A Brief Introduction to Olympiad Inequalities

Evan Chen

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The goal of this document is to provide a easier introduction to olympiad inequalities than the standard exposition Olympiad Inequalities, by Thomas Mildorf. I was motivated to write it by feeling guilty for getting free 7’s on problems by simply regurgitating a few tricks I happened to know, while other students were unable to solve the problem.

Warning: These are notes, not a full handout. Lots of the exposition is very minimal, and many things are left to the reader.

In a problem with \( n \) variables, these respectively mean to cycle through the \( n \) variables, and to go through all \( n! \) permutations. To provide an example, in a three-variable problem we might write

\[
\begin{align*}
\sum_{\text{cyc}} a^2 &= a^2 + b^2 + c^2 \\
\sum_{\text{cyc}} a^2b &= a^2b + b^2c + c^2a \\
\sum_{\text{sym}} a^2 &= a^2 + a^2 + b^2 + b^2 + c^2 + c^2 \\
\sum_{\text{sym}} a^2b &= a^2b + a^2c + b^2c + b^2a + c^2a + c^2b.
\end{align*}
\]

§1 Polynomial Inequalities

§1.1 AM-GM and Muirhead

Consider the following theorem.

**Theorem 1.1 (AM-GM)**

For nonnegative reals \( a_1, a_2, \ldots, a_n \) we have

\[
\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}.
\]

Equality holds if and only if \( a_1 = a_2 = \cdots = a_n \).

For example, this implies

\[
a^2 + b^2 \geq 2ab, \quad a^3 + b^3 + c^3 \geq 3abc.
\]

Adding such inequalities can give us some basic propositions.
Example 1.2
Prove that $a^2 + b^2 + c^2 \geq ab + bc + ca$ and $a^4 + b^4 + c^4 \geq a^2bc + b^2ca + c^2ab$.

Proof. By AM-GM,
\[
\frac{a^2 + b^2}{2} \geq ab \quad \text{and} \quad \frac{2a^4 + b^4 + c^4}{4} \geq a^2bc.
\]
Similarly,
\[
\frac{b^2 + c^2}{2} \geq bc \quad \text{and} \quad \frac{2b^4 + c^4 + a^4}{4} \geq b^2ca.
\]
\[
\frac{c^2 + a^2}{2} \geq ca \quad \text{and} \quad \frac{2c^4 + a^4 + b^4}{4} \geq c^2ab.
\]
Summing the above statements gives
\[
a^2 + b^2 + c^2 \geq ab + bc + ca \quad \text{and} \quad a^4 + b^4 + c^4 \geq a^2bc + b^2ca + c^2ab.
\]

Exercise 1.3. Prove that $a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a$.

Exercise 1.4. Prove that $a^5 + b^5 + c^5 \geq a^3bc + b^3ca + c^3ab \geq abc(ab + bc + ca)$.

The fundamental intuition is being able to decide which symmetric polynomials of a given degree are bigger. For example, for degree 3, the polynomial $a^3 + b^3 + c^3$ is biggest and $abc$ is the smallest. Roughly, the more "mixed" polynomials are the smaller. From this, for example, one can immediately see that the inequality
\[
(a + b + c)^3 \geq a^3 + b^3 + c^3 + 24abc
\]
must be true, since upon expanding the LHS and cancelling $a^3 + b^3 + c^3$, we find that the RHS contains only the piddling term $24abc$. That means a straight AM-GM will suffice.

A useful formalization of this is Muirhead’s Inequality. Suppose we have two sequences $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$ such that
\[
x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n,
\]
and for $k = 1, 2, \ldots, n - 1$
\[
x_1 + x_2 + \cdots + x_k \geq y_1 + y_2 + \cdots + y_k.
\]
Then we say that $(x_n)$ majorizes $(y_n)$, written $(x_n) \succ (y_n)$.

Using the above, we have the following theorem.

**Theorem 1.5 (Muirhead’s Inequality)**

If $a_1, a_2, \ldots, a_n$ are positive reals, and $(x_n)$ majorizes $(y_n)$ then we have the inequality.
\[
\sum_{\text{sym}} a_1^{x_1} a_2^{x_2} \cdots a_n^{x_n} \geq \sum_{\text{sym}} a_1^{y_1} a_2^{y_2} \cdots a_n^{y_n}.
\]
Example 1.6
Since \((5, 0, 0) \succ (3, 1, 1) \succ (2, 2, 1)\),
\[
a^5 + a^5 + b^5 + b^5 + c^5 + c^5 \geq a^3bc + a^3bc + b^3ca + b^3ca + c^3ab + c^3ab
\geq a^2b^2c + a^2b^2c + b^2c^2a + b^2c^2a + c^2a^2b + c^2a^2b.
\]
From this we derive \(a^5 + b^5 + c^5 \geq a^3bc + b^3ca + c^3ab \geq abc(ab + bc + ca)\).

Notice that Muirhead is symmetric, not cyclic. For example, even though \((3, 0, 0) \succ (2, 1, 0)\), Muirhead’s inequality only gives that
\[
2(a^3 + b^3 + c^3) \geq a^2b + a^2c + b^2c + b^2a + c^2a + c^2b
\]
and in particular this does not imply that \(a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a\). These situations must still be resolved by AM-GM.

§1.2 Non-homogeneous inequalities

Consider the following example.

Example 1.7
Prove that if \(abc = 1\) then \(a^2 + b^2 + c^2 \geq a + b + c\).

Proof. AM-GM alone is hopeless here, because whenever we apply AM-GM, the left and right hand sides of the inequality all have the same degree. So we want to use the condition \(abc = 1\) to force the problem to have the same degree. The trick is to notice that the given inequality can be rewritten as
\[
a^2 + b^2 + c^2 \geq a^{1/3}b^{1/3}c^{1/3} (a + b + c).\]
Now the inequality is homogeneous. Observe that if we multiply \(a, b, c\) by any real number \(k > 0\), all that happens is that both sides of the inequality are multiplied by \(k^2\), which doesn’t change anything. That means the condition \(abc = 1\) can be ignored now. Since \((2, 0, 0) \succ (\frac{4}{3}, \frac{1}{3}, \frac{1}{3})\), applying Muirhead’s Inequality solves the problem.

The importance of this problem is that it shows us how to eliminate a given condition by homogenizing the inequality; this is very important. (In fact, we will soon see that we can use this in reverse – we can impose an arbitrary condition on a homogeneous inequality.)

§1.3 Practice Problems

1. \(a^7 + b^7 + c^7 \geq a^4b^3 + b^4c^3 + c^4a^3\).
2. If \(a + b + c = 1\), then \(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq 3 + 2 \cdot \frac{(a^3+b^3+c^3)}{abc}\).
3. \(\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a + b + c\).
4. If \(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1\), then \((a + 1)(b + 1)(c + 1) \geq 64\).
5. (USA 2011) If \(a^2 + b^2 + c^2 + (a + b + c)^2 \leq 4\), then \(\frac{ab + 1}{(a + b)^2} + \frac{bc + 1}{(b + c)^2} + \frac{ca + 1}{(c + a)^2} \geq 3\).
6. If \(abcd = 1\), then \(a^4b + b^4c + c^4d + d^4a \geq a + b + c + d\).
§2 Inequalities in Arbitrary Functions

Let \( f : (u, v) \to \mathbb{R} \) be a function and let \( a_1, a_2, \ldots, a_n \in (u, v) \). Suppose that we fix \( \frac{a_1 + a_2 + \cdots + a_n}{n} = a \) (if the inequality is homogeneous, we will often insert such a condition) and we want to prove that

\[
f(a_1) + f(a_2) + \cdots + f(a_n)
\]

is at least (or at most) \( nf(a) \). In this section we will provide three methods for doing so.

We say that function \( f \) is **convex** if \( f''(x) \geq 0 \) for all \( x \); we say it is **concave** if \( f''(x) \leq 0 \) for all \( x \). Note that \( f \) is convex if and only if \( -f \) is concave.

§2.1 Jensen / Karamata

**Theorem 2.1** (Jensen’s Inequality)
If \( f \) is convex, then

\[
\frac{f(a_1) + \cdots + f(a_n)}{n} \geq f\left(\frac{a_1 + \cdots + a_n}{n}\right).
\]

The reverse inequality holds when \( f \) is concave.

**Theorem 2.2** (Karamata’s Inequality)
If \( f \) is convex, and \( (x_n) \) majorizes \( (y_n) \) then

\[
f(x_1) + \cdots + f(x_n) \geq f(y_1) + \cdots + f(y_n).
\]

The reverse inequality holds when \( f \) is concave.

**Example 2.3** (Shortlist 2009)
Given \( a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \), prove that

\[
\frac{1}{(2a + b + c)^2} + \frac{1}{(a + 2b + c)^2} + \frac{1}{(a + b + 2c)^2} \leq \frac{3}{16}.
\]

**Proof.** First, we want to eliminate the condition. The original problem is equivalent to

\[
\frac{1}{(2a + b + c)^2} + \frac{1}{(a + 2b + c)^2} + \frac{1}{(a + b + 2c)^2} \leq \frac{3}{16} \cdot \frac{1}{a + b + c}.
\]

Now the inequality is homogeneous, so we can assume that \( a + b + c = 3 \). Now our original problem can be rewritten as

\[
\sum_{\text{cyc}} \frac{1}{16a} - \frac{1}{(a + 3)^2} \geq 0.
\]

Set \( f(x) = \frac{1}{16x} - \frac{1}{(x+3)^2} \). We can check that \( f \) over \((0, 3)\) is convex so Jensen completes the problem. \( \square \)
Example 2.4
Prove that
\[
\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 2 \left( \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \right) \geq \frac{9}{a+b+c}.
\]

Proof. The problem is equivalent to
\[
\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{9}{a+b+c}.
\]
Assume WLOG that \(a \geq b \geq c\). Let \(f(x) = \frac{1}{x}\). Since \((a,b,c) \succ (a+b+c, a+b+c, a+b+c)\) the conclusion follows by Karamata.

Example 2.5 (APMO 1996)
If \(a, b, c\) are the three sides of a triangle, prove that
\[
\sqrt{a + b - c} + \sqrt{b + c - a} + \sqrt{c + a - b} \leq \sqrt{a} + \sqrt{b} + \sqrt{c}.
\]

Proof. Again assume WLOG that \(a \geq b \geq c\) and notice that \((a+b-c, c+a-b, b+c-a) \succ (a, b, c)\). Apply Karamata on \(f(x) = \sqrt{x}\).

§2.2 Tangent Line Trick
Again fix \(a = \frac{a_1 + \cdots + a_n}{n}\). If \(f\) is not convex, we can sometimes still prove the inequality
\[
f(x) \geq f(a) + f'(a) (x - a) .
\]
If this inequality manages to hold for all \(x\), then simply summing the inequality will give us the desired conclusion. This method is called the tangent line trick.

Example 2.6 (David Stoner)
If \(a + b + c = 3\), prove that
\[
18 \sum_{cyc} \frac{1}{(3-c)(4-c)} + 2(ab+bc+ca) \geq 15.
\]

Proof. We can rewrite the given inequality as
\[
\sum_{cyc} \left( \frac{16}{(3-c)(4-c)} - c^2 \right) \geq 6.
\]
Using the tangent line trick lets us obtain the magical inequality
\[
\frac{18}{(3-c)(4-c)} - c^2 \geq \frac{c+3}{2} \iff c(c-1)^2(2c-9) \leq 0
\]
and the conclusion follows by summing.
Example 2.7 (Japan)

Prove $\sum_{\text{cyc}} \frac{(b+c-a)^2}{a^2+(b+c)^2} \geq \frac{3}{5}$.

Proof. Since the inequality is homogeneous, we may assume WLOG that $a+b+c=3$. So the inequality we wish to prove is

$$\sum_{\text{cyc}} \frac{(3-2a)^2}{a^2+(3-a)^2} \geq \frac{3}{5}.$$  

With some computation, the tangent line trick gives away the magical inequality:

$$\frac{(3-2a)^2}{(3-a)^2+a^2} \geq \frac{1}{5} - \frac{18}{25} (a-1) \iff \frac{18}{25} (a-1)^2 \geq \frac{2a+1}{2a^2-6a+9}.$$  

\[\square\]

§ 2.3 $n-1$ EV

The last such technique is $n-1$ EV. This is a brute force method involving much calculus, but it is nonetheless a useful weapon.

Theorem 2.8 ($n-1$ EV)

Let $a_1, a_2, \ldots, a_n$ be real numbers, and suppose $a_1 + a_2 + \cdots + a_n$ is fixed. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with exactly one inflection point. If $f(a_1) + f(a_2) + \cdots + f(a_n)$ achieves a maximal or minimal value, then $n-1$ of the $a_i$ are equal to each other.

Proof. See page 15 of Olympiad Inequalities, by Thomas Mildorf. The main idea is to use Karamata to “push” the $a_i$ together.

Example 2.9 (IMO 2001 / APMOC 2014)

Let $a, b, c$ be positive reals. Prove $1 \leq \sum_{\text{cyc}} \frac{a}{\sqrt{a^2+8bc}} < 2$.

Proof. Set $e^x = \frac{bc}{a^2}, e^y = \frac{ca}{b^2}, e^z = \frac{ab}{c^2}$. We have the condition $x+y+z=0$ and want to prove

$$1 \leq f(x) + f(y) + f(z) < 2$$

where $f(x) = \frac{1}{\sqrt{1+8e^x}}$. You can compute

$$f''(x) = \frac{4e^x(4e^x-1)}{(8e^x+1)^2}$$

so by $n-1$ EV, we only need to consider the case $x=y$. Let $t = e^x$; that means we want to show that

$$1 \leq \frac{2}{\sqrt{1+8t}} + \frac{1}{\sqrt{1+8/t^2}} < 2.$$  

Since this a function of one variable, we can just use standard Calculus BC methods.  \[\square\]
Example 2.10 (Vietnam 1998)
Let $x_1, x_2, \ldots, x_n$ be positive reals satisfying $\sum_{i=1}^{n} \frac{1}{1998 + x_i} = \frac{1}{1998}$. Prove
\[
\frac{\sqrt[n]{x_1 x_2 \cdots x_n}}{n - 1} \geq 1998.
\]

Proof. Let $y_i = \frac{1998}{1998 + x_i}$. Since $y_1 + y_2 + \cdots + y_n = 1$, the problem becomes
\[
\prod_{i=1}^{n} \left( \frac{1}{y_i} - 1 \right) \geq (n - 1)^n.
\]
Set $f(x) = \ln \left( \frac{1}{x} - 1 \right)$, so the inequality becomes
\[
f(y_1) + \cdots + f(y_n) \geq nf \left( \frac{1}{n} \right).
\]
We can prove that
\[
f''(y) = \frac{1 - 2y}{(y^2 - y)^2}.
\]
So $f$ has one inflection point, we can assume WLOG that $y_1 = y_2 = \ldots = y_{n-1}$. Let this common value be $t$; we only need to prove
\[
(n - 1) \ln \left( \frac{1}{t} - 1 \right) + \ln \left( \frac{1}{1 - (n - 1)t} - 1 \right) \geq n \ln(n - 1).
\]
Again, since this is a one-variable inequality, calculus methods suffice. \hfill \Box

§2.4 Practice Problems
1. Use Jensen to prove AM-GM.
2. If $a^2 + b^2 + c^2 = 1$ then $\frac{1}{a^2 + 2} + \frac{1}{b^2 + 2} + \frac{1}{c^2 + 2} \leq \frac{1}{ab + c^2} + \frac{1}{bc + a^2} + \frac{1}{ca + b^2}$.
3. If $a + b + c = 3$ then
\[
\sum_{cyc} \frac{a}{2a^2 + a + 1} \leq \frac{3}{4}.
\]
4. (MOP 2012) If $a + b + c + d = 4$, then $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} \geq a^2 + b^2 + c^2 + d^2$.

§3 Eliminating Radicals and Fractions

§3.1 Weighted Power Mean
AM-GM has the following natural generalization.

Theorem 3.1 (Weighted Power Mean)
Let $a_1, a_2, \ldots, a_n$ and $w_1, w_2, \ldots, w_n$ be positive reals with $w_1 + w_2 + \cdots + w_n = 1$. For any real number $r$, we define
\[
P(r) = \begin{cases}
(w_1 a_1^r + w_2 a_2^r + \cdots + w_n a_n^r)^{1/r} & r \neq 0 \\
\prod_{i=1}^{n} a_i^{w_i} & r = 0.
\end{cases}
\]
If $r > s$, then $P(r) \geq P(s)$ equality occurs if and only if $a_1 = a_2 = \cdots = a_n$. 

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In particular, if \(w_1 = w_2 = \cdots = w_n = \frac{1}{n}\), the above \(P(r)\) is just

\[
P(r) = \begin{cases} 
\left( \frac{a_1^r + a_2^r + \cdots + a_n^r}{n} \right)^{1/r} & r \neq 0 \\
\sqrt[n]{a_1a_2\ldots a_n} & r = 0.
\end{cases}
\]

By setting \(r = 2, 1, 0, -1\) we derive

\[
\sqrt{\frac{a_1^2 + \cdots + a_n^2}{n}} \geq \frac{a_1 + \cdots + a_n}{n} \geq \sqrt[\lambda]{a_1a_2\ldots a_n} \geq \frac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}}
\]

which is QM-AM-GM-HM. Moreover, AM-GM lets us “add” roots, like

\[
\sqrt{a} + \sqrt{b} + \sqrt{c} \leq 3\sqrt{\frac{a + b + c}{3}}.
\]

**Example 3.2 (Taiwan TST Quiz)**

Prove \(3(a + b + c) \geq 8\sqrt[3]{abc} + 3\sqrt[3]{a^3+b^3+c^3}\).

**Proof.** By Power Mean with \(r = 1, s = \frac{1}{3}, w_1 = \frac{1}{9}, w_2 = \frac{8}{9}\), we find that

\[
\left( \frac{1}{9} \frac{a^3 + b^3 + c^3}{3} + \frac{8}{9} \sqrt[3]{abc} \right)^3 \leq \frac{1}{9} \left( \frac{a^3 + b^3 + c^3}{3} \right) + \frac{8}{9} \left( abc \right).
\]

so we want to prove \(a^3 + b^3 + c^3 + 24abc \leq (a + b + c)^3\), which is clear. \(\square\)

**§3.2 Cauchy and Hölder**

**Theorem 3.3 (Hölder’s Inequality)**

Let \(\lambda_a, \lambda_b, \ldots, \lambda_z\) be positive reals with \(\lambda_a + \lambda_b + \cdots + \lambda_z = 1\). Let \(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n, \ldots, z_1, z_2, \ldots, z_n\) be positive reals. Then

\[
(a_1 + \cdots + a_n)^{\lambda_a} (b_1 + \cdots + b_n)^{\lambda_b} \ldots (z_1 + \cdots + z_n)^{\lambda_z} \geq \sum_{i=1}^{n} a_1^{\lambda_a} b_1^{\lambda_b} \ldots z_i^{\lambda_z}.
\]

Equality holds if \(a_1 : a_2 : \cdots : a_n \equiv b_1 : b_2 : \cdots : b_n \equiv \cdots \equiv z_1 : z_2 : \cdots : z_n\).

**Proof.** WLOG \(a_1 + \cdots + a_n = b_1 + \cdots + b_n = \cdots = 1\) (note that the degree of the \(a_i\) on either side is \(\lambda_a\)). In that case, the LHS of the inequality is 1, and we just note

\[
\sum_{i=1}^{n} a_1^{\lambda_a} b_1^{\lambda_b} \ldots z_i^{\lambda_z} \leq \sum_{i=1}^{n} (\lambda_a a_i + \lambda_b b_i + \ldots) = 1.
\]

If we set \(\lambda_a = \lambda_b = \frac{1}{2}\), we derive what is called the Cauchy-Schwarz inequality,

\[
(a_1 + a_2 + \cdots + a_n) (b_1 + b_2 + \cdots + b_n) \geq \left( \sqrt{a_1b_1} + \sqrt{a_2b_2} + \cdots + \sqrt{a_nb_n} \right)^2.
\]
Cauchy can be rewritten as
\[
\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} + \cdots + \frac{x_n^2}{y_n} \geq \frac{(x_1 + x_2 + \cdots + x_n)^2}{y_1 + \cdots + y_n}.
\]

This form it is often called Titu’s Lemma in the United States.

Cauchy and Hölder have at least two uses:

1. eliminating radicals,
2. eliminating fractions.

Let us look at some examples.

**Example 3.4** (IMO 2001)

Prove
\[
\sum_{\text{cyc}} \frac{a}{\sqrt{a^2 + 8bc}} \geq 1.
\]

**Proof.** By Holder
\[
\left( \sum_{\text{cyc}} a(a^2 + 8bc) \right)^{\frac{1}{3}} \left( \sum_{\text{cyc}} \frac{a}{\sqrt{a^2 + 8bc}} \right)^{\frac{2}{3}} \geq (a + b + c)
\]

So it suffices to prove \((a + b + c)^3 \geq \sum_{\text{cyc}} a(a^2 + 8bc) = a^3 + b^3 + c^3 + 24abc\). Does this look familiar? 

In this problem, we used Hölder to clear the square roots in the denominator.

**Example 3.5** (Balkan)

Prove
\[
\frac{1}{a(b+c)} + \frac{1}{b(c+a)} + \frac{1}{c(a+b)} \geq \frac{27}{2(a+b+c)^2}.
\]

**Proof.** Again by Holder,
\[
\left( \sum_{\text{cyc}} a \right)^{\frac{1}{3}} \left( \sum_{\text{cyc}} b + c \right)^{\frac{1}{3}} \left( \sum_{\text{cyc}} \frac{1}{a(b+c)} \right)^{\frac{1}{3}} \geq 1 + 1 + 1 = 3.
\]

**Example 3.6** (JMO 2012)

Prove
\[
\sum_{\text{cyc}} \frac{a^3 + 5b^3}{3a^2 + b} \geq \frac{3}{2} \left( a^2 + b^2 + c^2 \right).
\]

**Proof.** We use Cauchy (Titu) to obtain
\[
\sum_{\text{cyc}} \frac{a^3}{3a + b} = \sum_{\text{cyc}} \frac{(a^2)^2}{3a^2 + ab} \geq \frac{(a^2 + b^2 + c^2)^2}{\sum_{\text{cyc}} 3a^2 + ab}.
\]

We can easily prove this is at least \(\frac{1}{4}(a^2 + b^2 + c^2)\) (recall \(a^2 + b^2 + c^2\) is the “biggest” sum, so we knew in advance this method would work). Similarly \(\sum_{\text{cyc}} \frac{5b^3}{3a + b} \geq \frac{5}{4}(a^2 + b^2 + c^2)\). 

Example 3.7 (USA TST 2010)
If \( abc = 1 \), prove \( \frac{1}{a^3(b+2c)^2} + \frac{1}{b^3(c+2a)^2} + \frac{1}{c^3(a+2b)^2} \geq \frac{1}{3} \).

Proof. We can use Hölder to eliminate the square roots in the denominator:

\[
\left( \sum_{\text{cyc}} ab + 2ac \right)^2 \left( \sum_{\text{cyc}} \frac{1}{a^3(b+2c)^2} \right) \geq \left( \sum_{\text{cyc}} \frac{1}{a} \right)^3 \geq 3(ab + bc + ca)^2. \quad \Box
\]

§3.3 Practice Problems

1. If \( a + b + c = 1 \), then \( \sqrt{ab} + \sqrt{bc} + \sqrt{ca} + b \geq 1 + \sqrt{ab} + \sqrt{bc} + \sqrt{ca} \).

2. If \( a^2 + b^2 + c^2 = 12 \), then \( a \cdot \sqrt{b^2 + c^2} + b \cdot \sqrt{c^2 + a^2} + c \cdot \sqrt{a^2 + b^2} \leq 12 \).

3. (ISL 2004) If \( ab + bc + ca = 1 \), prove \( \sqrt[3]{\frac{1}{a^3} + 6b + \sqrt[3]{\frac{1}{b^3} + 6c + \sqrt[3]{\frac{1}{c^3} + 6a}} \leq \frac{1}{abc} \).

4. (MOP 2011) \( \sqrt{a^2 - ab + b^2 + \sqrt{c^2 - ab + c^2} + \sqrt{c^2 - ca + a^2} + 9\sqrt{abc} \leq 4(a + b + c) }\).

5. (Evan Chen) If \( a^3 + b^3 + c^3 + abc = 4 \), prove

\[
\frac{(5a^2 + bc)^2}{(a + b)(a + c)} + \frac{(5b^2 + ca)^2}{(b + c)(b + a)} + \frac{(5c^2 + ab)^2}{(c + a)(c + b)} \geq \frac{(10 - abc)^2}{a + b + c}.
\]

When does equality hold?

§4 Problems

1. (MOP 2013) If \( a + b + c = 3 \), then

\[
\sqrt{a^2 + ab + b^2 + \sqrt{b^2 + bc + c^2} + \sqrt{c^2 + ca + a^2}} \geq \sqrt{3}.
\]

2. (IMO 1995) If \( abc = 1 \), then \( \frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2} \).

3. (USA 2003) Prove \( \sum_{\text{cyc}} \frac{(2a+b+c)^2}{2a^2+(b+c)^2} \leq 8 \).

4. (Romania) Let \( x_1, x_2, \ldots, x_n \) be positive reals with \( x_1 x_2 \ldots x_n = 1 \). Prove that

\[
\sum_{i=1}^{n} \frac{1}{x_i^{1/x_i}} \leq 1.
\]

5. (USA 2004) Let \( a, b, c \) be positive reals. Prove that

\[
(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a + b + c)^3.
\]

6. (Evan Chen) Let \( a, b, c \) be positive reals satisfying \( a + b + c = \sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} \). Prove \( a^\alpha b^\beta c^\gamma \geq 1 \).