41^{st} IMO Team Selection Test

Lincoln, Nebraska

Day I 1:00 p.m. - 5:30 p.m.

June 10, 2002

1. Let a, b, c be nonnegative real numbers. Prove that

$$\frac{a+b+c}{3} - \sqrt[3]{abc} \leq \max\{(\sqrt{a}-\sqrt{b})^2, (\sqrt{b}-\sqrt{c})^2, (\sqrt{c}-\sqrt{a})^2\}.$$

2. Let $ABCD$ be a cyclic quadrilateral and let E and F be the feet of perpendiculars from the intersection of diagonals AC and BD to $\overline{AB}$ and $\overline{CD}$, respectively. Prove that $\overline{EF}$ is perpendicular to the line through the midpoints of $\overline{AD}$ and $\overline{BC}$.

3. Let p be a prime number. For integers r, s such that $rs(r^2 - s^2)$ is not divisible by p, let $f(r, s)$ denote the number of integers $n \in \{1, 2, \ldots, p-1\}$ such that $\{rn/p\}$ and $\{sn/p\}$ are either both less than $1/2$ or both greater than $1/2$. Prove that there exists $N > 0$ such that for $p \geq N$ and all r, s,

$$\left\lceil \frac{p-1}{3} \right\rceil \leq f(r, s) \leq \left\lfloor \frac{2(p-1)}{3} \right\rfloor.$$

41^{st} IMO Team Selection Test

Lincoln, Nebraska

Day II   1:00 p.m. - 5:30 p.m.

June 11, 2002

4. Let n be a positive integer. Prove that

$$\binom{n}{0}^{-1} + \binom{n}{1}^{-1} + \cdots + \binom{n}{n}^{-1} = \frac{n+1}{2^{n+1}} \left(\frac{2}{1} + \frac{2^2}{2} + \cdots + \frac{2^{n+1}}{n+1} \right).$$

5. Let n be a positive integer. A *corner* is a finite set S of ordered n-tuples of positive integers such that if $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ are positive integers with $a_k \geq b_k$ for $k = 1, 2, \ldots, n$ and $(a_1, a_2, \ldots, a_n) \in S$, then $(b_1, b_2, \ldots, b_n) \in S$. Prove that among any infinite collection of corners, there exist two corners, one of which is a subset of the other one.

6. Let ABC be a triangle inscribed in a circle of radius R, and let P be a point in the interior of ABC. Prove that

$$\frac{PA}{BC^2} + \frac{PB}{CA^2} + \frac{PC}{AB^2} \geq \frac{1}{R}.$$