All you have to do is construct a parallelogram!

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As the name suggests, all of these problems can be solved by constructing the fourth vertex of a parallelogram. Most require just one point to be drawn in. A few will require constructing multiple points or even multiple parallelograms.

1 Samples

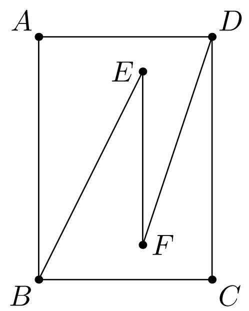
Problem 1. Let M and N be the midpoints of $\overline{AB}$ and $\overline{AC}$ in triangle ABC. Prove $MN = \frac{1}{2}BC$ without using similar triangles.

2 Appetizers

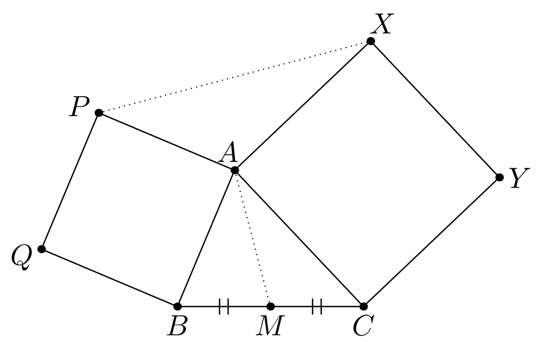
Problem 2. Let M be the midpoint of $\overline{BC}$ in a triangle ABC. Given that $AM = 2$, $AB = 3$, $AC = 4$, find the area of ABC.

Problem 3 (AIME 2011)**.** In rectangle $ABCD$, $AB = 12$ and $BC = 10$. Points E and F lie inside rectangle $ABCD$ so that $BE = 9$, $DF = 8$, $\overline{BE} \parallel \overline{DF}$, $\overline{EF} \parallel \overline{AB}$, and line BE intersects segment $\overline{AD}$. Find EF.

Problem 4. Let ABC be a triangle and M be the midpoint of $\overline{BC}$. Squares $ABQP$ and $ACYX$ are erected. Show that $PX = 2AM$.



Problem 3: AIME 2011.



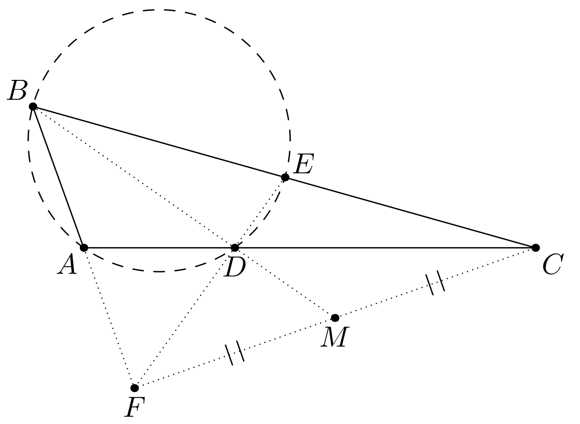
Problem 4: If $APQB$ and $AXYC$ are squares, prove $PX = 2AM$.

3 Meals

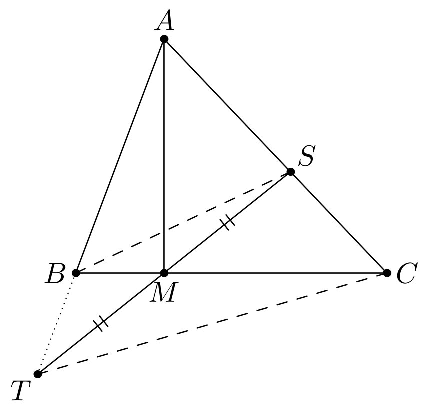
Problem 5. The area of triangle ABC is 4, and the length of the medians are m_a, m_b, and m_c. A second triangle has side lengths m_a, m_b, and m_c. What is its area?

Problem 6 (USAMO 2003)**.** Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Lines AB and DE intersect at F, while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

Problem 7 (NIMO 8.8)**.** The diagonals of convex quadrilateral $BSCT$ meet at the midpoint M of $\overline{ST}$. Lines BT and SC meet at A, and $AB = 91$, $BC = 98$, $CA = 105$. Given that $\overline{AM} \perp \overline{BC}$, find the positive difference between the areas of $\triangle SMC$ and $\triangle BMT$.

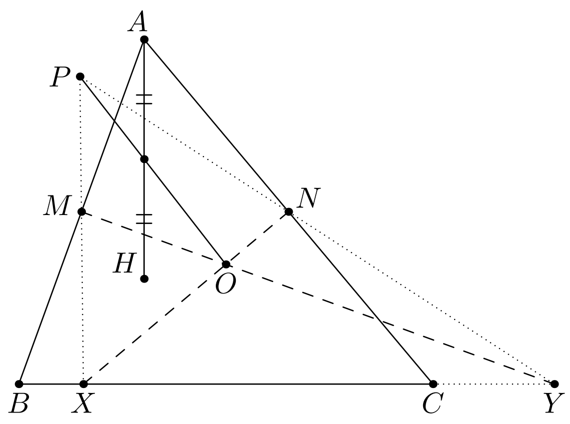


Problem 6: USAMO 2003.



Problem 7: NIMO 8.8.

Problem 8. Let ABC be a triangle with circumcenter O and orthocenter H, and let M and N be the midpoints of $\overline{AB}$ and $\overline{AC}$. Rays MO and NO meet line BC at Y and X, respectively. Lines MX and NY meet at P. Prove that $\overline{OP}$ bisects $\overline{AH}$.

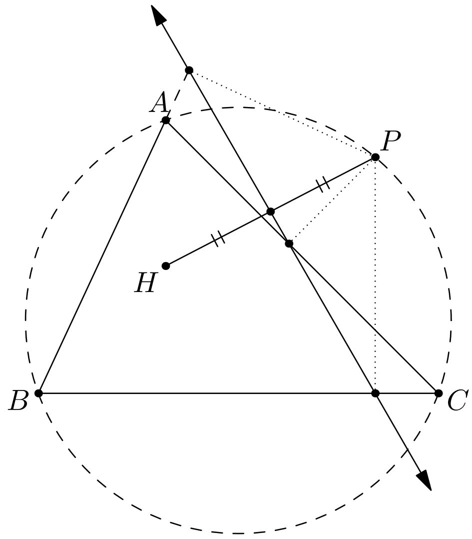


Problem 8: Show that $\overline{PO}$ bisects $\overline{AH}$.

4 Buffets

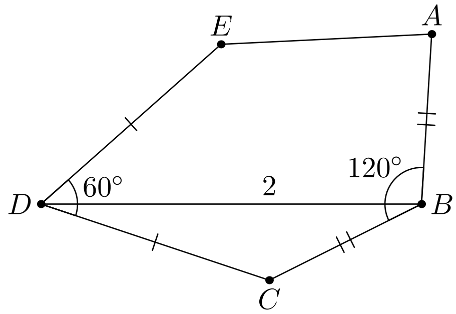
Problem 9. Let ABC be a triangle with orthocenter H and let P be a point on the circumcircle of ABC. The *Simson line* from P is the line passing through the feet of the altitudes from P to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$. Prove that it bisects $\overline{PH}$.

You may want to use this lemma: the reflection of H over $\overline{BC}$ lies on the circumcircle.



Problem 9: The Simson line bisects $\overline{PH}$.

Problem 10. Let $ABCDE$ be a convex pentagon with $AB = BC$ and $CD = DE$. If $\angle ABC = 2\angle CDE = 120^\circ$ and $BD = 2$, find the area of $ABCDE$.



Problem 10: If $\angle ABC = 2\angle CDE = 120^\circ$ and $BD = 2$, find the area of $ABCDE$.

Problem 11 (ELMO 2012)**.** Let ABC be an acute triangle with $AB < AC$, and let D and E be points on side BC such that $BD = CE$ and D lies between B and E. Suppose there exists a point P inside ABC such that $\overline{PD} \parallel \overline{AE}$ and $\angle PAB = \angle EAC$. Prove that $\angle PBA = \angle PCA$.