BMC Intermediate II: Set Theory

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This is adapted from `http://usamo.wordpress.com/tag/set-theory/`. For more on set theory, you might see my course notes from Math 145: `http://www.mit.edu/~evanchen/coursework.html`.

1 Setting Out

For those of you that don't already know: '$\forall$', '$\exists$' are shorthand for "for all" and "exists", respectively. '$\vee$' is short for "or" and '$\wedge$' is short for "and". Also, '$\in$' means "in".

Question 1. What is a set?

Axiom I (Extensionality)**.**

$$\forall A \forall B \forall x \, (x \in A \Leftrightarrow x \in B) \Rightarrow (A = B).$$

Question 2 (Russell's Paradox)**.** Define $X = \{\text{sets } S \mid S \notin S\}$. Does X contain itself?

2 Constructive ZF Axioms

Question 3. What are some ways to build sets?

Axiom II (Empty Set Exists)**.** $\exists \varnothing : \forall x \; (x \notin \varnothing)$.

Axiom III (Pairing)**.** $\forall x \forall y \exists A : \quad \forall z, \; z \in A \Leftrightarrow ((z = x) \vee (z = y))$.

Axiom IV (Union)**.** $\forall a \exists z \quad \forall x \; (x \in z) \Leftrightarrow (\exists y : x \in y \in z)$

Axiom V (Replacement)**.** Let f be a function on a set X defined by a formula in LST. Then the image of f is a function:

$$\exists Y : \quad \forall y, \; y \in Y \Leftrightarrow \exists x : f(x) = y.$$

Axiom VI (Restricted Comprehension)**.** Let ϕ be some property, and let S be a set. Then $\{x \in S \mid \phi(x)\}$ is a set.

(Actually, Comprehension follows from Replacement.)

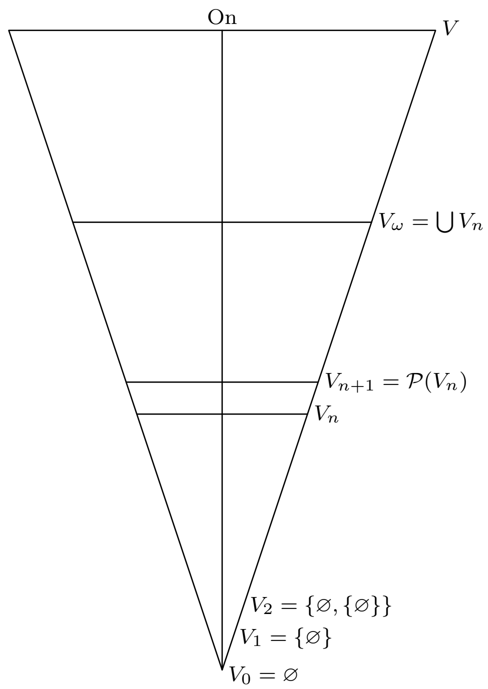
Axiom VII (Power Set)**.** We can construct the power set $\mathcal{P}(X)$. Formally,

$$\forall X \exists A : \quad \forall Y (Y \in A \Leftrightarrow Y \subseteq X)$$

where $Y \subseteq X$ is short for $\forall z (z \in Y \Rightarrow z \in X)$.

3 The von Neumann Universe

$$V_0 = \varnothing \qquad V_{n+1} = \mathcal{P}(V_n) \qquad V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$$



4 Foundation

Axiom VIII (Foundation)**.** The relation $\in$ is well-founded, meaning there are no infinite descending $\in$-chains:

$$y_0 \ni y_1 \ni y_2 \ni y_3 \ni \dots.$$

Question 4. Why can no set contain itself?

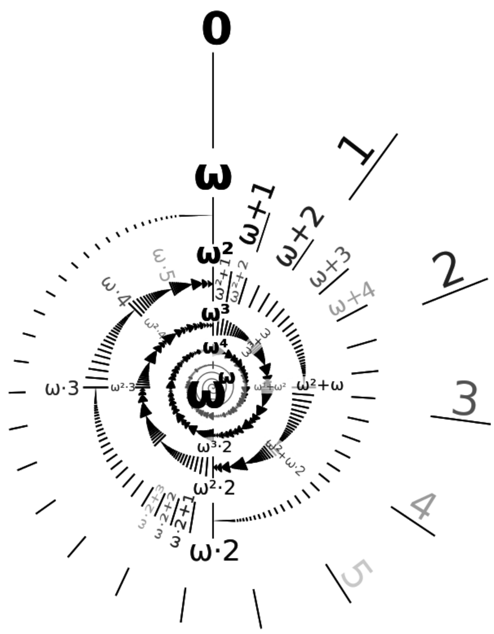
5 Encoding

- $(x, y) \stackrel{\text{def}}{=} \{\{x\}, \{x, y\}\}$.
- $\sim \stackrel{\text{def}}{=} \{(x, y) \mid x \sim y \text{ and } x, y \in X\}$.
- $f \stackrel{\text{def}}{=} \{(x, f(x)) \mid x \in X\}$.
- $0 \stackrel{\text{def}}{=} \varnothing$ and $5 \stackrel{\text{def}}{=} \{0, 1, 2, 3, 4\}$.

6 Ordinals

A set is *transitive* if $\in$ is transitive on it. We define an *ordinal* to be a transitive set which is *well-ordered* by $\in$.

Axiom IX (Infinity)**.** There exists a set $\omega = \{0, 1, 2, 3, \dots\}$.



Public domain: `http://commons.wikimedia.org/wiki/File:Omega-exp-omega-labeled.svg`

Note that there are no descending $\in$ chains. We can also distinguish between *successor ordinals* and *limit ordinals*. You can even induct on these ordinals.

Remark. Note that all the above ordinals are actually countable. In particular, there are bijections between all the infinite ordinals in the above spiral. Thus ordinals are not a great way of keeping track of "sizes". The way to fix this is to define a *cardinal* as an ordinal which is not in bijection with any smaller ordinal.

Question 5. Why do all sets appear in some V_α?

7 Ordinal Arithmetic

Addition:

$$\alpha + \beta \stackrel{\text{def}}{=} \operatorname{ot}\left(\left(\{0\} \times \alpha\right) \cup \left(\{1\} \times \beta\right), <_{\text{lex}}\right).$$

Alternatively, by transfinite induction,

$$\begin{aligned} \alpha + 0 &= \alpha \\ \alpha + (\beta + 1) &= (\alpha + \beta) + 1 \\ \alpha + \lambda &= \bigcup_{\beta < \lambda} (\alpha + \beta). \end{aligned}$$

Multiplication:

$$\alpha \cdot \beta = \operatorname{ot}\left(\beta \times \alpha, <_{\text{lex}}\right).$$

Alternatively, by transfinite induction,

$$\begin{aligned} \alpha \cdot 0 &= \alpha \\ \alpha \cdot (\beta + 1) &= (\alpha \cdot \beta) + \alpha \\ \alpha \cdot \lambda &= \bigcup_{\beta < \lambda} \alpha \cdot \beta. \end{aligned}$$

8 Cardinals

A *cardinal* is an ordinal κ which is not in bijection with any lesser ordinal. They can also be indexed by ordinals:

$$\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_\omega, \aleph_{\omega+1}, \ldots$$

- Cantor's diagonal argument states that $2^\kappa > \kappa$.
- Continuum Hypothesis: $2^{\aleph_0} = \aleph_1$.

9 The Axiom of Choice

Axiom X (Choice)**.** Given a set Y of nonempty sets, one can construction a *choice* function $f : Y \to \mathcal{P}(Y)$ such that $f(X) \in X$ for every $X \in Y$.

Problem. There are infinitely many people in a room, each wearing or red, green, or blue hat. At a signal, everyone simultaneously tries to guess the color of his or her hat. Prove that they can devise a strategy such that at most finitely many people guess incorrectly.