42^{nd} IMO Team Selection Test

Washington, D.C.

Day I 1:00 p.m. - 5:30 p.m.

June 9, 2001

1. Let $\{a_n\}_{n\geq 0}$ be a sequence of real numbers such that $a_{n+1} \geq a_n^2 + \dfrac{1}{5}$ for all $n \geq 0$. Prove that $\sqrt{a_{n+5}} \geq a_{n-5}^2$ for all $n \geq 5$.

2. Express
$$\sum_{k=0}^{n} (-1)^k (n-k)!(n+k)!$$
in closed form.

3. For a set S, let $|S|$ denote the number of elements in S. Let A be a set of positive integers with $|A| = 2001$. Prove that there exists a set B such that
   (i) $B \subseteq A$;
   (ii) $|B| \geq 668$;
   (iii) for any $u, v \in B$ (not necessarily distinct), $u + v \notin B$.

42^{nd} IMO Team Selection Test

Lincoln, Nebraska

Day II 1:00 p.m. - 5:30 p.m.

June 10, 2001

4. There are 51 senators in a senate. The senate needs to be divided into n committees so that each senator is on one committee. Each senator hates exactly three other senators. (If senator A hates senator B, then senator B does *not* necessarily hate senator A.) Find the smallest n such that it is always possible to arrange the committees so that no senator hates another senator on his or her committee.

5. In triangle ABC, $\angle B = 2\angle C$. Let P and Q be points on the perpendicular bisector of segment BC such that rays AP and AQ trisect $\angle A$. Prove that $PQ < AB$ if and only if $\angle B$ is obtuse.

6. Let a, b, c be positive real numbers such that

$$a + b + c \geq abc.$$

Prove that at least two of the inequalities

$$\frac{2}{a} + \frac{3}{b} + \frac{6}{c} \geq 6, \quad \frac{2}{b} + \frac{3}{c} + \frac{6}{a} \geq 6, \quad \frac{2}{c} + \frac{3}{a} + \frac{6}{b} \geq 6$$

are true.

42^{nd} IMO Team Selection Test

Washington, D.C.

Day III   1:00 p.m. - 5:30 p.m.

June 11, 2001

7. Let $ABCD$ be a convex quadrilateral such that $\angle ABC = \angle ADC = 135^\circ$ and

$$AC^2 \cdot BD^2 = 2AB \cdot BC \cdot CD \cdot DA.$$

Prove that the diagonals of quadrilateral $ABCD$ are perpendicular.

8. Find all pairs of nonnegative integers (m, n) such that

$$(m + n - 5)^2 = 9mn.$$

9. Let A be a finite set of positive integers. Prove that there exists a finite set B of positive integers such that $A \subseteq B$ and

$$\prod_{x \in B} x = \sum_{x \in B} x^2.$$