USA TSTST 2019 Solutions

United States of America — TST Selection Test

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§1 Solutions to Day 1

§1.1 Solution to TSTST 1, by Evan Chen

Find all binary operations $\diamondsuit\colon \mathbb{R}_{>0} \times \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ (meaning $\diamondsuit$ takes pairs of positive real numbers to positive real numbers) such that for any real numbers $a, b, c > 0$,

- the equation $a \diamondsuit (b \diamondsuit c) = (a \diamondsuit b) \cdot c$ holds; and
- if $a \ge 1$ then $a \diamondsuit a \ge 1$.

The answer is only multiplication and division, which both obviously work.

We present two approaches, one appealing to theorems on Cauchy's functional equation, and one which avoids it.

First solution using Cauchy FE We prove:

Claim — We have $a \diamondsuit b = af(b)$ where f is some involutive and totally multiplicative function. (In fact, this classifies all functions satisfying the first condition completely.)

Proof. Let $P(a, b, c)$ denote the assertion $a \diamondsuit (b \diamondsuit c) = (a \diamondsuit b) \cdot c$.

- Note that for any x, the function $y \mapsto x \diamondsuit y$ is injective, because if $x \diamondsuit y_1 = x \diamondsuit y_2$ then take $P(1, x, y_i)$ to get $y_1 = y_2$.
- Take $P(1, x, 1)$ and injectivity to get $x \diamondsuit 1 = x$.
- Take $P(1, 1, y)$ to get $1 \diamondsuit (1 \diamondsuit y) = y$.
- Take $P(x, 1, 1 \diamondsuit y)$ to get
$$x \diamondsuit y = x \cdot (1 \diamondsuit y).$$

Henceforth let us define $f(y) = 1 \diamondsuit y$, so $f(1) = 1$, f is involutive and

$$x \diamondsuit y = xf(y).$$

Plugging this into the original condition now gives $f(bf(c)) = f(b)c$, which (since f is an involution) gives f completely multiplicative. □

In particular, $f(1) = 1$. We are now interested only in the second condition, which reads $f(x) \ge 1/x$ for $x \ge 1$.

Define the function
$$g(t) = \log f(e^t)$$
so that g is additive, and also $g(t) \ge -t$ for all $t \ge 0$. We appeal to the following theorem:

Lemma

If $h\colon \mathbb{R} \to \mathbb{R}$ is an additive function which is not linear, then it is *dense* in the plane: for any point (x_0, y_0) and $\varepsilon > 0$ there exists (x, y) such that $h(x) = y$ and $\sqrt{(x-x_0)^2 + (y-y_0)^2} < \varepsilon$.

Applying this lemma with the fact that $g(t) \ge -t$ implies readily that g is linear. In other words, f is of the form $f(x) = x^r$ for some fixed real number r. It is easy to check $r = \pm 1$ which finishes.

Second solution manually As before we arrive at $a \diamondsuit b = af(b)$, with f an involutive and totally multiplicative function.

We prove that:

Claim — For any $a > 0$, we have $f(a) \in \{1/a, a\}$.

Proof. WLOG $b > 1$, and suppose $f(b) = a \geq 1/b$ hence $f(a) = b$.

Assume that $ab > 1$; we show $a = b$. Note that for integers m and n with $a^n b^m \geq 1$, we must have

$$a^m b^n = f(b)^m f(a)^n = f(a^n b^m) \geq \frac{1}{a^n b^m} \implies (ab)^{m+n} \geq 1$$

and thus we have arrived at the proposition

$$m + n < 0 \implies n \log_b a + m < 0$$

for all integers m and n. Due to the density of $\mathbb{Q}$ in the real numbers, this can only happen if $\log_b a = 1$ or $a = b$. □

Claim — The function f is continuous.

Proof. Indeed, it's equivalent to show $g(t) = \log f(e^t)$ is continuous, and we have that

$$|g(t) - g(s)| = \left|\log f(e^{t-s})\right| = |t - s|$$

since $f(e^{t-s}) = e^{\pm|t-s|}$. Therefore g is Lipschitz. Hence g continuous, and f is too. □

Finally, we have from f multiplicative that

$$f(2^q) = f(2)^q$$

for every rational number q, say. As f is continuous this implies $f(x) \equiv x$ or $f(x) \equiv 1/x$ identically (depending on whether $f(2) = 2$ or $f(2) = 1/2$, respectively).

Therefore, $a \diamondsuit b = ab$ or $a \diamondsuit b = a \div b$, as needed.

Remark. The Lipschitz condition is one of several other ways to proceed. The point is that if $f(2) = 2$ (say), and $x/2^q$ is close to 1, then $f(x)/2^q = f(x/2^q)$ is close to 1, which is enough to force $f(x) = x$ rather than $f(x) = 1/x$.

Remark. Compare to AMC 10A 2016 #23, where the second condition is $a \diamondsuit a = 1$.

§1.2 Solution to TSTST 2, by Merlijn Staps

Let ABC be an acute triangle with circumcircle Ω and orthocenter H. Points D and E lie on segments AB and AC respectively, such that $AD = AE$. The lines through B and C parallel to $\overline{DE}$ intersect Ω again at P and Q, respectively. Denote by ω the circumcircle of $\triangle ADE$.

(a) Show that lines PE and QD meet on ω.

(b) Prove that if ω passes through H, then lines PD and QE meet on ω as well.

We will give one solution to (a), then several solutions to (b).

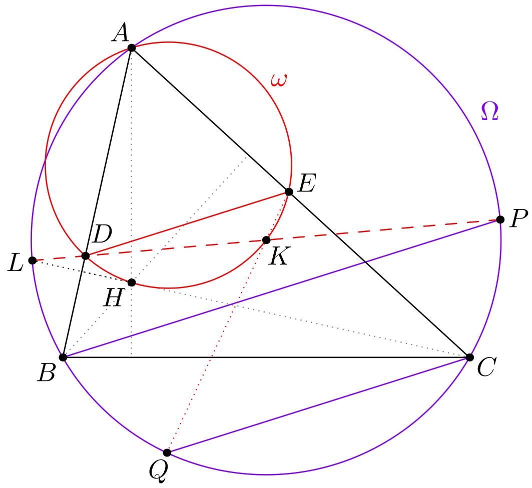
Solution to (a) Note that $\measuredangle AQP = \measuredangle ABP = \measuredangle ADE$ and $\measuredangle APQ = \measuredangle ACQ = \measuredangle AED$, so we have a spiral similarity $\triangle ADE \sim \triangle AQP$. Therefore, lines PE and QD meet at the second intersection of ω and Ω other than A.

Solution to (b) using angle chasing Let L be the reflection of H across $\overline{AB}$, which lies on Ω.

Claim — Points L, D, P are collinear.

Proof. This is just angle chasing:

$$\begin{aligned}\measuredangle CLD &= \measuredangle DHL = \measuredangle DHA + \measuredangle AHL = \measuredangle DEA + \measuredangle AHC \\ &= \measuredangle ADE + \measuredangle CBA = \measuredangle ABP + \measuredangle CBA = \measuredangle CBP = \measuredangle CLP.\end{aligned}$$ □



Now let $K \in \omega$ such that $DHKE$ is an isosceles trapezoid, i.e. $\measuredangle BAH = \measuredangle KAE$.

Claim — Points D, K, P are collinear.

Proof. Using the previous claim,

$$\measuredangle KDE = \measuredangle KAE = \measuredangle BAH = \measuredangle LAB = \measuredangle LPB = \measuredangle DPB = \measuredangle PDE.$$ □

By symmetry, $\overline{QE}$ will then pass through the same K, as needed.

> **Remark.** These two claims imply each other, so guessing one of them allows one to realize the other. It is likely the latter is easiest to guess from the diagram, since it does not need any additional points.

Solution to (b) by orthogonal circles (found by contestants) We define K as in the previous solution, but do not claim that K is the desired intersection. Instead, we note that:

> **Claim** — Point K is the orthocenter of isosceles triangle APQ.

Proof. Notice that $AH = AK$ and $BC = PQ$. Moreover from $\overline{AH} \perp \overline{BC}$ we deduce $\overline{AK} \perp \overline{PQ}$ by reflection across the angle bisector.

In light of the formula "$AH^2 = 4R^2 - a^2$", this implies the conclusion. □

Let M be the midpoint of $\overline{PQ}$. Since $\triangle APQ$ is isosceles, $\overline{AKM} \perp \overline{PQ}$ and we conclude that $MK \cdot MA = MP^2$. So the circle with diameter $\overline{PQ}$ is orthogonal to ω. Combined with (a), this implies the result by Brokard theorem.

Solution to (b) by complex numbers (Yang Liu and Michael Ma) Let M be the arc midpoint of $\widehat{BC}$. We use the standard arc midpoint configuration. We have that

$$A = a^2,\ B = b^2,\ C = c^2,\ M = -bc,\ H = a^2 + b^2 + c^2,\ P = \frac{a^2 c}{b},\ Q = \frac{a^2 b}{c},$$

where M is the arc midpoint of $\widehat{BC}$. By direct angle chasing we can verify that $\overline{MB} \parallel \overline{DH}$. Also, $D \in \overline{AB}$. Therefore, we can compute D as follows.

$$d + a^2 b^2 \bar{d} = a^2 + b^2 \text{ and } \frac{d-h}{\bar{d}-\bar{h}} = -mb^2 = b^3 c \implies d = \frac{a^2(a^2c + b^2c + c^3 - b^3)}{c(bc + a^2)}.$$

By symmetry, we have that

$$e = \frac{a^2(a^2b + bc^2 + b^3 - c^3)}{b(bc + a^2)}.$$

To finish, we want to show that the angle between $\overline{DP}$ and $\overline{EQ}$ is angle A. To show this, we compute $\frac{d-p}{e-q} \Big/ \overline{\frac{d-p}{e-q}}$. First, we compute

$$\begin{aligned} d - p &= \frac{a^2(a^2c + b^2c + c^3 - b^3)}{c(bc + a^2)} - \frac{a^2c}{b} \\ &= a^2 \left(\frac{a^2c + b^2c + c^3 - b^3}{c(bc + a^2)} - \frac{c}{b} \right) = \frac{a^2(a^2c - b^3)(b - c)}{bc(bc + a^2)}. \end{aligned}$$

By symmetry,

$$\frac{d-p}{e-q} = -\frac{a^2c - b^3}{a^2b - c^3} \implies \frac{d-p}{e-q} \Big/ \overline{\frac{d-p}{e-q}} = \frac{a^2b^3c}{a^2bc^3} = \frac{b^2}{c^2}$$

as desired.

Solution to (b) using moving points (Zack Chroman) We work in the real projective plane $\mathbb{RP}^2$, and animate C linearly on a fixed line through A.

First, define the *degree* of a moving point $(P(t) : Q(t) : R(t))$ to be the max degree of P, Q, R. Similarly we define the degree of a moving line $P(t)x + Q(t)y + R(t)z = 0$ in the same way.

Lemma

Suppose points A, B have degree d_1, d_2, and there are k values of t for which $A = B$. Then line AB has degree at most $d_1 + d_2 - k$. Similarly, if lines ℓ_1, ℓ_2 have degrees d_1, d_2, and there are k values of t for which $\ell_1 = \ell_2$, then the intersection $\ell_1 \cap \ell_2$ has degree at most $d_1 + d_2 - k$.

Proof. We show the first statement; the second follows from point-line duality. Note that the line through the points $A = (P_1(t) : Q_1(t) : R_1(t))$ and $B = (P_2(t) : Q_2(t) : R_2(t))$ is given by cross product $A \times B$; that is, the line

$$(Q_1R_2 - Q_2R_1)x + (R_1P_2 - R_2P_1)y + (P_1Q_2 - P_2Q_1)z = 0.$$

Clearly A and B lie on this line, so it is line AB. Then for every value t_0 for which $A = B$, $(t - t_0)$ factors out of each term. So the degree of the line is at most $d_1 + d_2 - k$. □

Now, note that H moves linearly in C on line BH. Furthermore, angles $\angle AHE$, $\angle AHF$ are fixed, we get that D and E have degree 2. One way to see this is using the lemma; D lies on line AB, which is fixed, and line HD passes through a point at infinity which is a constant rotation of the point at infinity on line AH, and therefore has degree 1. Then D, E have degree at most $1 + 1 - 0 = 2$.

Now, note that P, Q move linearly in C. Both of these are because the circumcenter O moves linearly in C, and P, Q are reflections of B, C in a line through O with fixed direction, which also moves linearly.

So by the lemma, the lines PD, QE have degree at most 3. I claim they actually have degree 2; to show this it suffices to give an example of a choice of C for which $P = D$ and one for which $Q = E$. But an easy angle chase shows that in the unique case when $P = B$, we get $D = B$ as well and thus $P = D$. Similarly when $Q = C$, $E = C$. It follows from the lemma that lines PD, QE have degree at most 2.

Let ℓ_∞ denote the line at infinity. I claim that the points $P_1 = PD \cap \ell_\infty$, $P_2 = QE \cap \ell_\infty$ are projective in C. Since ℓ_∞ is fixed, it suffices to show by the lemma that there exists some value of C for which $QE = \ell_\infty$ and $PD = \ell_\infty$. But note that as $C \to \infty$, all four points P, D, Q, E go to infinity. It follows that P_1, P_2 are projective in C.

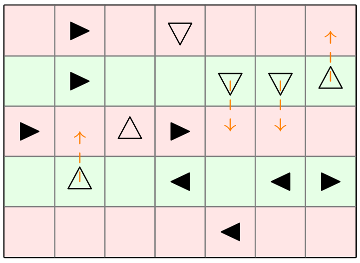
Then to finish, recall that we want to show that $\angle(PD, QE)$ is constant. It suffices then to show that there's a constant rotation sending P_1 to P_2. Since P_1, P_2 are projective, it suffices to verify this for 3 values of C.

We can take C such that $\angle ABC = 90$, $\angle ACB = 90$, or $AB = AC$, and all three cases are easy to check.

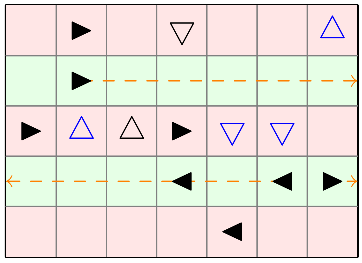
§1.3 Solution to TSTST 3, by Nikolai Beluhov

On an infinite square grid we place finitely many *cars*, which each occupy a single cell and face in one of the four cardinal directions. Cars may never occupy the same cell. It is given that the cell immediately in front of each car is empty, and moreover no two cars face towards each other (no right-facing car is to the left of a left-facing car within a row, etc.). In a *move*, one chooses a car and shifts it one cell forward to a vacant cell. Prove that there exists an infinite sequence of valid moves using each car infinitely many times.

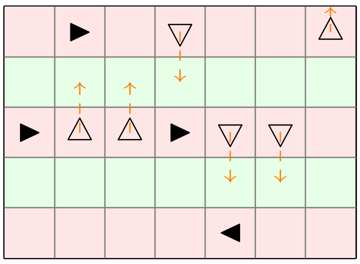
Let S be any rectangle containing all the cars. Partition S into horizontal strips of height 1, and color them red and green in an alternating fashion. It is enough to prove all the cars may exit S.



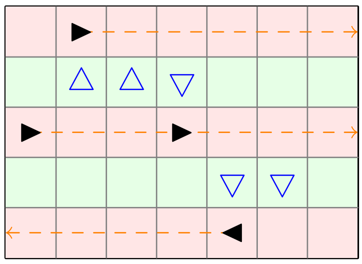
Step 1



Step 2



Step 3



Step 4

To do so, we outline a five-stage plan for the cars.

1. All vertical cars in a green cell may advance one cell into a red cell (or exit S altogether), by the given condition. (This is the only place where the hypothesis about empty space is used!)

2. All horizontal cars on green cells may exit S, as no vertical cars occupy green cells.

3. All vertical cars in a red cell may advance one cell into a green cell (or exit S altogether), as all green cells are empty.

4. All horizontal cars within red cells may exit S, as no vertical car occupy red cells.

5. The remaining cars exit S, as they are all vertical. The solution is complete.

Remark (Author's comments)**.** The solution I've given for this problem is so short and simple that it might appear at first to be about IMO 1 difficulty. I don't believe that's true! There are very many approaches that look perfectly plausible at first, and then fall apart in this or that twisted special case.

> **Remark** (Higher-dimensional generalization by author)**.** The natural higher-dimensional generalization is true, and can be proved in largely the same fashion. For example, in three dimensions, one may let S be a rectangular prism and partition S into horizontal slabs and color them red and green in an alternating fashion. Stages 1, 3, and 5 generalize immediately, and stages 2 and 4 reduce to an application of the two-dimensional problem. In the same way, the general problem is handled by induction on the dimension.

> **Remark** (Historical comments)**.** For $k > 1$, we could consider a variant of the problem where cars are $1 \times k$ rectangles (moving parallel to the longer edge) instead of occupying single cells. In that case, if there are $2k - 1$ empty spaces in front of each car, the above proof works (with the red and green strips having height k instead). On the other hand, at least k empty spaces are necessary. We don't know the best constant in this case.

§2 Solutions to Day 2

§2.1 Solution to TSTST 4, by Merlijn Staps

Consider coins with positive real denominations not exceeding 1. Find the smallest $C > 0$ such that the following holds: if we are given any 100 such coins with total value 50, then we can always split them into two stacks of 50 coins each such that the absolute difference between the total values of the two stacks is at most C.

The answer is $C = \frac{50}{51}$. The lower bound is obtained if we have 51 coins of value $\frac{1}{51}$ and 49 coins of value 1. We now present two (similar) proofs that this $C = \frac{50}{51}$ suffices.

First proof (original) Let $a_1 \le \dots \le a_{100}$ denote the values of the coins in ascending order. Since the 51 coins $a_{50}, \dots, a_{100}$ are worth at least $51a_{50}$, it follows that $a_{50} \le \frac{50}{51}$; likewise $a_{51} \ge \frac{1}{51}$.

We claim that choosing the stacks with coin values

$$a_1, a_3, \dots, a_{49}, \quad a_{52}, a_{54}, \dots, a_{100}$$

and

$$a_2, a_4, \dots, a_{50}, \quad a_{51}, a_{53}, \dots, a_{99}$$

works. Let D denote the (possibly negative) difference between the two total values. Then

$$\begin{aligned} D &= (a_1 - a_2) + \dots + (a_{49} - a_{50}) - a_{51} + (a_{52} - a_{53}) + \dots + (a_{98} - a_{99}) + a_{100} \\ &\le 25 \cdot 0 - \frac{1}{51} + 24 \cdot 0 + 1 = \frac{50}{51}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} D &= a_1 + (a_3 - a_2) + \dots + (a_{49} - a_{48}) - a_{50} + (a_{52} - a_{51}) + \dots + (a_{100} - a_{99}) \\ &\ge 0 + 24 \cdot 0 - \frac{50}{51} + 25 \cdot 0 = -\frac{50}{51}. \end{aligned}$$

It follows that $|D| \le \frac{50}{51}$, as required.

Second proof (Evan Chen) Again we sort the coins in increasing order $0 < a_1 \le a_2 \le \dots \le a_{100} \le 1$. A *large gap* is an index $i \ge 2$ such that $a_i > a_{i-1} + \frac{50}{51}$; obviously there is at most one such large gap.

Claim — If there is a large gap, it must be $a_{51} > a_{50} + \frac{50}{51}$.

Proof. If $i < 50$ then we get $a_{50}, \dots, a_{100} > \frac{50}{51}$ and the sum $\sum_1^{100} a_i > 50$ is too large. Conversely if $i > 50$ then we get $a_1, \dots, a_{i-1} < \frac{1}{51}$ and the sum $\sum_1^{100} a_i < 1/51 \cdot 51 + 49$ is too small. □

Now imagine starting with the coins a_1, a_3, ..., a_{99}, which have total value $S \le 25$. We replace a_1 by a_2, then a_3 by a_4, and so on, until we replace a_{99} by a_{100}. At the end of the process we have $S \ge 25$. Moreover, since we did not cross a large gap at any point, the quantity S changed by at most $C = \frac{50}{51}$ at each step. So at some point in the process we need to have $25 - C/2 \le S \le 25 + C/2$, which proves C works.

§2.2 Solution to TSTST 5, by Gunmay Handa

Let ABC be an acute triangle with orthocenter H and circumcircle Γ. A line through H intersects segments AB and AC at E and F, respectively. Let K be the circumcenter of $\triangle AEF$, and suppose line AK intersects Γ again at a point D. Prove that line HK and the line through D perpendicular to $\overline{BC}$ meet on Γ.

We present several solutions.

First solution (Andrew Gu) We begin with the following two observations.

Claim — Point K lies on the radical axis of (BEH) and (CFH).

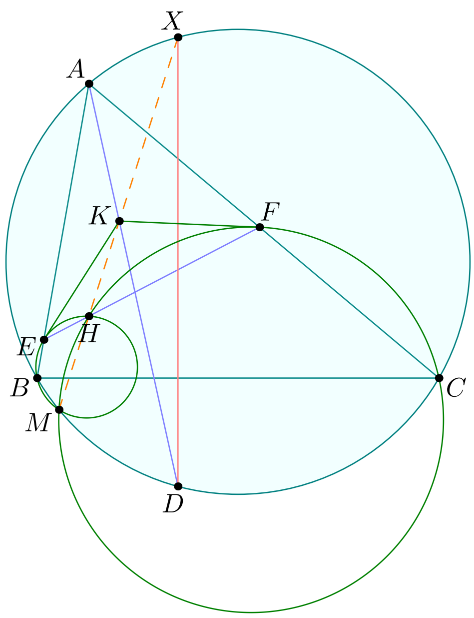
Proof. Actually we claim $\overline{KE}$ and $\overline{KF}$ are tangents. Indeed,

$$\measuredangle HEK = 90^\circ - \measuredangle EAF = 90^\circ - \measuredangle BAC = \measuredangle HBE$$

implying the result. Since $KE = KF$, this implies the result. □

Claim — The second intersection M of (BEH) and (CFH) lies on Γ.

Proof. By Miquel's theorem on $\triangle AEF$ with $H \in \overline{EF}$, $B \in \overline{AE}$, $C \in \overline{AF}$. □



In particular, M, H, K are collinear. Let X be on Γ with $\overline{DX} \perp \overline{BC}$; we then wish to show X lies on the line MHK we found. This is angle chasing: compute

$$\begin{aligned} \measuredangle XMB = \measuredangle XDB &= 90^\circ - \measuredangle DBC = 90^\circ - \measuredangle DAC \\ &= 90^\circ - \measuredangle KAF = \measuredangle FEA = \measuredangle HEB = \measuredangle HMB \end{aligned}$$

as needed.

Second solution (Ankan Bhattacharya) We let D' be the second intersection of $\overline{EF}$ with (BHC) and redefine D as the reflection of D' across $\overline{BC}$. We will first prove that this point D coincides with the point D given in the problem statement. The idea is that:

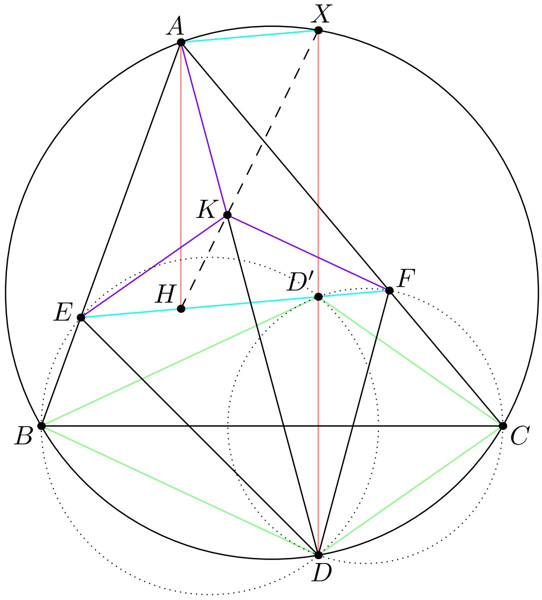
Claim — A is the D-excenter of $\triangle DEF$.

Proof. We contend $BED'D$ is cyclic. This follows by angle chasing:

$$\begin{aligned}\measuredangle D'DB = \measuredangle BD'D &= \measuredangle D'BC + 90^\circ = \measuredangle D'HC + 90^\circ \\ &= \measuredangle D'HC + \measuredangle(HC, AB) = \measuredangle(D'H, AB) = \measuredangle D'EB.\end{aligned}$$

Now as $BD = BD'$, we obtain $\overline{BEA}$ externally bisects $\angle DED' \cong \angle DEF$. Likewise $\overline{FA}$ externally bisects $\angle DFE$, so A is the D-excenter of $\triangle DEF$. $\square$

Hence, by the so-called "Fact 5", point K lies on $\overline{DA}$, so this point D is the one given in the problem statement.



Now choose point X on (ABC) satisfying $\overline{DX} \perp \overline{BC}$.

Claim — Point K lies on line HX.

Proof. Clearly $AHD'X$ is a parallelogram. By Ptolemy on $DEKF$,

$$\frac{KD}{KA} = \frac{KD}{KE} = \frac{DE + DF}{EF}.$$

On the other hand, if we let r_D denote the D-exradius of $\triangle DEF$ then

$$\frac{XD}{XD'} = \frac{[DEX] + [DFX]}{[XEF]} = \frac{[DEX] + [DFX]}{[AEF]} = \frac{DE \cdot r_D + DF \cdot r_D}{EF \cdot r_D} = \frac{DE + DF}{EF}.$$

Thus

$$[AKX] = \frac{KA}{KD} \cdot [DKX] = \frac{KA}{KD} \cdot \frac{XD}{XD'} \cdot [KD'X] = [D'KX].$$

This is sufficient to prove K lies on $\overline{HX}$. $\square$

The solution is complete: X is the desired concurrency point.

Third solution (Nikolai Beluhov, unedited) We are going to prove the following:

> Let ABC be a triangle with orthocenter H and circumcircle Γ. Let D be any point on arc BC of Γ that does not contain A. Let J lie on Γ so that line DJ is perpendicular to BC. Let lines AD and HJ meet at K. Let L be such that K is the midpoint of segment AL. Let E and F be the projections of L onto lines AB and AC, respectively. Then H lies on line EF.

This is the converse of the problem statement; clearly, if we prove this, then all is well. Let lines BC and DJ meet at M.

Claim — Point L lies on HM.

Proof. Let G be the midpoint of segment AH, let O be the circumcenter of triangle ABC, and let N be the projection of O onto line DJ. Then N is the midpoint of segment DJ, so K lies on GN. Also GH equals the distance from O to line BC, which equals MN; thus GN is parallel to HM. It follows that GK is parallel to both HL and HM. □

Let P and Q be the projections of D onto lines AB and AC, respectively. Let ℓ, the line through P, M, Q be the Simson line of D with respect to triangle ABC. Suppose ℓ meets line AH at R.

Claim — We have $DM = HR$.

Proof. This is a known property of the Simson line ℓ (that $DMHR$ is in fact a parallelogram as ℓ bisects $\overline{HD}$). □

Claim — Figures $ALEFH$ and $ADPQR$ are homothetic with center A.

Proof. All that we need to do to establish this is to verify that $AL : LD = AH : HR$. This is true by $AL : LD = AH : DM = AH : HR$. □

By the final claim, since R lies on line PQ, we get that H lies on line EF. This completes the solution.

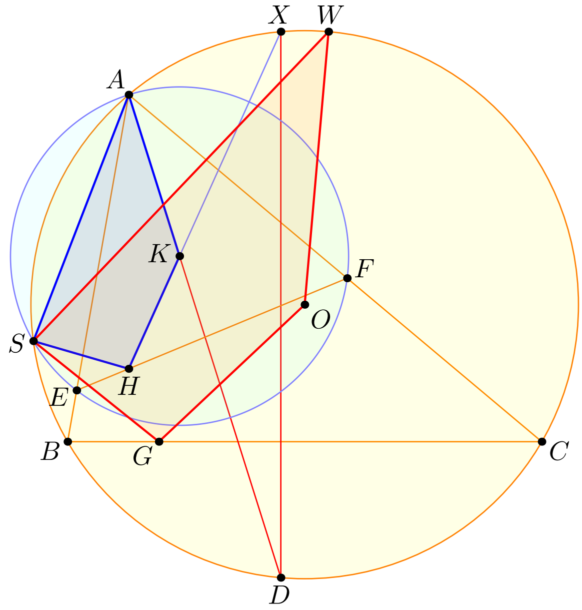
Fourth solution, complex numbers with spiral similarity (Evan Chen) First if $\overline{AD} \perp \overline{BC}$ there is nothing to prove, so we assume this is not the case. Let W be the antipode of D. Let S denote the second intersection of (AEF) and (ABC). Consider the spiral similarity sending $\triangle SEF$ to $\triangle SBC$:

- It maps H to a point G on line BC,
- It maps K to O.
- It maps the A-antipode of $\triangle AEF$ to D.
- Hence (by previous two observations) it maps A to W.
- Also, the image of line AD is line WO, which does not coincide with line BC (as O does not lie on line BC).

Therefore, K is the *unique* point on line $\overline{AD}$ for one can get a direct similarity

$$\triangle AKH \sim \triangle WOG \qquad (\heartsuit)$$

for some point G lying on line $\overline{BC}$.



On the other hand, let us re-define K as $\overline{XH} \cap \overline{AD}$. We will show that the corresponding G making ($\heartsuit$) true lies on line BC.

We apply complex numbers with Γ the unit circle, with a, b, c, d taking their usual meanings, $H = a+b+c$, $X = -bc/d$, and $W = -d$. Then point K is supposed to satisfy

$$\begin{aligned} k + ad\overline{k} &= a + d \\ \frac{k + \frac{bc}{d}}{a+b+c+\frac{bc}{d}} &= \frac{\overline{k} + \frac{d}{bc}}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{d}{bc}} \\ \iff \frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{d}{bc}}{a+b+c+\frac{bc}{d}}\left(k+\frac{bc}{d}\right) &= \overline{k} + \frac{d}{bc} \end{aligned}$$

Adding ad times the last line to the first line and cancelling $ad\overline{k}$ now gives

$$\left(ad \cdot \frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{d}{bc}}{a+b+c+\frac{bc}{d}} + 1\right) k = a + d + \frac{ad^2}{bc} - abc \cdot \frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{d}{bc}}{a+b+c+\frac{bc}{d}}$$

or

$$\begin{aligned} \left(ad\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{d}{bc}\right) + a+b+c+\frac{bc}{d}\right)k &= \left(a+b+c+\frac{bc}{d}\right)\left(a+d+\frac{ad^2}{bc}\right) \\ &\qquad - abc \cdot \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{d}{bc}\right). \end{aligned}$$

We begin by simplifying the coefficient of k:

$$\begin{aligned} ad\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{d}{bc}\right) + a+b+c+\frac{bc}{d} &= a+b+c+d+\frac{bc}{d}+\frac{ad}{b}+\frac{ad}{c}+\frac{ad^2}{bc} \\ &= a + \frac{bc}{d} + \left(1+\frac{ad}{bc}\right)(b+c+d) \\ &= \frac{ad+bc}{bcd}\left[bc+d(b+c+d)\right] \\ &= \frac{(ad+bc)(d+b)(d+c)}{bcd}. \end{aligned}$$

Meanwhile, the right-hand side expands to

$$\begin{aligned}
\text{RHS} &= \left(a+b+c+\frac{bc}{d}\right)\left(a+d+\frac{ad^2}{bc}\right) - abc\cdot\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{d}{bc}\right) \\
&= \left(a^2+ab+ac+\frac{abc}{d}\right) + (da+db+dc+bc) \\
&\quad + \left(\frac{a^2d^2}{bc}+\frac{ad^2}{c}+\frac{ad^2}{b}+ad\right) - (ab+bc+ca+ad) \\
&= a^2 + d(a+b+c) + \frac{abc}{d} + \frac{a^2d^2}{bc} + \frac{ad^2}{b} + \frac{ad^2}{c} \\
&= a^2 + \frac{abc}{d} + d(a+b+c)\cdot\frac{ad+bc}{bc} \\
&= \frac{ad+bc}{bcd}\left[abc + d^2(a+b+c)\right].
\end{aligned}$$

Therefore, we get

$$k = \frac{abc+d^2(a+b+c)}{(d+b)(d+c)}.$$

In particular,

$$\begin{aligned}
k-a &= \frac{abc+d^2(a+b+c)-a(d+b)(d+c)}{(d+b)(d+c)} \\
&= \frac{d^2(b+c)-da(b+c)}{(d+b)(d+c)} = \frac{d(b+c)(d-a)}{(d+b)(d+c)}.
\end{aligned}$$

Now the corresponding point G obeying ($\heartsuit$) satisfies

$$\begin{aligned}
\frac{g-(-d)}{0-(-d)} &= \frac{(a+b+c)-a}{k-a} \\
\implies g &= -d + \frac{d(b+c)}{k-a} \\
&= -d + \frac{(d+b)(d+c)}{d-a} = \frac{db+dc+bc+ad}{d-a}. \\
\implies bc\overline{g} &= \frac{bc\cdot\frac{ac+ab+ad+bc}{abcd}}{\frac{a-d}{ad}} = -\frac{ab+ac+ad+bc}{d-a}. \\
\implies g + bc\overline{g} &= \frac{(d-a)(b+c)}{d-a} = b+c.
\end{aligned}$$

Hence G lies on BC and this completes the proof.

Fifth solution by trigonometry (Ivan Borsenco, unedited) Let $\angle E = B - \theta$ and $\angle F = C - \theta$. Denote by H_a the intersection of AH with Γ and by D' the intersection of the line passing through D and perpendicular to BC with EF.

By angle-chasing, we get $\angle BAH = 90^\circ - B$, $\angle AHE = \angle H_aHD' = 90^\circ + \theta$. On the other hand, $\angle ADH_a = \angle ACH_a = 90^\circ - B + C$, $\angle AKB = 2(C+\theta)$, $\angle H_aAK = \angle HAK = \angle EAK - \angle EAH = 90^\circ - (C+\theta) - (90^\circ - B) = B - C - \theta$, and therefore $\angle AH_aD = 90^\circ + \theta$. Hence HH_aDD' is an isosceles trapezoid. Point D' is the reflection of D in BC, which implies that quadrilateral $BHD'C$ is cyclic, because $\angle BHC = \angle BD'C = 180^\circ - A$.

Choose point X on Γ satisfying $DX \perp BC$. Note that AH_aDX is an isosceles trapezoid. Hence $\angle HAX = 90^\circ + \theta$ and $\angle KAX = \angle HAX - \angle HAK = 90^\circ - (B - C - 2\theta)$. Denote by R and R' the circumradii of triangles ABC and AEF, respectively. It follows that $AH = 2R\cos A$, $AK = R'$, $AX = DH_a = 2R\sin(B-C-\theta)$.

In order to show that points H, K, and X are collinear, we will show that $[HAX] = [HAK] + [KAX]$, which is equivalent to

$$AH \cdot AX \cdot \sin(\angle HAX) = AK\left[AH \cdot \sin(\angle HAK) + AX \cdot \sin(\angle KAX)\right].$$

Using the Law of Sines in triangles AEH and AFH, we get

$$EF = EH + HF = AH \cdot \frac{\cos B}{\sin(B-\theta)} + AH \cdot \frac{\cos C}{\sin(C+\theta)}.$$

yielding

$$\begin{aligned} 2R' \sin A &= \frac{2R\cos A}{\sin(B-\theta)\sin(C+\theta)} \left(\cos B \sin(C+\theta) + \cos C \sin(B-\theta)\right) \\ &= \frac{2R\cos A}{\sin(B-\theta)\sin(C+\theta)} \cdot \cos\theta \cdot \sin(B+C). \end{aligned}$$

Using the fact that $2\sin(B-\theta)\sin(C+\theta) = \cos((B-\theta)-(C+\theta)) - \cos((B-\theta)+(C+\theta))$, we conclude that

$$R' = 2R \cdot \frac{\cos A \cos\theta}{\cos(B-C-2\theta) + \cos A}.$$

Denote by $\varphi = B - C - 2\theta$, then

$$AH = 2R\cos A, \quad AK = 2R \cdot \frac{\cos A \cos\theta}{\cos\varphi + \cos A}, \quad AX = 2R\sin(\varphi+\theta),$$

and

$$\angle HAK = \varphi + \theta, \quad \angle KAX = 90^\circ - \varphi, \quad \angle HAX = 90^\circ + \theta.$$

Returning back to proving the identity for areas, we have to show that

$$\cos A \cdot \sin(\varphi+\theta) \cdot \cos\theta = \cdot \frac{\cos A\cos\theta}{\cos\varphi + \cos A} \cdot \left[\cos A \cdot \sin(\varphi+\theta) + \sin(\varphi+\theta)\cdot\cos\varphi\right],$$

which is clearly true.

Sixth solution by moving points (Anant Mudgal, unedited) The meat of this solution is the following claim.

> **Claim** — In triangle AEF, with circumcenter K point H lies on $\overline{EF}$, points B and C lie on lines $\overline{AE}$ and $\overline{AF}$ respectively, such that $\overline{BH} \perp \overline{AF}$ and $\overline{CH} \perp \overline{AE}$. Line $\overline{AK}$ meets $\odot(KEF)$ again at point D. Then $ABCD$ is cyclic and reflection of D in $\overline{BC}$ lies on $\overline{EF}$.

Proof. Move H along $\overline{EF}$ and note that $B \mapsto H$ and $H \mapsto C$ are linear maps, hence $B \mapsto C$ is also linear. Suppose $\odot(DAB)$ meets line $\overline{AF}$ at C'. Then we need to show that $C = C'$. Since, by spiral similarity, $B \mapsto C'$ is linear; we need to check this for two choices of H.

- $H = E$. Then $B = E$ and we need to show that if $\odot(AED)$ meets $\overline{AF}$ at F', then $\angle AEF' = 90^\circ$. Apply inversion at A of radius $\sqrt{AE \cdot AF}$ followed by reflection in the bisector of angle EAF. Suppose $X \mapsto X^*$ under this transformation. Then $E^* = F, F^* = E$ and D^* is the orthocenter of $\triangle AEF$, so $(F')^* = \overline{FD^*} \cap \overline{AE}$ hence $\angle A(F')^*F = 90^\circ$ so $\angle AEF' = 90^\circ$, and we're done.

- $H = F$. Same proof as above works.

Finally, moving H, since $\triangle DBC$ has fixed shape, so the locus of the reflection of D in $\overline{BC}$ is a line.

- For $H = E$, we need to show that $\angle AED = 180^\circ - \angle AEF$ since $\angle FEF' = 90^\circ - \angle AEF$; this follows since $\angle AED = \angle AD^*F$ and D^* is the orthocenter of $\triangle AEF$.

- Similarly, $H = F$ case holds.

The lemma is proved. □

Now we go back to the original problem. Let L be the reflection of A in K and $N = \overline{EF} \cap \overline{AK}$, then, by our lemma, we have $(AL; ND) = -1$.

Suppose P lies on $\overline{EF}$ such that $\overline{DP} \perp \overline{BC}$ and $\overline{DP}$ meets $\overline{BC}$ at S and Γ again at Q. Reflect Q in S to get R. By the lemma, S is the midpoint of $\overline{DP}$. Let $S' = \overline{HL} \cap \overline{DP}$ and $Q' = \overline{HK} \cap \overline{DP}$.

Observe that $-1 = (AL; ND) \overset{H}{=} (\infty s', PD)$, so clearly, $\overline{HL}$ bisects $\overline{DP}$, so H, L, S are collinear. Finally, since $\overline{AK} \parallel \overline{RH}$ so $-1 = (AL; K\infty) \overset{H}{=} (\infty S; Q'R)$ so $\overline{HK}$ passes through Q, as desired.

Seventh solution using moving points (Zack Chroman) We state the converse of the problem as follows:

> Take a point D on Γ, and let $G \in \Gamma$ such that $\overline{DG} \perp \overline{BC}$. Then define K to lie on $\overline{GH}, \overline{AD}$, and take $L \in \overline{AD}$ such that K is the midpoint of $\overline{AL}$. Then if we define E and F as the projections of L onto $\overline{AB}$ and $\overline{AC}$ we want to show that E, H, F are collinear.

It's clear that solving this problem will solve the original. In fact we will show later that each line EF through H corresponds bijectively to the point D.

We work in the real projective plane $\mathbb{RP}^2$, and animate D on Γ. The point D has projective coordinates which are each quadratic polynomials in a real parameter t, and moves projectively on (ABC). We will state and prove some quick facts about animation. First, define the **degree** of a moving point $(P(t) : Q(t) : R(t))$ to be the max degree of P, Q, R. Similarly we define the degree of a moving line $P(t)x + Q(t)y + R(t)z = 0$ in the same way.

Lemma

Suppose points A, B have degree d_1, d_2, and there are k values of t for which $A = B$. Then line AB has degree at most $d_1 + d_2 - k$. Similarly, if lines ℓ_1, ℓ_2 have degrees d_1, d_2, and there are k values of t for which $\ell_1 = \ell_2$, then the intersection $\ell_1 \cap \ell_2$ has degree at most $d_1 + d_2 - k$.

Proof. We show the first statement; the second follows from point-line duality. Note that the line through the points $A = (P_1(t) : Q_1(t) : R_1(t))$ and $B = (P_2(t) : Q_2(t) : R_2(t))$ is given by cross product $A \times B$; that is, the line

$$(Q_1R_2 - Q_2R_1)x + (R_1P_2 - R_2P_1)y + (P_1Q_2 - P_2Q_1)z = 0.$$

Clearly A and B lie on this line, so it is line AB. Then for every value t_0 for which $A = B$, $(t - t_0)$ factors out of each term. So the degree of the line is at most $d_1 + d_2 - k$. □

Now, note that G is projective in D since it's a projection through the point at infinity on line AH. Now by the lemma, line HG has degree at most 2, and line AD has degree at most 1.

So by the lemma again, the point K has degree at most 3. However, note that when D lies on line AH, we have $G = A$, so lines HG and AD are the same. It follows that the point K actually has degree at most 2, thus so does L.

Let P_C be the point at infinity on the line perpendicular to AC, and similarly P_B. Then

$$F = \overline{AC} \cap \overline{P_C L}, \quad E = \overline{AB} \cap \overline{P_B L},$$

so E and F have degree at most 2, since lines AB and AC are fixed and $\deg(P_B L) \le \deg(P_B) + \deg(L) = 2$. In fact, note that if we can show that P_B, P_C lie on the locus of L, we'll show that E and F move with degree 1 (i.e. projectively) by the lemma again. To show that, we consider the case where L and K lie at infinity; that is, $\overline{HG} \parallel \overline{AD}$. In this case, $ADGH$ is a parallelogram as $AH \parallel DG$. Clearly $G = B$ and $G = C$ work; when $G = B$, D is the antipode of C in Γ.

Then, when $G = B$, we have $K = L$ is the point at infinity on line $\overline{GH} \equiv \overline{BH}$. This point is P_C, so we get that E, F are projective.

So it suffices to verify the problem for three distinct choices of D.

- If $D = A$, then line GH is line AH, and $L = \overline{AD} \cap \overline{AH} = A$. So $E = F = A$ and the statement is true.

- If $D = B$, G is the antipode of C on Γ. Then $K = \overline{HG} \cap \overline{AD}$ is the midpoint of $\overline{AB}$, so $L = B$. Then $E = B$ and F is the projection of B onto AC, so E, H, F collinear.

- We finish similarly when $D = C$.

Thus since the maps $D \mapsto E$ and $D \mapsto F$ are collinear, the map $E \mapsto F$ is projective as well. Since E, H, F are collinear for three values of E, they are in general. Moreover, since $D \to E$ is bijective, any line through H will correspond to some D, so we've solved the original problem as well.

Eighth solution by author using circumhyperbolas (Gunmay Handa, unedited) Let P be an arbitrary point on $\odot(ABC)$ with N as the midpoint of $\overline{HP}$, and define $\mathcal{H}_P = ABCHP$ as the rectangular circumhyperbola with center N passing through the aforementioned points. Moreover, define $D' \in \odot(ABC)$ with $\overline{PD'} \perp \overline{BC}$ and $P \neq D'$; observe that the line ℓ_P through O perpendicular to $\overline{AD'}$ is the isogonal conjugate of $\mathcal{H}_P$ with respect to $\odot(ABC)$, and so if we define U and V as the intersections of ℓ_P with $\overline{AB}$ and $\overline{AC}$, respectively, then N belongs to the pedal circles ω_U and ω_V of U and V with respect to $\triangle ABC$.

Let $\triangle RST$ be the orthic triangle of $\triangle ABC$ and M be the midpoint of $\overline{AH}$; angle chasing establishes that if $\{Q, N\} \equiv \omega_U \cap \omega_V$, then $Q \in \odot(ABC)$, and moreover $H \in \overline{QN}$ since it has equal power with respect to these circles. Suppose the line through H parallel to $\overline{AP}$ intersects $\overline{AB}$ and $\overline{AC}$ at E' and F', respectively, and observe that $\overline{E'F'}$ is antiparallel to $\overline{UV}$ in $\angle A$. If K' is the orthocenter of $\triangle AUV$, then $K' \in \overline{QN}$ by radical axes, and moreover $K' \in \odot(UD'V)$ since D' is the reflection of A across $\overline{UV}$. Further angle chasing establishes $Q \in \odot(UD'V)$; we now claim that $E', F' \in \odot(UD'V)$ as well. Suppose the line through N parallel to $\overline{AP}$ intersects $\overline{AB}$ at W, so that since $\triangle AST \cup \overline{MW} \sim \triangle ABC \cup \overline{OV}$, we have that $AK' \cdot AD'/2 = AW \cdot AU = AE'/2 \cdot AU$, and so $E', F' \in \odot(UD'V)$ as well. Finally, since $\angle EUK = \angle FVK = 90^\circ - \angle A$, we know that $\overline{DK}$ bisects $\angle EDF$, which implies that K' is the circumcenter of $\odot(AE'F')$ since $\overline{AK'} \perp \overline{UV}$ and lines $E'F'$ and UV are isogonal in $\angle A$, which finishes the problem.

§2.3 Solution to TSTST 6, by Nikolai Beluhov

Suppose P is a polynomial with integer coefficients such that for every positive integer n, the sum of the decimal digits of $|P(n)|$ is not a Fibonacci number. Must P be constant?

(A *Fibonacci number* is an element of the sequence F_0, F_1, … defined recursively by $F_0 = 0$, $F_1 = 1$, and $F_{k+2} = F_{k+1} + F_k$ for $k \ge 0$.)

The answer is yes, P must be constant. By $S(n)$ we mean the sum of the decimal digits of $|n|$.

We need two claims.

Claim — If $P(x) \in \mathbb{Z}[x]$ is nonconstant with positive leading coefficient, then there exists an integer polynomial $F(x)$ such that all coefficients of $P \circ F$ are positive except for the second one, which is negative.

Proof. We will actually construct a cubic F. We call a polynomial *good* if it has the property.

First, consider $T_0(x) = x^3 + x + 1$. Observe that in $T_0^{\deg P}$, every coefficient is strictly positive, except for the second one, which is zero.

Then, let $T_1(x) = x^3 - \frac{1}{D}x^2 + x + 1$. Using continuity as $D \to \infty$, it follows that if D is large enough (in terms of $\deg P$), then $T_1^{\deg P}$ is good, with $-\frac{3}{D}x^{3\deg f - 1}$ being the only negative coefficient.

Finally, we can let $F(x) = CT_1(x)$ where C is a sufficiently large multiple of D (in terms of the coefficients of P); thus the coefficients of $(CT_1(x))^{\deg P}$ dominate (and are integers), as needed. □

Claim — There are infinitely many Fibonacci numbers in each residue class modulo 9.

Proof. Easy. First note the Fibonacci sequence is periodic modulo 9 (indeed it is periodic modulo any integer). Moreover (allowing negative indices),

$$\begin{aligned}
F_0 &= 0 \equiv 0 \pmod 9 \\
F_1 &= 1 \equiv 1 \pmod 9 \\
F_3 &= 2 \equiv 2 \pmod 9 \\
F_4 &= 3 \equiv 3 \pmod 9 \\
F_7 &= 13 \equiv 4 \pmod 9 \\
F_5 &= 5 \equiv 5 \pmod 9 \\
F_{-4} &= -3 \equiv 6 \pmod 9 \\
F_9 &= 34 \equiv 7 \pmod 9 \\
F_6 &= 8 \equiv 8 \pmod 9.
\end{aligned}$$

□

We now show how to solve the problem with the two claims. WLOG P satisfies the conditions of the first claim, and choose F as above. Let

$$P(F(x)) = c_N x^N - c_{N-1}x^{N-1} + c_{N-2}x^{N-2} + \cdots + c_0$$

where $c_i > 0$ (and $N = 3\deg P$). Then if we select $x = 10^e$ for e large enough (say $x > 10 \max_i c_i$), the decimal representation $P(F(10^e))$ consists of the concatenation of

- the decimal representation of $c_N - 1$,
- the decimal representation of $10^e - c_{N-1}$
- the decimal representation of c_{N-2}, with several leading zeros,
- the decimal representation of c_{N-3}, with several leading zeros,
- ...
- the decimal representation of c_0, with several leading zeros.

(For example, if $P(F(x)) = 15x^3 - 7x^2 + 4x + 19$, then $P(F(1000)) = 14{,}993{,}004{,}019$.) Thus, the sum of the digits of this expression is equal to

$$S(P(F(10^e))) = 9e + k$$

for some constant k depending only on P and F, independent of e. But this will eventually hit a Fibonacci number by the second claim, contradiction.

> **Remark.** It is important to control the number of negative coefficients in the created polynomial. If one tries to use this approach on a polynomial P with $m > 0$ negative coefficients, then one would require that the Fibonacci sequence is surjective modulo $9m$ for any $m > 1$, which is not true: for example the Fibonacci sequence avoids all numbers congruent to $4 \pmod{11}$ (and thus $4 \pmod{99}$).

§3 Solutions to Day 3

§3.1 Solution to TSTST 7, by Ankan Bhattacharya

Let $f \colon \mathbb{Z} \to \{1, 2, \dots, 10^{100}\}$ be a function satisfying

$$\gcd(f(x), f(y)) = \gcd(f(x), x - y)$$

for all integers x and y. Show that there exist positive integers m and n such that $f(x) = \gcd(m + x, n)$ for all integers x.

Let $\mathcal{P}$ be the set of primes not exceeding 10^{100}. For each $p \in \mathcal{P}$, let $e_p = \max_x \nu_p(f(x))$ and let $c_p \in \operatorname{argmax}_x \nu_p(f(x))$.

We show that this is good enough to compute all values of x, by looking at the exponent at each individual prime.

Claim — For any $p \in \mathcal{P}$, we have

$$\nu_p(f(x)) = \min(\nu_p(x - c_p), e_p).$$

Proof. Note that for any x, we have

$$\gcd(f(c_p), f(x)) = \gcd(f(c_p), x - c_p).$$

We then take ν_p of both sides and recall $\nu_p(f(x)) \le \nu_p(f(c_p)) = e_p$; this implies the result. □

This essentially determines f, and so now we just follow through. Choose n and m such that

$$\begin{aligned} n &= \prod_{p \in \mathcal{P}} p^{e_p} \\ m &\equiv -c_p \pmod{p^{e_p}} \quad \forall p \in \mathcal{P} \end{aligned}$$

the latter being possible by Chinese remainder theorem. Then, from the claim we have

$$\begin{aligned} f(x) &= \prod_{p \in \mathcal{P}} p^{\nu_p(f(x))} = \prod_{p \mid n} p^{\min(\nu_p(x - c_p), e_p)} \\ &= \prod_{p \mid n} p^{\min(\nu_p(x+m), \nu_p(n))} = \gcd\left(x + m, n\right) \end{aligned}$$

for every $x \in \mathbb{Z}$, as desired.

Remark. The functions $f(x) = x$ and $f(x) = |2x - 1|$ are examples satisfying the gcd equation (the latter always being strictly positive). Hence the hypothesis f bounded cannot be dropped.

Remark. The pair (m, n) is essentially unique: every other pair is obtained by shifting m by a multiple of n. Hence there is not really any choice in choosing m and n.

§3.2 Solution to TSTST 8, by Ankan Bhattacharya

Let $\mathcal{S}$ be a set of 16 points in the plane, no three collinear. Let $\chi(\mathcal{S})$ denote the number of ways to draw 8 line segments with endpoints in $\mathcal{S}$, such that no two drawn segments intersect, even at endpoints. Find the smallest possible value of $\chi(\mathcal{S})$ across all such $\mathcal{S}$.

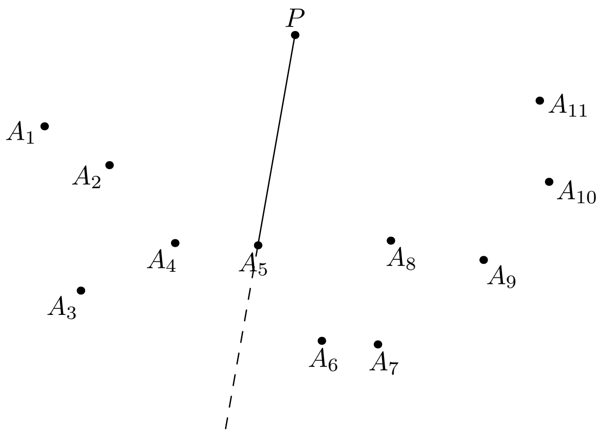
The answer is 1430. In general, we prove that with $2n$ points the answer is the n^{th} Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

First of all, it is well-known that if $\mathcal{S}$ is a convex $2n$-gon, then $\chi(\mathcal{S}) = C_n$.

It remains to prove the lower bound. We proceed by (strong) induction on n, with the base case $n = 0$ and $n = 1$ clear. Suppose the statement is proven for $0, 1, \dots, n$ and consider a set $\mathcal{S}$ with $2(n+1)$ points.

Let P be a point on the convex hull of $\mathcal{S}$, and label the other $2n+1$ points $A_1, \dots, A_{2n+1}$ in order of angle from P.

Consider drawing a segment $\overline{PA_{2k+1}}$. This splits the $2n$ remaining points into two halves $\mathcal{U}$ and $\mathcal{V}$, with $2k$ and $2(n-k)$ points respectively.



Note that by choice of P, no segment in $\mathcal{U}$ can intersect a segment in $\mathcal{V}$. By the inductive hypothesis,

$$\chi(\mathcal{U}) \ge C_k \quad \text{and} \quad \chi(\mathcal{V}) \ge C_{n-k}.$$

Thus, drawing $\overline{PA_{2k+1}}$, we have at least $C_k C_{n-k}$ ways to complete the drawing. Over all choices of k, we obtain

$$\chi(\mathcal{S}) \ge C_0 C_n + \dots + C_n C_0 = C_{n+1}$$

as desired.

> **Remark.** It is possible to show directly from the lower bound proof that convex $2n$-gons achieve the minimum: indeed, every inequality is sharp, and no segment $\overline{PA_{2k}}$ can be drawn (since this splits the rest of the points into two halves with an odd number of points, and no crossing segment can be drawn).
>
> Bobby Shen points out that in the case of 6 points, a regular pentagon with its center also achieves equality, so this is not the only equality case.

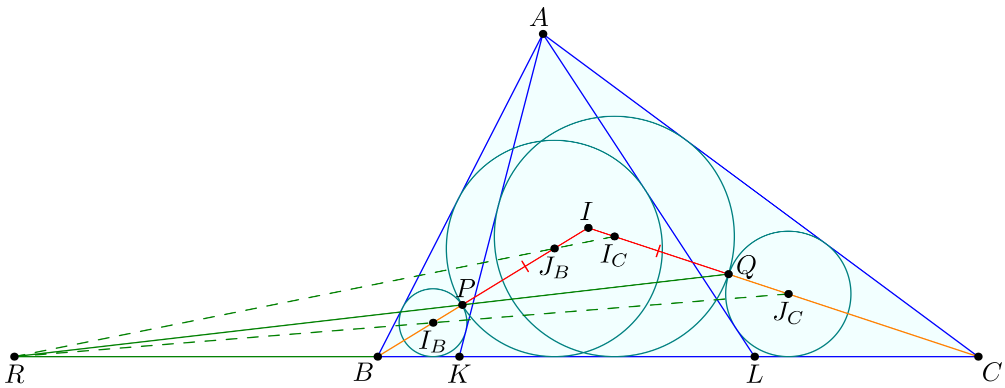
> **Remark.** The result that $\chi(S) \ge 1$ for all S is known (consider the choice of 8 segments with smallest sum), and appeared on Putnam 1979. However, it does not seem that knowing this gives an advantage for this problem, since the answer is much larger than 1.

§3.3 Solution to TSTST 9, by Ankan Bhattacharya

Let ABC be a triangle with incenter I. Points K and L are chosen on segment BC such that the incircles of $\triangle ABK$ and $\triangle ABL$ are tangent at P, and the incircles of $\triangle ACK$ and $\triangle ACL$ are tangent at Q. Prove that $IP = IQ$.

We present two solutions.

First solution, mostly elementary (original) Let I_B, J_B, I_C, J_C be the incenters of $\triangle ABK$, $\triangle ABL$, $\triangle ACK$, $\triangle ACL$ respectively.



We begin with the following claim which does not depend on the existence of tangency points P and Q.

> **Claim** — Lines BC, I_BJ_C, J_BI_C meet at a point R (possibly at infinity).

Proof. By rotating by $\frac{1}{2}\angle A$ we have the equality

$$A(BI; I_BJ_B) = A(IC; I_CJ_C).$$

It follows $(BI; I_BJ_B) = (IC; I_CJ_C) = (CI; J_CI_C)$. (One could also check directly that both cross ratios equal $\frac{\sin \angle BAK/2}{\sin \angle CAK/2} \div \frac{\sin \angle BAL/2}{\sin \angle CAL/2}$, rather than using rotation.)

Therefore, the concurrence follows from the so-called *prism lemma* on $\overline{IBI_BJ_B}$ and $\overline{ICJ_CI_C}$. □

> **Remark** (Nikolai Beluhov). This result is known; it appears as 4.5.32 in Akopyan's *Geometry in Figures*. The cross ratio is not necessary to prove this claim: it can be proven by length chasing with circumscribed quadrilaterals. (The generalization mentioned later also admits a trig-free proof for the analogous step.)

We now bring P and Q into the problem.

> **Claim** — Line PQ also passes through R.

Proof. Note $(BP; I_BJ_B) = -1 = (CQ; J_CI_C)$, so the conclusion again follows by prism lemma. □

We are now ready to complete the proof. Point R is the exsimilicenter of the incircles of $\triangle ABK$ and $\triangle ACL$, so $\frac{PI_B}{RI_B} = \frac{QJ_C}{RJ_C}$. Now by Menelaus,

$$\frac{I_BP}{PI} \cdot \frac{IQ}{QJ_C} \cdot \frac{J_CR}{RI_B} = -1 \implies IP = IQ.$$

Remark (Author's comments on drawing the diagram)**.** Drawing the diagram directly is quite difficult. If one draws $\triangle ABC$ first, they must locate both K and L, which likely involves some trial and error due to the complex interplay between the two points.

There are alternative simpler ways. For example, one may draw $\triangle AKL$ first; then the remaining points B and C are not related and the task is much simpler (though some trial and error is still required).

In fact, by breaking symmetry, we may only require one application of guesswork. Start by drawing $\triangle ABK$ and its incircle; then the incircle of $\triangle ABL$ may be constructed, and so point L may be drawn. Thus only the location of point C needs to be guessed. I would be interested in a method to create a general diagram without any trial and error.

Second solution, inversion (Nikolai Beluhov) As above, the lines BC, I_BJ_C, J_BI_C meet at some point R (possibly at infinity). Let ω_1, ω_2, ω_3, ω_4 be the incircles of $\triangle ABK$, $\triangle ACL$, $\triangle ABL$, and $\triangle ACK$.

Claim — There exists an inversion ι at R swapping $\{\omega_1, \omega_2\}$ and $\{\omega_3, \omega_4\}$.

Proof. Consider the inversion at R swapping ω_1 and ω_2. Since ω_1 and ω_3 are tangent, the image of ω_3 is tangent to ω_2 and is also tangent to BC. The circle ω_4 is on the correct side of ω_3 to be this image. □

Claim — Circles ω_1, ω_2, ω_3, ω_4 share a common radical center.

Proof. Let Ω be the circle with center R fixed under ι, and let k be the circle through P centered at the radical center of Ω, ω_1, ω_3.

Then k is actually orthogonal to Ω, ω_1, ω_3, so k is fixed under ι and k is also orthogonal to ω_2 and ω_4. Thus the center of k is the desired radical center. □

The desired statement immediately follows. Indeed, letting S be the radical center, it follows that $\overline{SP}$ and $\overline{SQ}$ are the common internal tangents to $\{\omega_1, \omega_3\}$ and $\{\omega_2, \omega_4\}$.

Since S is the radical center, $SP = SQ$. In light of $\angle SPI = \angle SQI = 90^\circ$, it follows that $IP = IQ$, as desired.

Remark (Nikolai Beluhov)**.** There exists a circle tangent to all four incircles, because circle k is orthogonal to all four, and line BC is tangent to all four; thus the inverse of line BC in k is a circle tangent to all four incircles.

The amusing thing here is that Casey's theorem is completely unhelpful for proving this fact: all it can tell us is that there is a line or circle tangent to these incircles, and line BC already satisfies this property.

Remark (Generalization by Nikolai Beluhov)**.** The following generalization holds:

> Let $ABCD$ be a quadrilateral circumscribed about a circle with center I. A line through A meets $\overrightarrow{BC}$ and $\overrightarrow{DC}$ at K and L; another line through A meets $\overrightarrow{BC}$ and $\overrightarrow{DC}$ at M and N. Suppose that the incircles of $\triangle ABK$ and $\triangle ABM$

are tangent at P, and the incircles of $\triangle ACL$ and $\triangle ACN$ are tangent at Q. Prove that $IP = IQ$.

The first approach can be modified to the generalization. There is an extra initial step required: by Monge, the exsimilicenter of the incircles of $\triangle ABK$ and $\triangle ADN$ lies on line BD; likewise for the incircles of $\triangle ABL$ and $\triangle ADM$. Now one may prove using the same trig approach that these pairs of incircles have a common exsimilicenter, and the rest of the solution plays out similarly. The second approach can also be modified in the same way, once we obtain that a common exsimilicenter exists. (Thus in the generalization, it seems we also get there exists a circle tangent to all four incircles.)