M$ be the witness. We derive the measure

$$U = \{X \subseteq \mathcal{P}_\kappa(\lambda) \mid j``\lambda \in j(X)\}$$

Because of our closure condition, we have $j``\lambda$ is in $M$, and

$$|j``\lambda| = \lambda < j(\kappa).$$

Moreover, $j``(\kappa) \subseteq j(\lambda)$ is clear (draw a picture). In other words, $j$ is shooting $X$'s... each $X \in \mathcal{P}_\kappa(\lambda)$ to a $j(X) \in \mathcal{P}_{j(\kappa)}(j(\lambda))$.

**Remark 16.5.** Compare this to the prior work with $U = \{x \subseteq \kappa \mid \kappa \in j(x)\}$ for measurable cardinals. The embedding $j$ is stretching our sets, so to decide whether $x$ was big, we wanted to see whether $j$ stretched it above the $\kappa$; in particular, $\kappa$ is big.

Exactly as before, we find that $U$ is $\kappa$-complete ultrafilter on $\mathcal{P}_\kappa(\lambda)$.

Next, let's check $U$ is fine. Observe that

$$\{x \in \mathcal{P}_\kappa(\lambda) \mid \alpha \in x\} \in U$$

if and only if

$$j``\lambda \in j\left(\{x \in \mathcal{P}_\kappa \lambda \mid \alpha \in x\}\right) = \{x \in \mathcal{P}_{j\kappa}(j\lambda) \mid j\alpha \in x\}.$$

So we just want $j\alpha \in j``\lambda$ for every $\alpha < \lambda$, which is duh.

Finally, let's check $U$ is normal. Fix $f : \mathcal{P}_\kappa(\lambda) \to \lambda$ such that

$$\{x \in \mathcal{P}_\kappa(\lambda) \mid f(x) \in x\} \in U$$

namely

$$j``\lambda \in j(\text{that}) = \{x \in \mathcal{P}_{j\kappa}(j\lambda) \mid (jf)x \in x\}$$

thus we are given

$$(jf)(j``\lambda) \in j``\lambda.$$

The thing we want is for some $\alpha < \lambda$,

$$\{x \in \mathcal{P}_\kappa \lambda \mid fx = \alpha\} \in U$$

*id est*

$$j``\lambda \in j(\text{that}) = \{x \in \mathcal{P}_{j\kappa}(j\lambda) \mid (jf)x = j(\alpha)\}$$

which is just All in all, we need to verify

$$(jf)(j``\lambda) \in j``\lambda \implies (jf)(j``\lambda) = j(\alpha).$$

This is *really* tautological. □

"Who knew to extract something like that? Answer: Reinhardt and Solovay."

*Proof of* ($\Leftarrow$). Suppose $U$ is a $\kappa$-complete normal fine ultrafilter on $_\kappa(\lambda)$ and let

$$j_U : V \to \text{Ult}(V, U)$$

be the ultrapower embedding. We have to show $j_U(\kappa) > \lambda$, the critical point is $\kappa$, and $\text{Ult}(V, U)$ is closed under $\lambda$-sequences.

This is again an analogue to what we did with normal measures. The point is

**Claim 16.6.** In the ultrapower construction, $j_U``\lambda = [\text{id}]_U$.

In the ultrapower construction, this mirrors $\kappa = [\text{id}]$.

*Proof.* First, we show $j_U``\lambda \subseteq [\text{id}]_U$. Suppose $[c_\alpha]_U = j_U\alpha \in j_U``\lambda$. By definition, $[c_\alpha]_U \in [\text{id}_U]$ exactly when

$$U \ni \{x \in \mathcal{P}_\kappa(\lambda) \mid c_\alpha(x) \in \text{id}(x)\} = \{x \in \mathcal{P}_\kappa(\lambda) \mid \alpha \in x\}$$

which is true since $U$ is fine.

Now, we show $[\mathrm{id}]_U \subseteq j``(\lambda)$. Suppose $f \in [\mathrm{id}_U]$, $f : \mathcal{P}_\kappa(\lambda)$, is in $[\mathrm{id}]_U$, which means

$$U \ni \{X \subseteq \mathcal{P}_\kappa(\lambda) \mid f(X) \in \mathrm{id}(X) = X\}.$$

By normality, this means for some $\alpha$ we have

$$U \ni \{X \subseteq \mathcal{P}_\kappa(\lambda) \mid f(X) = \alpha\}.$$

which reeds $[f] \in [c_\alpha]_U = j_U(\alpha)$. ■

First, we check $j_U(\kappa) > \lambda$. We have

$$\lambda = \mathrm{ordertype}(j_U``\lambda) = \mathrm{ordertype}([\mathrm{id}]) < [c_\kappa] = j_U(\kappa)$$

where the inequality follows by noting

$$\mathrm{ordertype}(\mathrm{id}X) < c_\kappa X = \kappa$$

holds measure one often (because it holds for *all* $X \in \mathcal{P}_\kappa\lambda$). Here we're using that order type is definable and then Łoś.

Next, to show that $\textsc{crt}(j_U) = \kappa$, we observe first that $j_U\alpha = \alpha$ for all $\alpha$ by $\kappa$-completeness. Moreover, $j_U(\kappa) > \lambda > \kappa$ by the previous condition. Done.

Finally, we wish to get closure for $\mathrm{Ult}(V,U)$ under $\lambda$-sequences. Suppose we have a sequence

$$\langle [f_\alpha]_U : \alpha < \lambda \rangle.$$

Let $g : \mathcal{P}_\kappa(\lambda) \to V$ by

$$x \mapsto \langle f_\alpha(x) : \alpha < \lambda \rangle.$$

Hence, for $\alpha < \lambda$ we have

$$[g](j_U(\alpha)) = [g](c_\alpha) = f_\alpha$$

since by Łoś, this holds if and only if

$$\{x \in \mathcal{P}_\kappa(\lambda) \mid (gx)(c_\alpha x) = f_\alpha(x)\} \in U$$

but now $c_\alpha x = \alpha$, so the $x$ which are in here are all of $\mathcal{P}_\kappa(\lambda)$. So we have $[g] \in \mathrm{Ult}(V,U)$ and

$$j_U``\lambda = [id] = \mathrm{Ult}(V,U)$$

so

$$\langle [f_\alpha] : \alpha < \lambda \rangle = \langle [g](j_U\alpha) : j_U\alpha \in j_U``\lambda \rangle \in \mathrm{Ult}(V,U).$$ □

**Corollary 16.7**

The following are equivalent.

(1) $\kappa$ is supercompact.

(2) For all $\lambda$ there is a $\kappa$-complete normal fine ultrafilter $U_\lambda$ on $\mathcal{P}_\kappa(\lambda)$.

## §16.3 Next time

Next time we will show a different formulation of supercompact, due to Magidor:

> A cardinal $\kappa$ is supercompact if and only if for every $\lambda > \kappa$ we can exhibit an elementary embedding
> $$j : V_{\overline{\lambda}} \to V_\lambda$$
> such that $\overline{\lambda} < \kappa$ and
> $$j(\textsc{crt}(j)) = \kappa.$$

This gives a lot of reflection across $\kappa$, governed by elementary embeddings.

# §17 April 9, 2015

Extra class is Friday, at 2 Arrow Street. Meet at room 414 at 2:15PM.

(Class starts by making an amendment to the section last time, repairing the proof of $\lambda$-closure.)

As a corollary, by pushing to extenders (i.e. all $\lambda \geq \kappa$)...

**Theorem 17.1**

Suppose $\kappa$ is a cardinal. The following are equivalent.

(1) For all $\lambda \geq \kappa$, there exists a $\kappa$-complete normal fine ultrafilter on $\mathcal{P}_\kappa(\lambda)$.

(2) For all $\lambda \geq \kappa$, there's an extender $E$ such that $\textsc{crt}(E) = \kappa$, $j_E(\kappa) > \lambda$, and $E$ has closure $\geq \lambda$.

By the closure theorem, supports of $E$ in (2) above get arbitrarily large. But inner models of supercompacts and extenders with longer and longer supports are harder to manage. So we would like a formulation with bounded supports (below $\kappa$). (The extenders are still, but...)

## §17.1 Magidor's Formulation

**Theorem 17.2** (Magidor)

The following are equivalent.

(1) $\kappa$ is supercompact.

(2) For all $\lambda \geq \kappa$, there is an elementary embedding

$$j : V_{\overline{\lambda}} \to V_\lambda$$

for some $\overline{\lambda} < \kappa$, such that

$$j(\textsc{crt}(j)) = \kappa.$$

This gives a lot of Reflection.

*Proof of* $\Rightarrow$. Suppose $\kappa$ is supercompact, and select $\lambda \geq \kappa$. Let

$$j : V \to M$$

witness that $\kappa$ is $|V_\lambda|$-supercompact (huge!). Thus $M$ is closed under $|V_\lambda|$-sequences.

In particular, the restriction of $j$ to $V_\lambda$ is *in* $M$ (which is why we needed so much closure). So $M$ thinks

> "There exists a $\lambda' < j(\kappa)$ and elementary embedding
>
> $$\tilde{\jmath} : V_{\lambda'} \to V_{j(\lambda)}$$
>
> such that $\tilde{\jmath}(\textsc{crt}(\tilde{j})) = j(\kappa)$".

as witnessed by $\lambda' = \lambda$, and $\tilde{\jmath} = j|_{V_\lambda}$. By elementarity, $V$ now thinks

There exists a $\overline{\lambda} < \kappa$ and

$$\tilde{\jmath} : V_{\overline{\lambda}} \to V_\lambda$$

such that $\tilde{\jmath}(\text{CRT}(\tilde{\jmath})) = \kappa$.

That's what we want. If this isn't clear, draw a picture. □

*Proof of* $\Leftarrow$. Fix $\lambda > \kappa$. By assumption, we may take

$$j : V_{\overline{\lambda}+2015} \to V_{\lambda+2015}$$

be an elementary embedding such that $\overline{\lambda} + 2015 < \kappa$ and

$$j(\text{CRT}(j)) = \kappa.$$

The "+2015" gives us some space.

Let $\overline{\kappa} = \text{CRT}(j)$. We want to pull out a $\kappa$-complete normal fine ultrafilter on $\mathcal{P}_\kappa(\lambda)$. We will do this in the shrunk universe $V_{\overline{\lambda}+2015}$ and then use $j$ to shoot it up.

Observe that

$$\mathcal{P}\mathcal{P}_{\overline{\kappa}}(\overline{\lambda}) \in V_{\overline{\lambda}+2015}$$

and

$$\overline{U} = \left\{ X \subseteq \mathcal{P}_{\overline{\kappa}}(\overline{\lambda}) \mid j``\overline{\lambda} \in j(X) \right\}$$

is a $\overline{\kappa}$-complete normal fine ultrafilter on $\mathcal{P}_\kappa(\lambda)$.. (The argument from before goes through because "+2015".) Then

$$U \overset{\text{def}}{=} j(\overline{U})$$

is a $j(\overline{\kappa}) = \kappa$-complete normal fine ultrafilter on $\mathcal{P}_{j\overline{\kappa}}(j\overline{\lambda}) = \mathcal{P}_\kappa(\lambda)$. □

There is an extender formulation of this.

**Theorem 17.3** (Magidor)
The following are equivalent.

(1) $\kappa$ is supercompact.

(2) For $\lambda \ge \kappa$ there is an extender $E$ such that the support of $E$ is bounded by $\kappa$, the strength of $E$ exceeds $\lambda$, and

$$j_E(\text{CRT}(E)) = \kappa.$$

*Proof.* The proof of (2) $\implies$ (1) is the same as before; generate the measure downstairs and shoot it above.

Suppose $\kappa$ is supercompact. Assume WLOG that $\lambda = |V_\lambda|$.

Sinec $\kappa$ is supercompact, we can take $j : V \to M$ such that $\text{CRT}(j) = \kappa$, $j(\kappa) > \lambda$, and $M$ has $|V_{\lambda+1}|$-closure (huge)! Let $E$ be the $(\kappa, j\lambda)$-extender derived from $j$. The supports of these $E$ is $\le \lambda + 1 \le j(\kappa)$.

Well we have

$$V^M_{\text{support}(E)+1} = V^V_{\text{support}(E)+1}$$

and since $\text{support}(E) \le \lambda + 1$, our closure gives us the entire extender $E$ is in $M$.

Now we use the Agreement theorem on extenders (Theorem 13.7). Hence $M$ thinks $E$ is a $(\kappa, j(\lambda))$-extender. Thus

- $j_E^M$ agrees with $j_E$ on $\lambda^+$.
- $V_{j_E(\lambda)+1}^{\mathrm{Ult}(M,E)} = V_{j_E(\lambda)+1}^{\mathrm{Ult}(V,E)}$.

But

$$j\lambda = j(|V_\lambda|) = \left|V_{j\lambda}^M\right|^M.$$

This means

$$V_{j(\lambda)}^{\mathrm{Ult}(V,E)} = V_{j\lambda}^M.$$

Putting everything together, we have

$$V_{j_E\lambda}^{\mathrm{Ult}(M,E)} = V_{j_E\lambda}^M.$$

Thus $M$ thinks

> "There is an extender $E$ such that support$(E) < j(\kappa)$ with strength $\ge j(\lambda)$ and $j_E(\textsc{crt}(j_E)) = j(\kappa)$".

Applying elementarity, $V$ thinks exactly what we want it to think mwahahaha. □

## §17.2 One Last Remark on Extenders

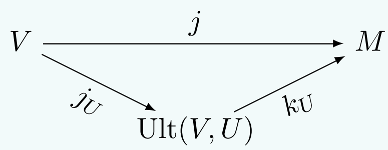
Seed representation for extenders.

**Theorem 17.4**

Suppose $j : V \to M$ witnesses that $\kappa$ is $\lambda$-supercompact, and $U$ is the derived measure on $P_\kappa\lambda$. Let

$$k_U : \mathrm{Ult}(V,U) \to M \text{ by } [f] \mapsto (jf)(j``\lambda).$$

Then $k_U$ is an elementary embedding and the diagram




commutes. Moreover, $k_U|_\gamma$ is the identity, where

$$\gamma = \left(|\mathcal{P}(\mathcal{P}_\kappa\lambda)|^+\right)^{\mathrm{Ult}(V,U)}.$$

## §17.3 Conclusions

Hence we've seen there are a ton of ways to formulate supercompact. We will in the future focus on two formulations:

(1) Normal fine $\kappa$-complete ultrafilters on $\mathcal{P}_\kappa\lambda$

(2) Magidor's extenders.

The question is

> We have $L[U]$ for $U$ witnessing $\kappa$ measurable. Is there an analog for supercompacts?

Next chapter: *weak extender models* for supercompactness. We'll introduce two notions for this, using normal fine ultrafilters then Magidor's extenders, in that order.

Some motivation for all this: let $\kappa$ be measurable with ultrafilter $U$. Form $L[U]$. Then

(1) (Concentration) $\kappa \cap L[U] \in U$ (since in fact $\kappa \cap L[U] = \kappa$; so this is trivial).

(2) (Inheritance) $U \cap L[U] \in L[U]$.

So $L[U]$ thinks $\kappa$ is a measurable cardinal as witnessed by by the measure $U \cap L[U]$. Thus the witness in $L[U]$ is *inherited* for $V$.

We try to get an analog now:

**Definition 17.5.** Suppose $N$ is an inner model of ZFC containing the ordinals. Then $N$ is a **weak extender model for the supercompactness of $\kappa$** if $\forall \lambda \geq \kappa$ there is a $\kappa$-complete normal fine ultrafilter $U$ on $\mathcal{P}_\kappa \lambda$ such that

(1) (Concentration) $\mathcal{P}_\kappa \lambda \cap N \in U$.

(2) (Inheritance) $U \cap N \in N$.

Note that concentration is no longer trivial.

Recall that $L[U]$ has exactly one measurable in it, so if $V$ has more than one measurable then $L[U]$ is completely mistaken about the nature of $V$. But weak extender models will not have this issue: Solovay's Theorem will tell us that *any* such model is "close to $V$". In particular, $N = V$ works as well; but we want $N$ to be "$L$-like". So we search through these type of models for ones that are $L$-like.

# §18 April 10, 2015

Make-up class.

## §18.1 Solovay's Theorem

**Theorem 18.1** (Solovay)
Suppose $U$ is a normal fine $\kappa$-complete ultrafilter on $\mathcal{P}_\kappa(\lambda)$ and $\lambda > \kappa$ is regular. Then there exists $X \in U$ such that the sup function

$$\sup : \mathcal{P}_\kappa(\lambda) \to \lambda \quad \text{by} \quad \sigma \mapsto \sup \sigma$$

is one-to-one on $X$. Moreover, $X$ is independent of $U$.

In other words, there are many (in the sense of $U$) sets with different sup's.

*Proof.* Just like the proof of Kunen's Theorem.

Let $\langle S_\alpha \mid \alpha < \lambda \rangle$ be a partition of

$$S^\lambda_\omega \overset{\text{def}}{=} \{\alpha < \lambda \mid \operatorname{cof} \alpha = \omega\}$$

into stationary sets.

For $\beta < \lambda$ such that $\omega < \operatorname{cof}(\beta) < \kappa$, set

$$\sigma_\beta = \{\alpha < \beta \mid S_\alpha \cap \beta \text{ stationary in } \beta\}.$$

We leave $\sigma_\beta$ undefined otherwise.

**Claim 18.2.** For $\beta$ with $\omega < \operatorname{cof} \beta < \kappa$, we have $\sigma_\beta \in \mathcal{P}_\kappa \lambda$.

*Proof.* Notice that

$$\langle S_\alpha \cap \beta : \alpha \in \sigma_\beta \rangle$$

is a partition of $\beta$ into stationary sets (since the $S_\alpha$ are a partition). If $|\sigma_\beta| \ge \kappa$, we get a contradiction as we've partitioned $\beta$ into $\ge \kappa$ many sets. Yet there is a club $C$ in $\beta$ such that $\operatorname{ordtype} C = \operatorname{cof} \beta$, which is impossible since each stationary $S_\alpha \cap \beta$ would have to hit $C$ at a different point. (The assumption $\operatorname{cof} \beta > \omega$ lets us make sense of clubs in the first place.) ■

Now we can set

$$X = \{\sigma_\beta \in \mathcal{P}_\kappa(\lambda) \mid \sup \sigma_\beta = \beta\}.$$

Clearly sup is injective on $X$, and moreover $X$ does not depend on $U$ since it was defined only on the partition.

So it remains to see $X$ is in $U$. Let $j_U : V \to \operatorname{Ult}(V, U)$ be the ultrapower embedding, Set

$$\langle T_\alpha : \alpha < j_U(\lambda) \rangle = j_U(\langle S_\alpha : \alpha < \lambda \rangle).$$

(Note that the former sequence is much longer than the latter.) We will show that $j_U``\lambda$, a $\lambda$-sequence contained inside the ultrapower, is in fact *definable* from (i) $\langle T_\alpha : \alpha < j_U(\lambda) \rangle$ and (ii) $\lambda^* \overset{\text{def}}{=} \sup j_U``\lambda$.

We have by the usual definition that

$$j_U``\lambda \in j_U(x).$$

We can compute

$$j_U(x) = \left\{\sigma_\beta \in \mathcal{P}_{j_U(\kappa)}(j_U\lambda) \mid \text{Ult}(V,U) \models \text{“}\sup\sigma_\beta = \beta\text{”}\right\}$$

where

$$\sigma_\beta = \{\alpha < \beta \mid T_\alpha \cap \beta \text{ stationary in } \beta\}.$$

Thus we are done if we can show in the ultrapower that

**Claim 18.3.**

$$j_U\text{“}\lambda = \sigma_{\lambda^*}.$$

(Certainly $j_U\text{“}\lambda \in \mathcal{P}_{j_U(\kappa)}j_U(\lambda)$ since $\lambda < j_U(\kappa)$.)

*Proof.* First, we want to show $j_U(\alpha) \in \sigma_{\lambda^*}$ for any $\alpha < \lambda$, meaning that

$$T_{j_U(\alpha)} \cap \lambda^* = j_\alpha(S_\alpha) \cap \lambda^*$$

is stationary in $\lambda^*$. Suppose $C \in \text{Ult}(V,U)$ be club in $\lambda^*$. Let

$$D = \{\alpha < \lambda \mid j_U(\alpha) \in C\} \subseteq V.$$

Since $j_U\text{“}\lambda$ is an $\omega$-club, so is the set $D$ (in $V$). Thus there exists $\xi$ such that $\xi \in D \cap S_\alpha$, meaning

$$j_U(\xi) \in C \cap S_{j_U}(\alpha) = T_{j_U}(\alpha)$$

and thus $j_U(\alpha) \in \sigma_{\lambda^*}$.

Conversely, we wish to show that $\sigma^*_\lambda \subseteq j_U\text{“}\lambda$. Suppose $\alpha \in \sigma_{\lambda^*}$, which means

$$\text{Ult}(V,U) \models \text{“}T_\alpha \cap \lambda \text{ is stationary in } \lambda^*\text{”}.$$

We wish to show $\alpha = j_U(\overline{\alpha})$ for some $\overline{\alpha}$. But $j_U\text{“}\lambda$ is an $\omega$-club in $\text{Ult}(V,U)$ (we're just repeating the proof downstairs from before, but now upstairs). So $\exists \xi < \lambda$ such that

$$j_U(\xi) \in T_\alpha \cap j_U\text{“}\lambda.$$

Now,

$$\langle j_U\text{“}S_{\overline{\alpha}} \cap \lambda^* : \overline{\alpha} < \lambda\rangle$$

is a partition of $j_U\text{“}\lambda$. Let $\overline{\alpha} < \lambda$ be such that

$$j_U(\xi) \in j_U\text{“}S_{\overline{\alpha}} \cap \lambda^*.$$

Hence

$$j_U(\xi) \in j_U\text{“}S_{\overline{\alpha}} \subseteq j_U(S_{\overline{\alpha}}) = T_{j_U(\overline{\alpha})}$$

and $\alpha = j_U(\overline{\alpha})$. ■

This completes the proof of the theorem. □

From this we can extract the following.

**Theorem 18.4**

Suppose $\kappa$ is a $\lambda$-supercompact. Suppose $j : V \to M$ is the associated embedding. Suppose $\nu \le \lambda$ is a regular cardinal, and set

$$\nu^* \overset{\text{def}}{=} \sup j``\nu.$$

Let $\langle S_\alpha : \alpha < \nu \rangle$ be a partition of $S^\nu_\omega$ as before, and let $\langle T_\alpha : \alpha < j(\nu) \rangle$ be the $j$-image of the sequence. Then

$$j``\nu = \{\alpha < \nu^\kappa \mid T_\alpha \cap \nu^* \text{ stationary in } \nu^*\}.$$

For Solovay's Lemma, we applied this with $\lambda$ regular, which gave the extra fact that $X \in U$ if and only if $j_U``\lambda \in j_U(X)$ (we took $X$ to be the $j``\nu$ computed downstairs).

## §18.2 Weak Extender Models

Suppose $N$ is an inner model of ZFC. Recall that $N$ is a *weak extender model for the supercompactness of* $\kappa$ if $\forall \lambda \ge \kappa$ there is a $\kappa$-complete normal fine ultrafilter $U$ on $\mathcal{P}_\kappa\lambda$ such that

(1) (Concentration) $\mathcal{P}_\kappa\lambda \cap N \in U$.

(2) (Inheritance) $U \cap N \in N$.

**Lemma 18.5**

Suppose $N$ is an inner model of ZFC. Suppose that $\forall \lambda \ge \kappa$ there is a $\kappa$-complete normal fine ultrafilter on $U$ such that $\mathcal{P}_\kappa\lambda$ satisfying Concentration ($\mathcal{P}_\kappa\lambda \cap N \in U$). Then $N$ has the **$\kappa$-cover property**, meaning that if $\tau \subseteq N$ has size $< \kappa$, then in fact there is a set $\tau' \in N$ which is *in* $N$, size less than $\kappa$, such that $\tau \subseteq \tau'$.

The proof of this will be straightforward; this gives a "closeness by covering" situation.

If $N$ also satisfies inheritance (hence is a weak extender model), we will see that $N$ correctly computes singular cardinals in $V$ above $\kappa$.

Later on we'll see a third condition which means we're *very* close: given a large cardinal $\kappa$, if $\kappa + \varepsilon$ is in $V$ then $\kappa$ is in $N$.

# §19 April 14, 2015

Suppose $N$ is a transitive inner model of ZFC containing the ordinals. Recall that $N$ is a *weak extender model for the supercompactness of* $\kappa$ if $\forall \lambda \ge \kappa$ there is a $\kappa$-complete normal fine ultrafilter $U$ on $\mathcal{P}_\kappa \lambda$ such that

(1) (Concentration) $\mathcal{P}_\kappa \lambda \cap N \in U$.

(2) (Inheritance) $U \cap N \in N$.

Recall also that $N$ has the *$\kappa$-cover property*, meaning that if $\tau \subseteq N$ has size $< \kappa$, then in fact there is a set $\tau' \in N$ which is *in* $N$, size less than $\kappa$, such that $\tau \subseteq \tau'$.

## §19.1 The $\kappa$-covering property

If $N = L[U]$, then concentration is immediate, as

$$\kappa \cap L[U] = \kappa \in U.$$

For $U \subseteq \mathcal{P}_\kappa(\lambda)$ this is not trivial, as we're about to see. We prove the lemma from last time:

**Lemma 19.1**

Suppose $N$ is an inner model of ZFC and $\kappa$ is such that for all $\lambda \ge \kappa$, there is a normal fine ultrafilter on $\mathcal{P}_\kappa \lambda$ such that $\mathcal{P}_\kappa \lambda \cap N \in U$. Then $N$ has the $\kappa$-covering property.

Thus $N$ is close to $V$ in the sense that it is really fat.

*Proof.* By coding, it suffices to verify the result for $\tau \subseteq \mathrm{On}$. Let $\lambda > \sup \tau$, so $\tau \subseteq \lambda$; hen let $U$ be the filter on $\mathcal{P}_\kappa \lambda$ specified in the condition.

Since $U$ is fine, we have that for each $\alpha < \lambda$,

$$A_\alpha \overset{\text{def}}{=} \{\sigma \in \mathcal{P}_\kappa \lambda \mid \alpha \in \sigma\} \in U.$$

Thus by completeness,

$$A_\tau \overset{\text{def}}{=} \bigcap_{\alpha \in \tau} A_\alpha = \{\sigma \in \mathcal{P}_\kappa \lambda \mid \tau \subseteq \sigma\} \in U.$$

Also, $\mathcal{P}_\kappa \tau \cap N \in U$. Thus, as $U$ is an ultrafilter,

$$U \ni A_\tau \cap (\mathcal{P}_\kappa \lambda \cap N)$$

is nonempty. Then any $\tau'$ in this set works, since it extends $\tau$, has size less than $\kappa$, and is in $N$. $\square$

## §19.2 Amenability / Inheritance

Now we include the second condition that $U \cap N \in N$.

**Lemma 19.2**

Suppose $N$ is a weak extender model for $\kappa$, and $\lambda \geq \kappa$ is a regular cardinal in $N$. Then

$$\operatorname{cof} \lambda = |\lambda|$$

as computed in $V$.

*Proof.* This is where the Solovay lemma gets used, but it gets used in $N$.

Let $U$ be a $\kappa$-complete normal fine ultrafilter satisfying Concentration and Inheritance, so $\mathcal{P}_\kappa \lambda \cap N \in U$ and $U \cap N \in U$. So we can now apply Solovay's Theorem *inside* $N$, to obtain $X \in U \cap N$ such that

$$\sup : \mathcal{P}_\kappa \lambda \in \lambda \text{ by } \sigma \mapsto \sup \gamma$$

is injective on $X$.

Note that $\operatorname{cof}(\lambda) \geq \kappa$ as computed in $V$, because the lemma implies that for all $\sigma \in \mathcal{P}_\kappa \lambda$, there is a covering $\tau$.

Let $C \subseteq \lambda$ be a club, $\operatorname{ordtype}(C) = \operatorname{cof}(\lambda)$. Let

$$X_C = \{\sigma \in X \mid \sup \sigma \in C\}.$$

We claim that $X_C \in U$. Indeed, this occurs if and only if $j_U``\lambda \in j_U(X_C)$, which is true if and only if

$$j_U``\lambda \in \{\sigma \in j_U(X) \mid \sup \sigma \in j_U(C)\}.$$

which must be true since $j_U``C \subseteq j_U(C)$, and $j_U(C)$ is a club. Then $\sup j_U``\lambda = \sup j_U``C$, and so the supremum is in $j_U(C)$.

Now,

$$|X_C| \leq \kappa \cdot |C| = \kappa \cdot \operatorname{cof}(\lambda) = \operatorname{cof}(\lambda)$$

(since $\operatorname{cof}(\lambda) \geq \kappa$). But by fine-ness of $U$, we have

$$\bigcup X_C = \lambda.$$

Thus

$$\kappa \leq |\lambda| \leq |X_C| \cdot \kappa = \operatorname{cof}(\lambda) \cdot \kappa.$$

By the rules of cardinal arithmetic, $|\lambda| = \operatorname{cof}(\lambda)$. □

**Corollary 19.3**

Let $N$, $\kappa$, etc. be as above and suppose $\gamma > \kappa$ is a singular cardinal in $V$. Then

(1) $\gamma$ is singular in $N$, and,

(2) $\gamma^+$ is correctly computed in $N$, meaning $(\gamma^+)^N = \gamma^+$.

Thus $N$ is close to $V$ in a cardinal computation sense.

*Proof.* Assume towards contradiction that $\gamma$ is regular in $N$. Then

$$\gamma < \operatorname{cof}(\gamma) = |\gamma| = \gamma$$

where the first inequality follows since $\gamma$ is singular, and the second equality occurs since $\gamma$ is a cardinal in $V$.

For the second part, assume for contradiction that

$$(\gamma^+)^N < (\gamma^+)^V.$$

Let $\lambda$ be the result computed in $N$, so $\lambda$ is a regular cardinal in $N$ (since it's a successor). So by the lemma,

$$\operatorname{cof}^V(\lambda) = |\lambda|^V = \gamma.$$

But $|\lambda| = \gamma$, meaning $\operatorname{cof}(\lambda) < \gamma$, which is a contradiction. □

## §19.3 Summary

Weak extender models are "close to $V$" above $\kappa$. This leads to the expectation that if $N$ is a generalization of $L[U]$ for small cardinals, then $N$ should contain all large cardinals, since as we saw earlier missing large cardinals obliterate lower models.

We will soon prove

**Theorem 19.4** (Universality)

Suppose $N$ is a weak extender model for the supercompactness of $\kappa$. Let $\gamma > \kappa$ be such that

$$j : N \cap V_{\gamma+1} \to N \cap V_{j(\gamma)+1}$$

is an elementary embedding with $\textsc{crt}(j) \ge \kappa$. Then $j \in N$.

**Corollary 19.5**

Suppose $N$ is a weak extender model for the supercompactness of $\kappa$. Then there is no $j : N \to N$ with $\textsc{crt}(j) \ge \kappa$.

Then we will focus on HOD. They are quite different; you can change HOD by forcing, but not $L$. One can code arbitrary sets...

> "Suppose you had an infinite Social Security Number..."

Thus everything is definable from ordinals.

We can have any large cardinal in HOD that we like. In fact,

**Lemma 19.6**

Suppose $X$ is ordinal-definable. Then $L[X] \subseteq \mathrm{HOD}$.

*Proof.* $X$ is ordinal-definable, so $Y \overset{\text{def}}{=} X \cap \mathrm{HOD} \in \mathrm{HOD}$, so $L[X] = L[Y]$ is in HOD. □

**Corollary 19.7**

Suppose $U$ is a $\kappa$-complete normal uniform ultrafilter on $\kappa$. Then $L[U] \subseteq \mathrm{HOD}$.

Thus the model we built before is in HOD.

*Proof.* Let $\overline{U} = U \cap L[U]$. Then $\overline{U}$ is ordinal definable, since for any $W$ a $\kappa$-complete normal uniform ultrafilter we have

$$W \cap L[W] = U \cap L[U].$$

Thus $\overline{U} = U \cap L[U]$ is ordinal definale. Hence $L[U] = L[\overline{U}] \subseteq \text{HOD}$. $\square$

This leads to the expectation that (assuming large cardinals) there is a weak extender model $N$ for $\kappa$ such that $N \subseteq \text{HOD}$. So now we want to see if HOD itself is close to $V$, which will lead to the HOD Dichotomoy Theorem.

# §20 April 16, 2015

## §20.1 Lemma

**Lemma 20.1**

Suppose $N$ is a proper class model of ZFC and $U$ is a $\kappa$-complete normal fine ultrafilter on $\mathcal{P}_\kappa\lambda$, where

$$\lambda = |N \cap V_\lambda|^N$$

which satisfies $\mathcal{P}_\kappa\lambda \cap N \in U$ and $U \cap N \in N$. Then if

$$j_U : V \to \text{Ult}(V, U)$$

is the ultrapower embedding, we have

$$j_U(N \cap V_\kappa) \cap V_\lambda = N \cap V_\lambda.$$

Hence the ultrapower in $V$ is somehow preserving the entire big chunk $N \cap V_\kappa$

*Proof.* First we check $j_U(N \cap V_\kappa) \cap V_\lambda \supseteq N \cap V_\lambda$. Recall that $X \in U$ if and only if $j_U``\lambda \in j_U(X)$. So, since

$$\mathcal{P}_\kappa(\lambda) \cap N \in U$$

we have

$$j_U``\lambda \in j_U(\mathcal{P}_\kappa(\lambda) \cap N) \subseteq j_U(N).$$

It follows that $j_U``(N \cap V_\lambda) \in j_U(N)$, since we're given a bijection

$$(e : \lambda \to N \cap V_\lambda) \in N$$

thus giving

$$j_U``(N \cap V_\lambda) = j_U(e)``(j_U``\lambda) \in j_U(N).$$

(Fix $x \in N \cap V_\lambda$, and let $\alpha < \lambda$ be such that $e(\alpha) = x$, then $j_U(x) = j_U(e(\alpha)) = j_U(e)j_U(\alpha)$.)

But $N \cap V_\lambda$ is the transitive collapse of $j_U``(N \cap V_\lambda)$, thus

$$N \cap V_\lambda = \text{TransCollapse}(j_U(e)``(j_U``\lambda)).$$

Hence $N \cap V_\lambda \subseteq j_U(N)$. Hence $N \cap V_\lambda \subseteq j_U(N \cap V_\kappa) \cap V_\lambda$.

For the other direction we have to show $j_U(N \cap V_\kappa) \cap V_\lambda \subseteq N \cap V_\lambda$. Using

$$N \cap V_\lambda = \text{TransCollapse}(j_U(e)``(j_U``\lambda))$$

it suffices to show that

$$j_U(N \cap V_\kappa) \cap V_\lambda = \text{TransCollapse}(j_U(e)``(j_U``\lambda)).$$

(At this point there's a brief interjection about the typography of “. Apparently some people who don't know LaTeX do *ZFC* in their code.)

Let

$$X = \left\{\sigma \in \mathcal{P}_\kappa\lambda \mid e``(\sigma) \prec N \cap V_\lambda \text{ and } \text{TransCollapse}(e``\sigma) = N \cap V_{\text{ordtype}(\sigma)}\right\}.$$

Notice that

$$j_U(X) = \big\{\sigma \in \mathcal{P}_{j_U(\kappa)}(j_U(\lambda)) \mid (j_U(e))``(\sigma) \prec j_U(N) \cap V_\lambda$$
$$\text{and TransCollapse}(j_U(e)``\sigma) = N \cap V_{\text{ordtype}(\sigma)}\big\}.$$

If $j_U``\lambda \in j_U(X)$ then

$$\text{TransCollapse}((j_U(e))``(j_U``\lambda)) = j_U(N) \cap V_\lambda = j_U(N \cap V_\kappa) \cap V_\lambda.$$

So it suffices to show

$$j_U``\lambda \in j_U(X)$$

or equivalently

$$X \in U.$$

Let $\overline{X} = X \cap N$; we need to show $\overline{X} \in U$. Since $\overline{U} = U \cap N$ is a $\kappa$-complete normal fine ultrafilter on $\mathcal{P}_\kappa\lambda \cap N$. Thus we have

$$j_{\overline{U}} : N \to \text{Ult}(N, \overline{U}).$$

Notice that

(1) $(j_{\overline{U}}(e))``(j_{\overline{U}}``\lambda) \prec j_{\overline{U}}(N) \cap V_{j_{\overline{U}}\lambda}$.

(2) The transitive collapse of $(j_{\overline{U}}(e))``(j_{\overline{U}}``\lambda)$ is $N \cap V_\lambda = N \cap V_{\text{ordtype}(j_U``\lambda)}$.

(3) $\text{Ult}(N, \overline{U}) \cap V_\lambda = N \cap V_\lambda$.

In other words, $j_{\overline{U}}``\lambda \in j_{\overline{U}}(X \cap N)$.

The point is that since the internal ultrapower weak extender model doesn't "overshoot" we have

$$\text{TransCollapse}(j_{\overline{U}}(e)``(j_U``\lambda)) = j_{\overline{U}}(N) \cap V_\lambda = \text{Ult}(N, \overline{U}) \cap V_\lambda.$$

Thus $X \cap N \in U$, and so $X \in U$. □

## §20.2 Magidor Formulations

We now prove a souped-up version of Magidor's result.

**Theorem 20.2**

Let $N$ be a proper class model of ZFC. The following are equivalent.

(1) $N$ is a weak extender model for the supercompactness of $\kappa$.

(2) $\forall \lambda > \kappa \forall a \in V_\lambda \exists \overline{\kappa} < \overline{\lambda} < \kappa \exists \overline{a} \in V_{\overline{\lambda}} \exists j : V_{\overline{\lambda}+1} \to V_{\lambda+1}$ such that $\textsc{crt}(j) = \overline{\kappa}$, $j(\overline{\kappa}) = \kappa$, $j(\overline{a}) = a$ and $j_U(N \cap V_{\overline{\lambda}}) = N \cap V_\lambda$, and $j \restriction (N \cap V_{\overline{\lambda}+1}) \in N$.


Picture of things living below other things. $\overline{*}$ should all be below $\overline{\kappa}$ on the left.

*Proof that (2) $\implies$ (1).* First, assume the second. Fix $\gamma > \kappa$ such that $\gamma = |V_\gamma|$. Apply (2) (ignoring $a$) to $\lambda \overset{\text{def}}{=} \gamma + \omega$ to get $\overline{\kappa} < \overline{\lambda} < \kappa$ and $j : V_{\overline{\lambda}+1} \to V_{\lambda+1}$ with the requested properties.

Since $\gamma$ is definable in $V_{\lambda+1}$, we may let $\overline{\gamma}$ have the same definition in $V_{\overline{\lambda}+1}$, so

$$j(\overline{\gamma}) = \gamma.$$

Also, we have $j``\overline{\gamma} \in V_{\lambda+1}$.

Now we consider the measure generated downstairs. Let

$$\overline{U} = \left\{ X \in \mathcal{P}_{\overline{\kappa}}\overline{\lambda} \mid j``\overline{\gamma} \in j(X) \right\}$$

and let $U = j(\overline{U})$ As before, we have that $\overline{U}$ is a $\overline{\kappa}$-complete normal fine ultrafilter on $\mathcal{P}_{\overline{\kappa}}(\overline{\lambda})$ and so, by elementarity, $U$ is a $\kappa$-complete normal fine ultrafilter on $\mathcal{P}_\kappa\lambda$. But now, using the additional properties

(a) $j(N \cap V_{\overline{\lambda}}) = N \cap V_\lambda$, and

(b) $j \restriction (N \cap V_{\overline{\lambda}+1}) \in N$.

we want to get

(a') $\mathcal{P}_\kappa\gamma \cap n \in U$.

(b') $U \cap N \in N$.

For (a'), we just need

**Claim 20.3.** $\mathcal{P}_{\overline{\kappa}}(\overline{\gamma}) \cap N \in \overline{U}$.

*Proof.* By (a), $j(N \cap V_{\overline{\lambda}}) = N \cap V_\lambda$. So for all $\overline{a} \in V_{\overline{\lambda}}$, we have

$$j(a \cap N) = j(\overline{a}) \cap j(N) = j(\overline{a}) \cap N.$$

In particular, since $\mathcal{P}_{\overline{\kappa}}(\gamma) \in V_{\overline{\lambda}}$, we have that

$$j(\mathcal{P}_{\overline{\kappa}}(\overline{\gamma}) \cap N) = \mathcal{P}_\kappa\gamma \cap N.$$

By (b),

$$j``\overline{\gamma} \in N.$$

Thus

$$j``\overline{\gamma} \in \mathcal{P}_\kappa(\gamma) \cap N = j(\mathcal{P}_{\overline{\kappa}}(\overline{\gamma}) \cap N)$$

and so, by the definition of $\overline{U}$, we have

$$\mathcal{P}_{\overline{\kappa}}(\overline{\gamma}) \cap N \in \overline{U}. \qquad \blacksquare$$

It follows that

$$\mathcal{P}_\kappa(\gamma) \cap N = j\left(P_{\overline{\kappa}}(\overline{\gamma}) \cap N\right) \in j(\overline{U}) = U$$

which gives (a').

For (b'), we have from (b) that $j \restriction (N \cap V_{\overline{\lambda}} + 1) \in N$. Thus

$$j(\overline{U} \cap N) \in N.$$

So

$$j(\overline{U} \cap N) = j(\overline{U}) \cap j(N) = U \cap N. \qquad \square$$

*Proof that (1) $\implies$ (2).* Assume $N$ is a weak extender model. We seek $\overline{\kappa} < \overline{\lambda} < \kappa$ and $\overline{a} \in V_{\overline{\lambda}}$, $j : V_{\overline{\lambda}+1} \to V_{\lambda+1}$, with the desired properties. Let $\gamma > \lambda$ be such that $\gamma = |V_\gamma|$, and use the lemma to get that for suitable $\lambda'' \gg \gamma$ we have

$$j_U : V \to \text{Ult}(V, U).$$

Thus we have the four properties

(a) $j_U(\kappa) > \lambda$.

(b) $\text{Ult}(V, U)$ is closed under $|V_\gamma + 1|$ sequences.

(c) $j_U(N \cap V_\kappa) \cap V_\lambda = N \cap V_\lambda$.

(d) $j_U``\lambda \in j_U(N)$.

Consider

$$j_U \restriction (V_{\lambda+1}) : V_{\lambda+1} \to V_{j_U(\lambda)+1}^{\text{Ult}(V,U)}.$$

Yet $V_{\lambda+1} = V_{\lambda+1}^{\text{Ult}(V,U)}$ by our closure. So we have for some fixed $\lambda$ and $a \in V_\lambda$:

> $\text{Ult}(V, U) \vDash$ "$\exists \tilde{\kappa}, \tilde{\lambda}, \tilde{a} \exists k : V_{\tilde{\lambda}+1} \to V_{j_U(\lambda)+1}$ such that:
> $\tilde{\kappa}, \tilde{\lambda} < j_U(\kappa)$ $\text{CRT}(\kappa) = \tilde{\kappa}$, $k(\tilde{\kappa}) = j_U(\kappa)$, $k(\tilde{a}) = j_U(a)$.
> Also, $k(j_U(N) \cap V_{\overline{\lambda}}) = j_U(N) \cap V_{j(\lambda)}$, and $k \restriction (j_U(N) \cap V_{\overline{\lambda}+1}) \in j_U(N)$. "

Indeed, $k = j_U \restriction (V_{\lambda+1})$ is the witness. So pulling back through by $j_U$, elementarity gives

> $\text{Ult}(V, U) \vDash$ "$\exists \tilde{\kappa}, \tilde{\lambda}, \tilde{a} \exists k : V_{\tilde{\lambda}+1} \to V_{\lambda+1}$ such that:
> $\tilde{\kappa}, \tilde{\lambda} < j_U(\kappa)$ $\text{CRT}(\kappa) = \tilde{\kappa}$, $k(\tilde{\kappa}) = \kappa$, $k(\tilde{a}) = a$.
> Also, $k(N \cap V_{\overline{\lambda}}) = N \cap V_\lambda$, and $k \restriction (N \cap V_{\overline{\lambda}+1}) \in N$. " $\square$

The whole basic idea is that we can get embeddings as high as you want, and so we can get the ultrapower to pick the big initial segment of its own piece. So the ultrapower has all the nice properties that we want. Then we use $j_U^{-1}$ to get the ditto properties downstairs.

# §21 April 21, 2015

Today we're gong to finish the main results on weak extender models and then move on towards the HOD dichotomy.

## §21.1 A Weird Theorem – Universality

**Theorem 21.1** (Universality)

Suppose $N$ is a weak extender model for the supercompactness of $\kappa$. Let $\gamma$ be a cardinal in $N$ and

$$j : H(\gamma^+)^N \to M$$

is an elementary embedding with $\operatorname{CRT}(j) \ge \kappa$, and $M \subseteq N$. Then $j \in N$.

Here $H(-)$ is a Skolem hull; hence $j(\gamma^+)^N$ is pretty big. This is kind of surprising, since $\gamma$ can be anything.

It follows from this theorem we can't get $N \to N$ nontrivial, since otherwise we can iterate such a $j$ and get a $\lambda$; now $j : V_{\lambda+2} \to V_{\lambda+2}$, but $N$ can see $j$, contradicting Kunen. More later.

*Proof.* Fix $\gamma > \lambda$ large enough so that $\lambda = |V_\lambda|$, and $j \in V_\lambda$. By the Magidor-like reformulation,

$$\exists \overline{\kappa}, \overline{\gamma}, \overline{\lambda}, \overline{\jmath} \in V_{\overline{\lambda}}, \overline{\pi} : V_{\overline{\lambda}+1} \to V_{\lambda+1}$$

such that $\operatorname{CRT}(\pi) = \overline{\kappa}$, $\pi(\overline{\kappa}) = \kappa$, $\pi(\overline{\gamma}) = \gamma$, $\pi(\overline{\jmath}) = j$, and

(1) $\pi(N \cap V_{\overline{\lambda}}) = N \cap V_\lambda$.

(2) $\pi \restriction (N \cap V_{\overline{\lambda}+1}) \in N$.

(Thus we've reflected $\gamma$, $j$, $\gamma$, $\kappa$ down.) So we have that

$$\overline{\jmath} : H(\overline{\gamma}^+)^N \to \overline{M}$$

such that $\overline{M} \subseteq N$.

To show $j \in N$ it suffices to show $\overline{\jmath} \in N$ since $\pi \restriction (N \cap V_{\overline{\lambda}+1}) \in N$, an $j = \pi(\overline{\jmath})$, so from $\overline{\jmath}$ the model $N$ can exhibit $j$. The key point is that

$$\pi \restriction (H(\overline{\gamma}^+)^N) \in N$$

since the bigger restriction $\pi \restriction (N \cap V_{\overline{\gamma}+!}) \in N$. Thus

$$\pi \restriction (H(\overline{\gamma}^+)^N) \in H(\gamma^+)^N.$$

We want to compute

$$\overline{\jmath} : H(\overline{\gamma}^+)^N \to \overline{M}$$

in $N$. Let's try and see what happens: Fix an $\overline{a} \in H(\overline{\gamma}^+)^N$, and $b \in \overline{M}$. Since stuff lives in $V_{\overline{\lambda}+1}$, we have $\overline{b} = \overline{\jmath}(\overline{a})$ This is true fi and only if

$$\pi(\overline{b}) = (\pi(\overline{\jmath}))(\pi(\overline{a})) = pi(\overline{b}) = j(\pi(\overline{a}))$$

since $\pi$ is in the hull. But for both $\overline{b}$ and $\overline{a}$ are in the hull, so the right-hand side is

$$j(\pi(\overline{a})) = j\left((\pi \restriction ((H\overline{\gamma}^+)^N))(j(\overline{a}))\right) = \underbrace{j\left(\underbrace{\pi \restriction ((H\overline{\gamma}^+)^N)}_{\in H(\gamma^+)^N}\right)}_{\in N}(\overline{a})$$

In other words, all the stuff we want is actually in $N$. Specifically, both the left and right hand side of

$$\pi(\overline{b}) = j(\pi(\overline{a}))$$

are in $N$.

Hence $\overline{\jmath} \in N$. □

## §21.2 Corollaries of Universality

Two special cases are the following.

**Theorem 21.2**

Suppose $N$ is a weak extender model for the supercompactness of $\kappa$. Suppose $\gamma$ is a cardinal of $N$ and

$$j : H(\gamma^+)^N \to H(j(\gamma)^+)^N$$

is an elementary embedding with $\text{CRT}(j) \ge \kappa$. Then $j \in N$.

Hence moving big enough pieces guarantees $j \in N$.

**Theorem 21.3**

Suppose $N$ is a weak extender model for the supercompactness of $\kappa$. Suppose $\gamma$ is an ordinal and

$$j : N \cap V_{\gamma+1} \to N \cap V_{j(\gamma)}$$

is an elementary embedding with $\text{CRT}(j) \ge \kappa$. Then $j \in N$.

*Proof.* For $\gamma \ge \omega$, we have

$$V_{\gamma+1} \sim H\left(|V_\gamma|^+\right)$$

in the sense that an elementary embedding on one translates to an elementary embedding on the other. □

**Corollary 21.4**

Suppose $N$ is a weak extender model for the supercompactness of $\kappa$. Then there does not exist a nontrivial elementary embedding $j : N \to N$ with $\text{CRT}(j) \ge \kappa$.

*Proof.* Suppose for contradiction such $j$ exists. Let $\lambda = \sup_n \kappa_n$ be the supremum of the critical sequence defined by $\kappa_0 = \kappa$, and $\kappa_{n+1} = j(\kappa_n)$. So, by the theorem,

$$j \in N.$$

So

$$N \vDash \exists j : V_{\lambda+2} \to V_{\lambda+2}, j \neq \text{id}.$$

This contradicts Kunen's Theorem. □

The upshot of the Universality Theorem is that we can use it to show (assuming large cardinal axioms in $V$) that "all" large cardinals are absorbed by $N$.

Compare $L[U]$ for measurable cardinals $\kappa$ with measure $U$ with $N$ a weak extender model. .

- Both satisfy concentration and amenability/inheritance. They both get their witness from measures in $V$ in the most natural way.
- But in the measurable case, we saw that $L[U]$ can only have one measuarble. It was built to inherit the measurable and succeeded, but nothing.
- Yet $N$ can have everything! What's happening is that the interaction between the two embeddings $j$ and $\pi$ is doing the great absorption. The issue with this model $N$ is that it isn't $L$-like; we don't understand it well the same way we understand $L$.

So first, we have to decide whether $N$ is in HOD, since if not there's no way we have it close to $L$. (Note for example that $L[U]$ is in HOD an hence satisfies GCH.)

Then, we'd like to ask if $N$ satisfies GCH. And so on... eventually we want fine structure.

## §21.3 HOD Dichotomy Overview

We want to get $N \subseteq \text{HOD}$. Actually, there's a conjecture that $N = \text{HOD}$ works!

Recall Jensen's covering lemma. If $0^\sharp$ doesn't exist, then $L$ is close to $V$. Otherwise, $L$ is far from $V$, and every uncountable cardinal is inaccessible, Mahlo, ....

Now we're interested in whether there's a dichotomy theorem for HOD in the same way. And in fact there is. Assume there's an extendible. If $\text{HOD}^\sharp$ doesn't exist, then HOD is close to $V$ in that it computes successors of singular cardinals correctly. Otherwise, it is far from $V$, and all sufficiently large cardinals are measurable. The HOD conjecture says we're in the good half of the dichotomy and HOD is close to $V$.

Note the Ultimate $L$ conjecture implies the HOD conjecture.

## §21.4 HOD Dichotomy

**Definition 21.5.** Suppose $\kappa > \omega$ is a regular carinal. Then $\kappa$ is **$\omega$-strongly measurable in HOD** if there exists $\lambda < \kappa$ such that

(1) $(2^\lambda)^{\text{HOD}} < \kappa$.

(2) There is no partition of

$$(S^\kappa_\omega)^V = \{\alpha < \kappa \mid \text{cof}(\alpha) = \omega\}$$

in HOD of length $\lambda$ into stationary sets (in $V$).

It's important that the stationary condition is in $V$, since HOD has Choice and without it it's trivial. But $V$ could have tons of club sets that HOD doesn't see. Indeed,

$$(S^\kappa_\omega)^V \in \text{HOD}$$

but it's possible that

$$(S^\kappa_\omega)^V = (S^\kappa_\omega)^{\text{HOD}}.$$

In fact, HOD can see which subsets of $\kappa$ in it are stationary using the definition in $V$.

In other words, HOD is built using definability in $V$ which could be quite different from definability in HOD. (Indeed, how are you going to get a *definable* partition? There's a reason we used Choice...)

Let's prove a lemma that tells us why these conditions are here.

**Lemma 21.6**

Suppose $\kappa$ is an $\omega$ strongly measurable on HOD. Then

$$\text{HOD} \vDash \text{“}\kappa \text{ is measurable”}.$$

*Proof.* We claim that there exists a stationary set (in the sense of $V$) $S \subseteq (S^\kappa_\omega)^V$ such that $S \in \text{HOD}$ and there is no partition in HOD of $S$ into stationary (in $V$) sets.

Let's see this claim is sufficient. Let $F$ be the club filter restricted to $S$, that is

$$F = \{X \subseteq S \mid \exists \text{club } C : C \cap S \subseteq X\}.$$

(We have aeedd the extra condition "$\subseteq S$".) Let $\overline{F} = F \cap \text{HOD}$. Since $F$ and HOD are ordinal definable, so is $\overline{F}$, but its elements are in HOD. Thus $\overline{F} \in \text{HOD}$. Also,

$$\text{HOD} \vDash \text{“}\overline{F} \text{ is a } \kappa\text{-complete filter on } \mathcal{P}(S)\text{”}$$

because the club filter is $\kappa$-complete. (To see this: if we have $\kappa$ many sets, they contain a club in $V$, so by $\kappa$-completeness they intersect in $V$, and we can push this back to HOD. Note that we keep having to switch between $V$ and HOD, since the intersections live in HOD but the certificates live in $V$.)

But

$$\text{HOD} \vDash \text{“}\overline{F} \text{ is a } \kappa\text{-complete non-principal ultrafilter on } \mathcal{P}(S)\text{”}$$

by the claim.

Briefly: you show this by repeatedly splitting. Starting with a stationary set $S$, we can split it. Keep splitting. We get a tree. Eventually we have $2^\omega$ branches. Since $2^\omega$ is small, at least one branch has to be stationary, or we could put together all these guys to show $S$ wasn't stationary to begin with it. Keep splitting. Go all the way up to $\lambda$.

We can construct this tree, because HOD can scan whether things are stationary, and it can also well-order all the stationary sets (Choice) to pick things. Then when we get to the $\lambda$th level, we break the condition that one can't do a $\lambda$ partition. $\square$

On Thursday we'll talk about the HOD dichotomy, and then on the last class on Tuesday, we'll stand back and talk about the two futures.

# §22 April 23, 2015

> *HOD has been found to be useful in that it is an inner model that can accommodate essentially all known large cardinals.*

We finish the proof from last time, and then discuss the HOD dichotomy.

## §22.1 Completion of Proof

Our goal is to show:

> Suppose $\kappa$ is an $\omega$ strongly measurable on HOD. Then
> $$\text{HOD} \vDash \text{“}\kappa \text{ is measurable”}.$$

As we said last time, it suffices to show the following lemma.

**Claim 22.1.** There exists a stationary set $S \subseteq S^\kappa_\omega$ in HOD such that it is not possible to write a disjoint union $S = S_0 \cup S_1$ of stationary sets such that $S_0$ and $S_1$ are also in HOD.

*Proof.* Assume the claim fails. Since $\kappa$ is $\omega$-strongly measurable there exists $\lambda < \kappa$. such that

(1) $(2^\lambda)^{\text{HOD}} < \kappa$.

(2) There is no partition of
$$(S^\kappa_\omega)^V = \{\alpha < \kappa \mid \operatorname{cof}(\alpha) = \omega\}$$
in HOD of length $\lambda$ into stationary sets (in $V$).

We will contradiction (2).

We build a binary branching tree $T$ of height $\lambda + 1$, viewed as a function $\le \lambda \to 2$, and a sequence
$$\langle S_t \mid t \in T \rangle$$
such that the following conditions hold.

1. $S_{\langle\rangle} = (S^\kappa_\omega)^V$. (Here $\langle\rangle$ is the empty sequence.)
2. We then want
   (a) $S_t$ is a stationary subset of $(S^\kappa_\omega)^V$ (in $V$).
   (b) $t + \langle 0 \rangle$ and $t + \langle 1 \rangle$ are in $T$.
   (c) $S_t$ is the disjoint of $S_{t+\langle 0 \rangle}$ and $S_{t \cup \langle 1 \rangle}$.
   (d) At the limit stages (meaning $\operatorname{dom}(t)$ is a limit ordinal), we have
   $$S_t = \bigcap \left\{ S_{t \restriction (\alpha)} \mid \alpha \in \operatorname{dom}(t) \right\}.$$
3. We finally require that for all limit ordinals $\beta \le \lambda$ and for all $t : \beta \to 2$ not in $T$, then
   $$(\forall \alpha \le \beta : t \restriction (\alpha) \in T) \implies \bigcap_{\alpha < \beta} S_{t \restriction (\alpha)} \text{ isn't stationary.}$$
   In other words, if you're at a limit stage and you're not in the tree, the reason is that the intersection along the way wasn't stationary.

Notice that

(A) $(S^\kappa_\omega)^V \in \text{HOD}$.

(B) We have
$$\left\{ S \subseteq (S^\kappa_\omega)^V \mid S \in \text{HOD stationary in } V \right\} \in \text{HOD}.$$

(C) There's a well-ordering of the above set in HOD.

Here's how we build the tree now. At the successor stage, by the (failure of the) claim, we can conjure a splitting; then take the minimal guy via the well-ordering. (At every stage we want to keep the tree in HOD!)

At the limit stage, we use the fact that the tree isn't too large. Suppose $\beta \le \lambda$ is a limit stage. Assume we have $T \cap ({}^{<\beta}2)$ and
$$\left\langle S_t \mid t \in T \cap {}^{<\beta}2 \right\rangle.$$
By (3) we have that modulo a non-stationary set, $(S^\kappa_\omega)^V$ equals
$$\bigcup \left\{ \bigcap \left\{ S_{t\restriction(\alpha)} \mid \alpha \le \beta \right\} \mid t \text{ is a } \beta\text{-branch of } T \cap {}^{<\beta}2,\ t \in \text{HOD} \right\}.$$
The club filter is $\kappa$-complete and $\left|2^\beta\right|^{\text{HOD}} < \kappa$ so for some $\beta$-branches $t$ of $T \cap {}^{<\beta}2$ the set
$$\bigcap \left\{ S_{t\restriction(\alpha)} : \alpha \le \beta \right\}$$
is stationary. Keep all the stationary ones, and put
$$S_t = \bigcap \left\{ S_{t\restriction(\alpha)} : \alpha \le \beta \right\}.$$
We keep going until we get $T$ a tree of height $\lambda+1$ in HOD. Let $t$ be a $\lambda+1$ branch of $T$, so $S_t$ is a disjoint union
$$\bigcup \left\{ S_{t\restriction(\alpha)} \setminus S_{t\restriction(\alpha+1)} \right\}$$
except for a non-stationary set. This gives a partition in HOD of $(S^\kappa_\omega)^V$ into stationary sets in $V$. Contradiction. □

## §22.2 The HOD Dichotomy

Recall Jensen covering theorem.

**Theorem 22.2** (The $L$ Dichotomy Theorem, Jensen)
Exactly one of the following holds.

(1) For all singular cardinals $\gamma$,
   (a) $L \vDash$ "$\gamma$ is singular".
   (b) $(\gamma^+)^L = \gamma^+$.

   Colloquially, "$L$ is close to $V$."

(2) Every uncountable cardinal is inaccessible in $L$. Colloquially, "$L$ is far from $V$."

It is a pretty nasty proof, done in cases based on $0^\sharp$.

Surprisingly, the corresponding claim for HOD has a much less difficult proof, and does not require cases (there is no "missing cardinal").

**Definition 22.3.** A cardinal $\kappa$ is **extendible** if for all $\alpha > \kappa$ there exists an elementary embedding

$$j : V_\alpha + 1 \to V_{j(\alpha)+1}$$

such that $\textsc{crt}(j) = \kappa$ and $j(\kappa) > \alpha$.

This is stronger than supercompact.

**Theorem 22.4** (HOD Dichotomy Theorem)
Let $\kappa$ be an extendible cardinal. Then exactly one of the following holds.

(1) HOD correctly computes singular cardinals and their successors.

(2) All regular cardinals greater than $\kappa$ are $\omega$-strongly measurable in HOD.

> Woodin firmly believes (1). I'm hoping (2) is true, because I just like disaster.

This is quite remarkable. Solovay said this was a remarkable theorem.

> Actually, Solovay probably said "this is a remarkable theorem if it's true". (Apparently he is very careful.)

*Proof.* Assume (2) is false. The strategy for this proof is really very simple: We will show that HOD is a weak extender model for the supercompactness of $\kappa$, which will directly give (1) by an earlier theorem.

> In the $L$ Dichotomy, we're hoping $V = L$ because then we could go home. If $0^\sharp$ exists (which it does) (cough cough). . .

Assume the existence of $\gamma_0 > \kappa$ which is regular and *not* $\omega$-strongly measurable in HOD.

**Claim 22.5.** There is a *proper class* of regular $\gamma$ which are *not* $\omega$-strongly measurable in HOD.

*Proof.* Fix $\lambda_0 > \gamma_0$ such that $V_{\lambda_0}$ is a $\Sigma_2$ substructure of $V$ and moreover

$$\lambda_0 = |V_{\lambda_0}|.$$

Thus

$$(\mathrm{HOD})^{V_{\lambda_0}} = \mathrm{HOD} \cap V_{\lambda_0}.$$

(We always have $\subseteq$ but if we can pull the $\Sigma_2$ definition down.)

Fix $\alpha > \lambda_0$. We produce a regular cardinal above $\alpha$ which is not $\omega$-strongly measurable in HOD. We know there exists an elementary embedding

$$j : V_{\alpha+1} \to V_{j(\alpha)+1}$$

such that $\textsc{crt}(j) = \kappa$ and $j(\kappa) > \alpha$.

We claim that $j(\gamma_0)$ is not $\omega$-strongly measurable in HOD. We have

$$\forall \lambda < \omega_0 \left( (2^\lambda)^{\mathrm{HOD}} < \gamma \implies \exists \text{ partition } \langle S_\alpha : \alpha < \lambda \rangle \in \mathrm{HOD} \right) \qquad (*_{\gamma_0})$$

where the partition is of $(S^{\gamma_0}_\omega)^V$ into stationary sets in $V$. Since $(\mathrm{HOD})^{V_\lambda} = \mathrm{HOD} \cap V_{\lambda_0}$ and $\lambda_0 > \gamma_0$ is such that $\lambda_0 = |V_{\lambda_0}|$,

Hence $(*_{\gamma_0})$ holds relative to $(\mathrm{HOD})^{V_{\lambda_0}}$, *id est*

$$V_{\lambda_0} \models (*_{\gamma_0})$$

we obtain

$$V_{j(\lambda_0)} \models (*_{j(\gamma_0)}).$$

■

**Remark 22.6.** I think it's not necessarily the case that $(\mathrm{HOD})^{V_{j(\lambda_0)}} = \mathrm{HOD} \cap V_{j(\lambda_0)}$.

Thus, we have a large number of $\omega$-strongly measurable cardinals just from the mere existence of one. Now we want to show HOD is a weak extender model for the supercompactness of $\kappa$; thus we want to get measures.

Fix $\lambda > \kappa$ now. We have to show that there exists a $\kappa$-complete normal fine ultrafilter $U$ on $\mathcal{P}_\kappa \lambda$ such that

(1) (Concentration) $\mathcal{P}_\kappa \lambda \cap \mathrm{HOD} \in U$.

(2) (Inheritance) $U \cap \mathrm{HOD} \in \mathrm{HOD}$.

Since $\lambda$ was arbitrary, this will give us what we want.

Fix $\lambda_0 > \lambda$ such that $\lambda_0 = |V_{\lambda_0}|$. By the claim we can find a regular $\gamma_0 > (2^\lambda_0)^{\mathrm{HOD}}$ which is not $\omega$-strongly measurable in HOD. We are going to use $\gamma_0$ to get our partition.

Since $2^{\gamma_0} < \gamma_0$, we have $(2^{\gamma_0})^{\mathrm{HOD}} < \gamma_0$, and so there exists a partition

$$\langle S_\alpha : \alpha < \lambda_0 \rangle \in \mathrm{HOD}.$$

Choose $\lambda_1 > \gamma_0$ such that $\lambda_1 = |V_{\lambda_1}|$ and $V_{\lambda_1} \prec_{\Sigma_2} V$, so

$$(\mathrm{HOD})^{V_{\lambda_1}} = \mathrm{HOD} \cap V_{\lambda_1}.$$

Thus it picks up the sequence:

$$\langle S_\alpha : \alpha < \lambda_0 \rangle \in (\mathrm{HOD})^{V_{\lambda_1}}.$$

Since $\kappa$ is extendible, we can embed

$$j : V_{\lambda_1 + 1} \to V_{j(\lambda_1)+1}$$

such that $\textsc{crt}(j) = \kappa$ and $j(\kappa) > \lambda_1$.

Let

$$\langle T_\alpha : \alpha < j(\lambda_0) \rangle = j(\langle S_\alpha : \alpha < \lambda_0 \rangle).$$

By elementarity of $j$, this sequence is a partition of $j\left( (S^{\lambda_0}_\omega)^V \right) = \left( S^{j(\lambda_0)}_\omega \right)^V$ into stationary (in $V$) sets. Moreover, it is in $(\mathrm{HOD})^{V_{j(\lambda_0)}} \subseteq \mathrm{HOD} \cap V_{j(\lambda)}$ (might not be equality here, but we don't care).

In summary, we have the following ordinals:

$$\kappa < \lambda < \lambda_0 < \gamma_0 < \lambda_1$$

in $V$, and on the $j$-side
$$\lambda_1 < j(\gamma_0).$$
Since $j(\gamma_0) > \gamma_0$ is regular, we know
$$j(\gamma_0) > \sup j``\gamma_0.$$
we will use
$$\langle T_\alpha : \alpha < j(\lambda_0)\rangle \in (\mathrm{HOD})^{V_{j(\lambda_1)}} \subseteq \mathrm{HOD}$$
to compute $j``\lambda_0$.

Let
$$Z = \{\alpha < j(\lambda_0) \mid T_\alpha \text{ stationarily reflects to } \sup j``\gamma_0\}.$$

**Claim 22.7.** $Z = j``\lambda_0$.

*Proof.* To see $Z \supseteq j``\lambda_0$, fix $\alpha < \lambda_0$ with $j(\alpha) \in j``\lambda_0$. Let $C \subseteq \sup ``\gamma_0$ be a club. We need to hit the club with $T_{j(\alpha)}$.

Let
$$D = \{\alpha < \gamma_0 \mid j(\alpha) \in C\}.$$
The set $D$ is certainly unbounded, though it need not be a club. However, its closure $\overline{D}$ is a club! Moreover, cutting it back to
$$\overline{D} \cap (S^{\gamma_0}_\omega)^V = D \cap (S^{\gamma_0}_\omega)^V$$
since $j$ is continuous at those points.

But $S_\alpha \subseteq (S^{\gamma_0}_\omega)^V$ is stationary in $V$. Thus $\exists \beta$ such that
$$\beta \in S_\alpha \cap \overline{D} = S_\alpha \cap D$$
thus
$$j(\beta) \in j(S_\alpha) \cap C = T_{j(\alpha)} \cap C.$$
So $T_{j(\alpha)}$ stationarily reflects to $\sup j``\gamma_0$, giving the first direction.

For the other way, fix $\alpha \in Z$. Thus $T_\alpha \cap \sup j``\gamma_0$ is stationary, meaning it hits every club. In particular, it hits the club
$$C = \overline{j``\gamma_0} \cap \sup j``\gamma_0.$$
Thus there exists $\beta \in T_\alpha \cap C$. The point is that
$$C \cap j(S^{\gamma_0}_\omega) = j``(S^{\gamma_0}_\omega)^V$$
thus $\beta \in j``(S^{\gamma_0}_\omega)^V$. Choose $\overline{\beta} \in (S^{\gamma_0}_\omega)^V$ such that $\beta = j(\overline{\beta})$ so $j(\overline{\beta}) \in T_\alpha$. Choose the $\overline{\alpha} < \lambda_0$ such that $\overline{\beta} \in S_{\overline{\alpha}}$ (since the $S_*$ are a partition). Then $j(\overline{\beta}) \in j(S_{\overline{\alpha}}) = T_{j(\overline{\alpha})}$. But $j(\overline{\beta}) \in T_\alpha$. Thus the $T_*$ are a partition meaning $\alpha = j(\overline{\alpha})$. This completes the proof. ■

In summary, $j``\lambda_0 \in \mathrm{HOD}$.

We have
$$\lambda_0 = |V_{\lambda_0}|$$
so
$$\lambda_0 = |\mathrm{HOD} \cap V_{\lambda_0}|^{\mathrm{HOD}}.$$
Now fix a bijection $\pi : \lambda_0 \to \mathrm{HOD} \cap V_{\lambda_0}$ (so that $\pi \in \mathrm{HOD}$).

We have

$$j``(\mathrm{HOD} \cap V_{\lambda_0}) = j(\pi)``(j``\lambda_0).$$

Since both $j(\pi)$ and $j``\lambda_0 \in \mathrm{HOD}$, we obtain the above is in HOD: that is,

$$j``(\mathrm{HOD} \cap V_{\lambda_0}) \in \mathrm{HOD}.$$

Let $U$ be the $\kappa$-complete normal fine ultrafilter on $\mathcal{P}_\kappa(\lambda)$ given by

$$X \in U \iff j``\lambda \in j(X).$$

This is definable in HOD using the previous thing. This gives

(1) $\mathcal{P}_\kappa \lambda \cap \mathrm{HOD} \in U$ (just check that the seed is in there) and

(2) $U \cap \mathrm{HOD} \in \mathrm{HOD}$, which happens since we defined it in HOD. (All HOD needs to see $U \cap \mathrm{HOD}$ is to have the embedding, which it does.)

Thus HOD is a weak extender model for the supercompactness of $\kappa$. □

# §23 April 28, 2015

## §23.1 HOD Dichotomy

We stated the HOD Dichotomy Theorem. Perhaps worth mentioning is that

**Corollary 23.1**

Suppose $\kappa$ is an extendible cardinal. Then there exists a measurable cardinal in HOD.

*Proof.* Look at the two cases of the HOD Dichotomy Theorem, and recall we can actually strengthen (1) to read "HOD is a weak extender model for the supercompactness of $\kappa$". Thus $\kappa$ is a measurable in HOD.

In the second case, of the HOD Dichotomy, well... not much to prove! □

HOD Hypothesis (once called the Silly Hypothesis): There is a proper class of cardinals $\kappa$ such that $\kappa$ is *not* strongly measurable in HOD. This is equivalent to the good case of the HOD Dichotomy.

The **HOD Conjecture** states that HOD Hypothesis is provable in ZFC.

## §23.2 Which side of the dichotomy are we on?

Well, the first thing we could worry about is whether there's even an extendible cardinal to begin with. It could be that extendibles are inconsistent with ZFC – but that seems highly unlikely. So let's put this aside, and assume large cardinal axioms for granted.

The other thing we might worry about is independence. For example, Continuum Hypothesis is independent of not only ZFC but all the large cardinal axioms we are aware of. So we might also worry that the HOD hypothesis is as intractable as CH. However,

**Theorem 23.2** (Woodin)

Assume that there is a proper class of extendible cardinals. Then $V$ satisfies the HOD Hypothesis if and only for every boolean algebra $\mathfrak{B}$, $V^{\mathfrak{B}} \vDash \text{HOD}$.

In other words, *the HOD Hypothesis is immune to forcing.* This is strong evidence that the HOD Conjecture might be true; why else would forcing fail to break it?

> "I hope I did say that all the theorems which are not attributed are due to Woodin."

## §23.3 Equivalent Formulations of the HOD Hypothesis

To better understand which side of the dicohtomy we are in, we now give several equivalent formulations of the HOD Hypothesis.

**Theorem 23.3**

Assume $\kappa$ is an extendible cardinal. The following are equivalent.

(1) HOD is a weak extender model for the supercompactness of $\kappa$.

(2) There exists $N \subseteq \text{HOD}$ which is a weak extender model for the supercompactness of $\kappa$.

(3) The HOD Hypothesis.

(4) For all singular cardinals $\gamma > \kappa$,

  (a) HOD $\vDash$ "$\gamma$ singular

  (b) $(\gamma^+)^{\text{HOD}} = \gamma^+$.

(5) There exists *one* cardinal $\gamma > \kappa$ satisfying

$$(\gamma^+)^{\text{HOD}} = \gamma^+.$$

(6) There's a regular cardinal $\gamma \geq \kappa$ such that $\gamma$ is not $\omega$-strongly measuarble in HOD.

This comes out of the proof: our proof was (6) $\implies$ (1) and (1) $\implies$ (4). Also (4) $\implies$ (6), giving an equivalence. Moreover (1) $\implies$ (2) $\implies$ (4) is easy.

Clearly, (4) $\implies$ (3) $\implies$ (6). For (5), we have (4) $\implies$ (5) and (5) $\implies$ (6).

The one of interest is (2).

## §23.4 Magidor Formulation

Another concern: perhaps the expectation of (2) is problematic because of Concentration. This is alleviated by the following result.

**Theorem 23.4**

The following are equivalent.

(1) $N$ is weak extender model for the supercompactness of $\kappa$.

(2) $N$ is a proper class model of ZFC and

$$\forall \lambda > \kappa \exists \text{extender } E$$

which witnesses the Magidor formulation:

  (a) $j_E(\text{CRT}(E)) = \kappa$.

  (b) The support of $E$ is less than $\kappa$.

  (c) The length of $E$ exceeds the strength $\lambda$.

  (d) $E \cap N \in N$.

## §23.5 The Ultimate $L$

Things that put you in the "close to $V$" half are this existence of weak extender models inside HOD. An axiom Woodin introduced will show this.

**Definition 23.5.** A set of reals $A \subseteq \mathbb{R}$ is **universally Baire** if for all topological spaces $\Omega$ and for all continuous function

$$\pi : \Omega \to \mathbb{R}$$

the premiage of $A$ under $\pi$ has the property of Baire.

Here the **property of Baire** means that there is an open set $U$ such that $A \triangle U$ is meager, where $\triangle$ is the symmetric difference, and a meager set is the union of countably many nowhere dense sets.

Let

$$\Gamma^\infty = \{A \subseteq \mathbb{R} \mid A \text{ universally Baire}\}.$$

**Theorem 23.6**

Assume there is a proper class of Woodin cardinals. Then

(1) The $\Gamma^\infty$ sets have the regularity properties (are "nice").

(2) Nice closure: for any $A \in \Gamma^\infty$, the set

$$\mathcal{P}(\mathbb{R}) \cap L(A, \mathbb{R}) \subseteq \Gamma^\infty$$

has $L(A, R) \vDash \text{AD}$, the Axiom of Determinacy (?).

**Definition 23.7.** We define an ordinal

$$\theta^{L(A,\mathbb{R})} = \sup \{\alpha \in \text{On} \mid \exists (\pi : \mathbb{R} \twoheadrightarrow \alpha) \in L(A, \mathbb{R})\}.$$

Also, let

$$\theta_0^{L(A,\mathbb{R})} = \sup \{\alpha \in \text{On} \mid \exists (\pi : \mathbb{R} \twoheadrightarrow \alpha) \text{ is On-definable in } L(A, \mathbb{R})\}.$$

Assume ZFC and the existence of a proper class of Woodin cardinals. Let $A \in \Gamma^\infty$. Then

$$\text{HOD}^{L(A,\mathbb{R})}$$

is a fine structural model (i.e. is very $L$-like). Moreover, it satisfies ZFC and thinks $\Theta^{L(A,\mathbb{R})}$ is a Woodin cardinal.

Moreover, $\text{HOD}^{L(A,\mathbb{R})}$ restricted to $\Theta_0^{L(A,\mathbb{R})}$ is a Mitchell-Steel model. Without the restriction, $\text{HOD}^{L(A,\mathbb{R})}$ is not such a model, but a *strategic extender* model.

Best result so far.

**Theorem 23.8**

Assume there is a Woodin limit of Woodin cardinals. Then $\exists A \in \Gamma^\infty$ such that

$$\text{HOD}^{L(A,\mathbb{R})} \restriction (\theta^{L(A,\mathbb{R})})$$

thinks there exists a strong cardinal which is a limit of Woodin cardinals.

Also, the model

$$M_A \overset{\text{def}}{=} \text{HOD}^{L(A,\mathbb{R})} \restriction (\Theta^{L(A,\mathbb{R})})$$

is comparable, in the sense that either $M_A = M_B$, or one is a substructure of the other. (Assuming ZFC and proper class of Woodin carinals.)

motivation: for each $\alpha$ we let $M_\alpha$ denote the intersection of all transitive models $N$ of ZFC such that $(\mathrm{On})^N = \alpha$. (We let $M_\alpha = \varnothing$ if no such $N$ exist.)

## §23.6 Ultimate $L$

**Definition 23.9.** Assume ZFC and a proper class of Woodin cardinals. We say $\Gamma \prec \Gamma^\alpha$ if

(a) $\Gamma \subseteq \Gamma^\infty$ and $\Gamma = \mathcal{P}(\mathbb{R}) \cap L(\Gamma, \mathbb{R})$.

(b) $L(\Gamma, \mathbb{R}) \vDash \neg \mathrm{AD}_{\mathbb{R}}$.

**Definition 23.10.** ($V = \mathrm{Ult}\ L$)

(1) There is a proper class of Woodin cardinals

(2) There is a proper class of strong cardinals.

(3) For all $\phi \in \Sigma_4$ if $V \vDash \phi$ then $\exists \Gamma \prec \Gamma^\infty$ such that

$$\mathrm{HOD}^{L(\Gamma,\mathbb{R})} \restriction (\Theta^{L(\Gamma,\mathbb{R})}) \Vdash \phi.$$

The Ultimate $L$ Conjecture in ZFC says that, given $\kappa$ is extendible, there exists $N$ such that

(1) $N$ is a weak extender model for the supercompactness of $\kappa$.

(2) $N \subseteq \mathrm{HOD}$.

(3) $N \vDash$ "$V = \text{Ultimate } L$".

If the Ultimate $L$ Conjecture is true and there is an extendible cardinal, then HOD hypothesis holds. Moreover,

**Theorem 23.11**

Assume $V$ equals ultimate $L$. Then

(1) Continuum Hypothesis holds.

(2) $V = \mathrm{HOD}$.

(3) The $\Omega$ conjecture

## §23.7 Chaos

The hierarchy of large cardinals beyond Choice started at a super Reinhardt cardinal.

**Theorem 23.12**

(ZF) Suppose $\kappa$ is a super-Reinhardt cardinal. Then there is a definable, homogeneous class forcing $\mathbb{P}_{\text{AC}}$ such that if $G$ is a $V$-generic in this poset, then

(1) $V[G] \vDash \text{ZFC}$

(2) $V[G] \vDash \kappa\text{extendible}$

(3) $V[G] \vDash \forall\text{singular } \gamma > \kappa\text{:}(\gamma^+)^{\text{HOD}} < \gamma^+$

(4) $V[G]$ thinks every regular cardinal $\gamma \ge \kappa$ is $\omega$-strongly measurable in HOD.

(5) $V[G]$ thinks there exists $j : \text{HOD} \to \text{HOD}$ such that $j \neq \text{id}$.

(6) $V[G]$ thinks that $\forall \lambda \exists j : \text{HOD} \to \text{HOD}$ such that $\textsc{crt}(j) = \kappa$ and $j(\kappa) > \lambda$.

So some people really hope that the super Reinhardt cardinal is not compatible with ZF!

Summary: in the "Pattern" future,

- HOD Conjecture
- Ultimate $L$ Conjecture
- $\Omega$ Conjecture
- Super-Reinhardt and friends are inconsistent
- Inner model theory for supercomapact

In the other future, large cardinals beyond Choice rule, and we obtain

- $\neg$ HOD Conjecture
- $\neg$ Ultimate $L$ Conjecture
- $\neg$ $\Omega$ Conjecture
- Super-Reinhardt and friends are consistent with ZF
- No inner model theory for supercomapact