USA TSTST 2017 Solutions

United States of America — TST Selection Test

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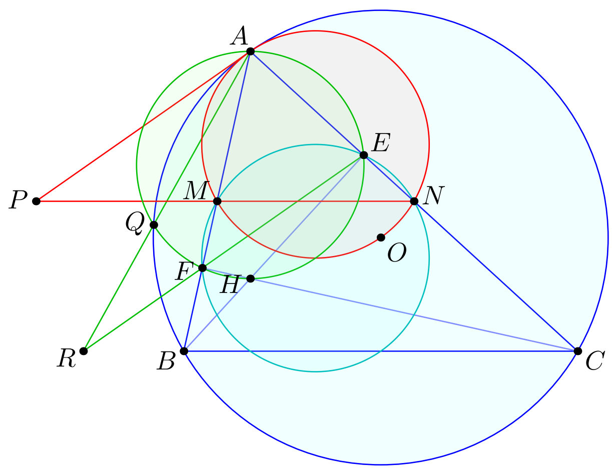
§1 Solutions to Day 1

§1.1 Solution to TSTST 1, by Ray Li

Let ABC be a triangle with circumcircle Γ, circumcenter O, and orthocenter H. Assume that $AB \neq AC$ and $\angle A \neq 90^\circ$. Let M and N be the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively, and let E and F be the feet of the altitudes from B and C in $\triangle ABC$, respectively. Let P be the intersection point of line MN with the tangent line to Γ at A. Let Q be the intersection point, other than A, of Γ with the circumcircle of $\triangle AEF$. Let R be the intersection point of lines AQ and EF. Prove that $\overline{PR} \perp \overline{OH}$.

First solution (power of a point) Let γ denote the nine-point circle of ABC.



Note that

- $PA^2 = PM \cdot PN$, so P lies on the radical axis of Γ and γ.
- $RA \cdot RQ = RE \cdot RF$, so R lies on the radical axis of Γ and γ.

Thus $\overline{PR}$ is the radical axis of Γ and γ, which is evidently perpendicular to $\overline{OH}$.

Remark. In fact, by power of a point one may also observe that R lies on $\overline{BC}$, since it is on the radical axis of $(AQFHE)$, $(BFEC)$, (ABC). Ironically, this fact is not used in the solution.

Second solution (barycentric coordinates) Again note first $R \in \overline{BC}$ (although this can be avoided too). We compute the points in much the same way as before. Since $\overline{AP} \cap \overline{BC} = (0 : b^2 : -c^2)$ we have

$$P = \left(b^2 - c^2 : b^2 : -c^2\right)$$

(since $x = y + z$ is the equation of line $\overline{MN}$). Now in Conway notation we have

$$R = \overline{EF} \cap \overline{BC} = (0 : S_C : -S_B) = \left(0 : a^2 + b^2 - c^2 : -a^2 + b^2 - c^2\right).$$

Hence

$$\overrightarrow{PR} = \frac{1}{2(b^2 - c^2)} \left(b^2 - c^2, c^2 - a^2, a^2 - b^2\right).$$

On the other hand, we have $\overrightarrow{OH} = \vec{A} + \vec{B} + \vec{C}$. So it suffices to check that

$$\sum_{\text{cyc}} a^2 \left((a^2 - b^2) + (c^2 - a^2) \right) = 0$$

which is immediate.

Third solution (complex numbers) Let ABC be the unit circle. We first compute P as the midpoint of A and $\overline{AA} \cap \overline{BC}$:

$$\begin{aligned} p &= \frac{1}{2} \left(a + \frac{a^2(b+c) - bc \cdot 2a}{a^2 - bc} \right) \\ &= \frac{a(a^2 - bc) + a^2(b+c) - 2abc}{2(a^2 - bc)}. \end{aligned}$$

Using the remark above, R is the inverse of D with respect to the circle with diameter $\overline{BC}$, which has radius $\left|\frac{1}{2}(b-c)\right|$. Thus

$$\begin{aligned} r - \frac{b+c}{2} &= \frac{\frac{1}{4}(b-c)\left(\frac{1}{b} - \frac{1}{c}\right)}{\overline{\frac{1}{2}\left(a - \frac{bc}{a}\right)}} \\ r &= \frac{b+c}{2} + \frac{-\frac{1}{2}\frac{(b-c)^2}{bc}}{\frac{1}{a} - \frac{a}{bc}} \\ &= \frac{b+c}{2} + \frac{a(b-c)^2}{2(a^2 - bc)} \\ &= \frac{a(b-c)^2 + (b+c)(a^2 - bc)}{2(a^2 - bc)}. \end{aligned}$$

Expanding and subtracting gives

$$p - r = \frac{a^3 - abc - ab^2 - ac^2 + b^2c + bc^2}{2(a^2 - bc)} = \frac{(a+b+c)(a-b)(a-c)}{2(a^2 - bc)}$$

which is visibly self-conjugate once the factor of $a + b + c$ is deleted.

(Actually, one can guess this factorization ahead of time by noting that if $A = B$, then $P = B = R$, so $a - b$ must be a factor; analogously $a - c$ must be as well.)

§1.2 Solution to TSTST 2, by Kevin Sun

Ana and Banana are playing a game. First Ana picks a word, which is defined to be a nonempty sequence of capital English letters. Then Banana picks a nonnegative integer k and challenges Ana to supply a word with exactly k subsequences which are equal to Ana's word. Ana wins if she is able to supply such a word, otherwise she loses. For example, if Ana picks the word "TST", and Banana chooses $k = 4$, then Ana can supply the word "TSTST" which has 4 subsequences which are equal to Ana's word. Which words can Ana pick so that she can win no matter what value of k Banana chooses?

First we introduce some notation. Define a *block* of letters to be a maximal contiguous subsequence of consecutive letters. Throughout the solution, we fix the word A that Ana picks, and introduce the following notation for its m blocks:

$$A = A_1 A_2 \dots A_m = \underbrace{a_1 \dots a_1}_{x_1} \underbrace{a_2 \dots a_2}_{x_2} \dots \underbrace{a_m \dots a_m}_{x_m}.$$

A *rainbow* will be a subsequence equal to Ana's initial word A (meaning Ana seeks words with exactly k rainbows). Finally, for brevity, let $A_i = \underbrace{a_i \dots a_i}_{x_i}$, so $A = A_1 \dots A_m$.

We prove two claims that resolve the problem.

Claim. If $x_i = 1$ for some i, then for any $k \ge 1$, the word

$$W = A_1 \dots A_{i-1} \underbrace{a_i \dots a_i}_{k} A_{i+1} \dots A_m$$

obtained by repeating the ith letter k times has exactly k rainbows.

Proof. Obviously there are at least $\binom{k}{k-1} = k$ rainbows, obtained by deleting $k-1$ choices of the letter a_i in the repeated block. We show they are the only ones.

Given a rainbow, consider the location of this singleton block in W. It cannot occur within the first $|A_1|+\dots+|A_{i-1}|$ letters, nor can it occur within the final $|A_{i+1}|+\dots+|A_m|$ letters. So it must appear in the ith block of W. That implies that all the other a_i's in the ith block of W must be deleted, as desired. (This last argument is actually nontrivial, and has some substance; many students failed to realize that the upper bound requires care.) □

Claim. If $x_i \ge 2$ for all i, then no word W has exactly two rainbows.

Proof. We prove if there are two rainbows of W, then we can construct at least three rainbows.

Let $W = w_1 \dots w_n$ and consider the two rainbows of W. Since they are not the same, there must be a block A_p of the rainbow, of length $\ell \ge 2$, which do not occupy the same locations in W.

Assume the first rainbow uses $w_{i_1}, \dots, w_{i_\ell}$ for this block and the second rainbow uses $w_{j_1}, \dots, w_{j_\ell}$ for this block. Then among the letters w_q for $\min(i_1, j_1) \le q \le \max(i_\ell, j_\ell)$, there must be at least $\ell + 1$ copies of the letter a_p. Moreover, given a choice of ℓ copies of the letter a_p in this range, one can complete the subsequence to a rainbow. So the number of rainbows is at least $\binom{\ell+1}{\ell} \ge \ell + 1$.

Since $\ell \ge 2$, this proves W has at least three rainbows. □

In summary, Ana wins if and only if $x_i = 1$ for some i, since she can duplicate the isolated letter k times; but if $x_i \ge 2$ for all i then Banana only needs to supply $k = 2$.

§1.3 Solution to TSTST 3, by Calvin Deng and Linus Hamilton

Consider solutions to the equation

$$x^2 - cx + 1 = \frac{f(x)}{g(x)}$$

where f and g are nonzero polynomials with nonnegative real coefficients. For each $c > 0$, determine the minimum possible degree of f, or show that no such f, g exist.

First, if $c \ge 2$ then we claim no such f and g exist. Indeed, one simply takes $x = 1$ to get $f(1)/g(1) \le 0$, impossible.

For $c < 2$, let $c = 2\cos\theta$, where $0 < \theta < \pi$. We claim that f exists and has minimum degree equal to n, where n is defined as the smallest integer satisfying $\sin n\theta \le 0$. In other words

$$n = \left\lceil \frac{\pi}{\arccos(c/2)} \right\rceil.$$

First we show that this is necessary. To see it, write explicitly

$$g(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-2}x^{n-2}$$

with each $a_i \ge 0$, and $a_{n-2} \neq 0$. Assume that n is such that $\sin(k\theta) \ge 0$ for $k = 1, \dots, n-1$. Then, we have the following system of inequalities:

$$\begin{aligned} a_1 &\ge 2\cos\theta \cdot a_0 \\ a_0 + a_2 &\ge 2\cos\theta \cdot a_1 \\ a_1 + a_3 &\ge 2\cos\theta \cdot a_2 \\ &\vdots \\ a_{n-5} + a_{n-3} &\ge 2\cos\theta \cdot a_{n-4} \\ a_{n-4} + a_{n-2} &\ge 2\cos\theta \cdot a_{n-3} \\ a_{n-3} &\ge 2\cos\theta \cdot a_{n-2}. \end{aligned}$$

Now, multiply the first equation by $\sin\theta$, the second equation by $\sin 2\theta$, et cetera, up to $\sin((n-1)\theta)$. This choice of weights is selected since we have

$$\sin(k\theta) + \sin((k+2)\theta) = 2\sin((k+1)\theta)\cos\theta$$

so that summing the entire expression cancels nearly all terms and leaves only

$$\sin((n-2)\theta)\, a_{n-2} \ge \sin((n-1)\theta) \cdot 2\cos\theta \cdot a_{n-2}$$

and so by dividing by a_{n-2} and using the same identity gives us $\sin(n\theta) \le 0$, as claimed.

This bound is best possible, because the example

$$a_k = \sin((k+1)\theta) \ge 0$$

makes all inequalities above sharp, hence giving a working pair (f, g).

Remark. Calvin Deng points out that a cleaner proof of the lower bound is to take $\alpha = \cos\theta + i\sin\theta$. Then $f(\alpha) = 0$, but by condition the imaginary part of $f(\alpha)$ is apparently strictly positive, contradiction.

Remark. Guessing that $c < 2$ works at all (and realizing $c \ge 2$ fails) is the first part of the problem.

The introduction of trigonometry into the solution may seem magical, but is motivated in one of two ways:

- Calvin Deng points out that it's possible to guess the answer from small cases: For $c \le 1$ we have $n = 3$, tight at $\frac{x^3+1}{x+1} = x^2 - x + 1$, and essentially the "sharpest $n = 3$ example". A similar example exists at $n = 4$ with $\frac{x^4+1}{x^2+\sqrt{2}x+1} = x^2 - \sqrt{2}x + 1$ by the Sophie-Germain identity. In general, one can do long division to extract an optimal value of c for any given n, although c will be the root of some polynomial.

  The thresholds $c \le 1$ for $n = 3$, $c \le \sqrt{2}$ for $n = 4$, $c \le \frac{1+\sqrt{5}}{2}$ for $n = 5$, and $c \le 2$ for $n < \infty$ suggest the unusual form of the answer via trigonometry.

- One may imagine trying to construct a polynomial recursively / greedily by making all inequalities above hold (again the "sharpest situation" in which f has few coefficients). If one sets $c = 2t$, then we have

$$a_0 = 0, \quad a_1 = 2t, \quad a_2 = 4t^2 - 1, \quad a_3 = 8t^2 - 4t, \quad \dots$$

which are the Chebyshev polynomials of the second type. This means that trigonometry is essentially mandatory. (One may also run into this when by using standard linear recursion techniques, and noting that the characteristic polynomial has two conjugate complex roots.)

Remark. Mitchell Lee notes that an IMO longlist problem from 1997 shows that if $P(x)$ is any polynomial satisfying $P(x) > 0$ for $x > 0$, then $(x+1)^n P(x)$ has nonnegative coefficients for large enough n. This show that f and g at least exist for $c \le 2$, but provides no way of finding the best possible $\deg f$.

Meghal Gupta also points out that showing f and g exist is possible in the following way:

$$\left(x^2 - 1.99x + 1\right)\left(x^2 + 1.99x + 1\right) = \left(x^4 - 1.9601x^2 + 1\right)$$

and so on, repeatedly multiplying by the "conjugate" until all coefficients become positive. To my best knowledge, this also does not give any way of actually minimizing $\deg f$, although Ankan Bhattacharya points out that this construction is actually optimal in the case where n is a power of 2.

§2 Solutions to Day 2

§2.1 Solution to TSTST 4, Mark Sellke

Find all nonnegative integer solutions to $2^a + 3^b + 5^c = n!$.

For $n \le 4$, one can check the only solutions are:

$$\begin{aligned} 2^2 + 3^0 + 5^0 &= 3! \\ 2^1 + 3^1 + 5^0 &= 3! \\ 2^4 + 3^1 + 5^1 &= 4!. \end{aligned}$$

Now we prove there are no solutions for $n \ge 5$.

A tricky way to do this is to take modulo 120, since

$$\begin{aligned} 2^a &\pmod{120} \in \{1, 2, 4, 8, 16, 32, 64\} \\ 3^b &\pmod{120} \in \{1, 3, 9, 27, 81\} \\ 5^c &\pmod{120} \in \{1, 5, 25\} \end{aligned}$$

and by inspection one notes that no three elements have vanishing sum modulo 120.

I expect most solutions to instead use casework. Here is one possible approach with cases (with $n \ge 5$). First, we analyze the cases where $a < 3$:

- $a = 0$: No solutions for parity reasons.
- $a = 1$: since $3^b + 5^c \equiv 6 \pmod 8$, we find b even and c odd (hence $c \neq 0$). Now looking modulo 5 gives that $3^b + 5^c \equiv 3 \pmod 5$,
- $a = 2$: From $3^b + 5^c \equiv 4 \pmod 8$, we find b is odd and c is even. Now looking modulo 5 gives a contradiction, even if $c = 0$, since $3^b \in \{2, 3 \pmod 5\}$ but $3^b + 5^c \equiv 1 \pmod 5$.

Henceforth assume $a \ge 3$. Next, by taking modulo 8 we have $3^b + 5^c \equiv 0 \pmod 8$, which forces both b and c to be odd (in particular, $b, c > 0$). We now have

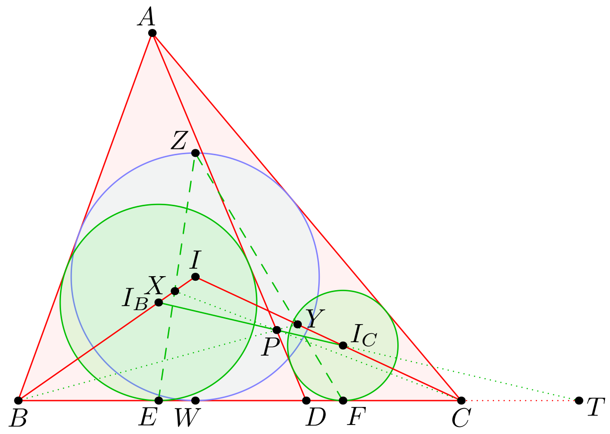
$$\begin{aligned} 2^a + 5^c &\equiv 0 \pmod 3 \\ 2^a + 3^b &\equiv 0 \pmod 5. \end{aligned}$$

The first equation implies a is even, but the second equation requires a to be odd, contradiction. Hence no solutions with $n \ge 5$.

§2.2 Solution to TSTST 5, Ray Li

Let ABC be a triangle with incenter I. Let D be a point on side BC and let ω_B and ω_C be the incircles of $\triangle ABD$ and $\triangle ACD$, respectively. Suppose that ω_B and ω_C are tangent to segment BC at points E and F, respectively. Let P be the intersection of segment AD with the line joining the centers of ω_B and ω_C. Let X be the intersection point of lines BI and CP and let Y be the intersection point of lines CI and BP. Prove that lines EX and FY meet on the incircle of $\triangle ABC$.

First solution (homothety) Let Z be the diametrically opposite point on the incircle. We claim this is the desired intersection.



Note that:

- P is the insimilicenter of ω_B and ω_C
- C is the exsimilicenter of ω and ω_C.

Thus by Monge theorem, the insimilicenter of ω_B and ω lies on line CP.

This insimilicenter should also lie on the line joining the centers of ω and ω_B, which is $\overline{BI}$, hence it coincides with the point X. So $X \in \overline{EZ}$ as desired.

Second solution (harmonic) Let $T = \overline{I_B I_C} \cap \overline{BC}$, and W the foot from I to $\overline{BC}$. Define $Z = \overline{FY} \cap \overline{IW}$. Because $\angle I_B D I_C = 90^\circ$, we have

$$-1 = (I_B I_C; PT) \overset{B}{=} (II_C; YC) \overset{F}{=} (I\infty; ZW)$$

So I is the midpoint of $\overline{ZW}$ as desired.

Third solution (outline, barycentric, Andrew Gu) Let $AD = t$, $BD = x$, $CD = y$ (so with Stewart). We then have $D = (0 : y : x)$ and so

$$\overline{AI_B} \cap \overline{BC} = \left(0 : y + \frac{tx}{c+t} : \frac{cx}{c+t}\right)$$

hence intersection with BI gives

$$I_B = (ax : cy + at : cx).$$

Similarly,

$$I_C = (ay : by : bx + at).$$

Then, we can compute

$$P = (2axy : y(at + bx + cy) : x(at + bx + cy))$$

since $P \in \overline{I_B I_C}$, and clearly $P \in \overline{AD}$. Intersection now gives

$$X = (2ax : at + bx + cy : 2cx)$$
$$Y = (2ay : 2by : at + bx + cy).$$

Finally, we have $BE = \frac{1}{2}(c + x - t)$, and similarly for CF. From here one can check that the antipode

$$Q = \left(4a^2 : -a^2 + 2ab - b^2 + c^2 : -a^2 + 2ac - c^2 + b^2\right)$$

lies on each of lines EX and FY (using Stewart's Theorem).

§2.3 Solution to TSTST 6, Ivan Borsenco

A sequence of positive integers $(a_n)_{n\ge 1}$ is of *Fibonacci type* if it satisfies the recursive relation $a_{n+2} = a_{n+1} + a_n$ for all $n \ge 1$. Is it possible to partition the set of positive integers into an infinite number of Fibonacci type sequences?

Yes, it is possible. The following solutions were written for me by Kevin Sun and Mark Sellke.

First solution (Kevin Sun) We are going to appeal to the so-called Zeckendorf theorem:

Theorem

Every positive integer can be uniquely expressed as the sum of nonconsecutive Fibonacci numbers.

The idea is to take the following so-called *Wythoff array*:

- $1, 2, 3, 5, \dots$
- $1+3,\ 2+5,\ 3+8, \dots$
- $1+5,\ 2+8,\ 3+13, \dots$
- $1+8,\ 2+13,\ 3+21, \dots$
- $1+3+8,\ 2+5+13,\ 3+8+21, \dots$
- $\dots$

We write the details below.

Let $\{F_i\}$ denote the Fibonacci numbers with $F_1 = 1, F_2 = 2$. Say $n = \overline{a_k \cdots a_1}_{\text{Fib}}$ with $a_k = 1$ is a Fibonacci base representation of n if a_i is 0 or 1,

$$n = \sum_{i=1}^{k} a_i F_i$$

and a_i, a_{i+1} are not both 1 for any i. Equivalently, it is a representation of n as a sum of nonconsecutive Fibonacci numbers.

We begin by outlining a proof of Zeckendorf's theorem, which implies the representation above is unique. Note that if F_k is the greatest Fibonacci number at most n, then

$$n - F_k < F_{k+1} - F_k = F_{k-1}.$$

In particular, repeatedly subtracting off the largest F_k from n will produce one such representation with no two consecutive Fibonacci numbers. On the other hand, this F_k must be used, as

$$n \ge F_k > F_{k-1} + F_{k-3} + F_{k-5} \cdots$$

This shows, by a simple inductive argument, that such a representation exists and unique.

Now for each $\overline{a_k \cdots a_1}_{\text{Fib}}$ with $a_1 = 1$, consider the sequence

$$\overline{a_k \cdots a_1}_{\text{Fib}}, \overline{a_k \cdots a_1 0}_{\text{Fib}}, \overline{a_k \cdots a_1 00}_{\text{Fib}}, \dots$$

These sequences are Fibonacci-type by definition, and partition the positive integers since each positive integer has exactly one Fibonacci base representation.

Second solution Call an infinite set of integers S *sandwiched* if there exist increasing sequences $\{a_i\}_{i=0}^{\infty}, \{b_i\}_{i=0}^{\infty}$ such that the following are true:

- $a_i + a_{i+1} = a_{i+2}$ and $b_i + b_{i+1} = b_{i+2}$.
- The intervals $[a_i + 1, b_i - 1]$ are disjoint and are nondecreasing in length.
- $S = \bigcup_{i=0}^{\infty} [a_i + 1, b_i - 1]$.

We claim that if S is any nonempty sandwiched set, then S can be partitioned into a Fibonacci-type sequence (involving the smallest element of S) and two smaller sandwiched sets. If this claim is proven, then we can start with $\mathbb{N} \setminus \{1, 2, 3, 5, \dots\}$, which is a sandwiched set, and repeatedly perform this partition, which will eventually sort each natural number into a Fibonacci-type sequence.

Let S be a sandwiched set given by $\{a_i\}_{i=0}^{\infty}, \{b_i\}_{i=0}^{\infty}$, so the smallest element in S is $x = a_0 + 1$. Note that $y = a_1 + 1$ is also in S and $x < y$. Then consider the Fibonacci-type sequence given by $f_0 = x, f_1 = y$, and $f_{k+2} = f_{k+1} + f_k$. We can then see that $f_i \in [a_i + 1, b_i - 1]$, as the sum of numbers in the intervals $[a_k + 1, b_k - 1]$, $[a_{k+1} + 1, b_{k+1} - 1]$ lies in the interval

$$[a_k + a_{k+1} + 2, b_k + b_{k+1} - 2] = [a_{k+2} + 2, b_{k+2} - 2] \subset [a_{k+2} + 1, b_{k+2} - 1].$$

Therefore, this gives a natural partition of S into this sequence and two sets:

$$S_1 = \bigcup_{i=0}^{\infty} [a_i + 1, f_i - 1]$$
$$\text{and } S_2 = \bigcup_{i=0}^{\infty} [f_i + 1, b_i - 1].$$

(For convenience, $[x, x - 1]$ will be treated as the empty set.)

We now show that S_1 and S_2 are sandwiched. Since $\{a_i\}$, $\{f_i\}$, and $\{b_i\}$ satisfy the Fibonacci recurrence, it is enough to check that the intervals have nondecreasing lengths. For S_1, that is equivalent to $f_{k+1} - a_{k+1} \ge f_k - a_k$ for each k. Fortunately, for $k \ge 1$, the difference is $f_{k-1} - a_{k-1} \ge 0$, and for $k = 0$, $f_1 - a_1 = 1 = f_0 - a_0$. Similarly for S_2, checking $b_{k+1} - f_{k+1} \ge b_k - f_k$ is easy for $k \ge 1$ as $b_{k-1} - f_{k-1} \ge 0$, and

$$(b_1 - f_1) - (b_0 - f_0) = (b_1 - a_1) - (b_0 - a_0),$$

which is nonnegative since the lengths of intervals in S are nondecreasing.

Therefore we have shown that S_1 and S_2 are sandwiched. (Note that some of the $[a_i + 1, f_i - 1]$ may be empty, which would shift some indices back.) Since this gives us a procedure to take a set S and produce a Fibonacci-type sequence with its smallest element, along which two other sandwiched types, we can partition $\mathbb{N}$ into an infinite number of Fibonacci-type sequences.

Third solution We add Fibonacci-type sequences one-by-one. At each step, let x be the smallest number that has not been used in any previous sequence. We generate a new Fibonacci-type sequence as follows. Set $a_0 = x$ and for $i \ge 1$, set

$$a_i = \left\lfloor \varphi a_{i-1} + \frac{1}{2} \right\rfloor.$$

Equivalently, a_i is the closest integer to φa_{i-1}.

It suffices to show that this sequence is Fibonacci-type and that no two sequences generated in this way overlap. We first show that for a positive integer n,

$$\left\lfloor \varphi \left\lfloor \varphi n + \frac{1}{2} \right\rfloor + \frac{1}{2} \right\rfloor = n + \left\lfloor \varphi n + \frac{1}{2} \right\rfloor.$$

Indeed,

$$\begin{aligned} \left\lfloor \varphi \left\lfloor \varphi n + \frac{1}{2} \right\rfloor + \frac{1}{2} \right\rfloor &= \left\lfloor (1+\varphi^{-1}) \left\lfloor \varphi n + \frac{1}{2} \right\rfloor + \frac{1}{2} \right\rfloor \\ &= \left\lfloor \varphi n + \frac{1}{2} \right\rfloor + \left\lfloor \varphi^{-1} \left\lfloor \varphi n + \frac{1}{2} \right\rfloor + \frac{1}{2} \right\rfloor. \end{aligned}$$

Note that $\left\lfloor \varphi n + \frac{1}{2} \right\rfloor = \varphi n + c$ for some $|c| \le \frac{1}{2}$; this implies that $\varphi^{-1} \left\lfloor \varphi n + \frac{1}{2} \right\rfloor$ is within $\varphi^{-1} \cdot \frac{1}{2} < \frac{1}{2}$ of n, so its closest integer is n, proving the claim.

Therefore these sequences are Fibonacci-type. Additionally, if $a \neq b$, then $|\varphi a - \varphi b| \ge \varphi > 1$. Then

$$a \neq b \implies \left\lfloor \varphi a + \frac{1}{2} \right\rfloor \neq \left\lfloor \varphi b + \frac{1}{2} \right\rfloor,$$

and since the first term of each sequence is chosen to not overlap with any previous sequences, these sequences are disjoint.

Remark. Ankan Bhattacharya points out that the same sequence essentially appears in IMO 1993, Problem 5 — in other words, a strictly increasing function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ with $f(1) = 2$, and $f(f(n)) = f(n) + n$.

Fourth solution (Mark Sellke) For later reference let

$$f_1 = 0, f_2 = 1, f_3 = 1, \dots$$

denote the ordinary Fibonacci numbers. We will denote the Fibonacci-like sequences by F^i and the elements with subscripts; hence F^2_1 is the first element of the second sequence. Our construction amounts to just iteratively add new sequences; hence the following claim is the whole problem.

Lemma

For any disjoint collection of Fibonacci-like sequences $F^1, \dots, F^k$ and any integer m contained in none of them, there is a new Fibonacci-like sequence F^{k+1} beginning with $F^{k+1}_1 = m$ which is disjoint from the previous sequences.

Observe first that for each sequence F^j there is $c^j \in \mathbb{R}^n$ such that

$$F^j_n = c^j \phi^n + o(1)$$

where

$$\phi = \frac{1+\sqrt{5}}{2}.$$

Collapse the group $(\mathbb{R}^+, \times)$ into the half-open interval $J = \{x \mid 1 \le x < \phi\}$ by defining $T(x) = y$ for the unique $y \in J$ with $x = y\phi^n$ for some integer n.

Fix an interval $I = [a, b] \subseteq [1.2, 1.3]$ (the last condition is to avoid wrap-around issues) which contains none of the c^j, and take $\varepsilon < 0.001$ to be small enough that in fact each c^j

has distance at least 10ε from I; this means any c_j and element of I differ by at least a $(1+10\varepsilon)$ factor. The idea will be to take $F_1^{k+1} = m$ and F_2^{k+1} to be a large such that the induced values of F_j^{k+1} grow like $k\phi^j$ for $j \in T^{-1}(I)$, so that F_n^{k+1} is separated from the c^j after applying T. What's left to check is the convergence.

Now let

$$c = \lim_{n\to\infty} \frac{f_n}{\phi^n}$$

and take M large enough that for $n > M$ we have

$$\left| \frac{f_n}{c\phi^n} - 1 \right| < \varepsilon.$$

Now $\frac{T^{-1}(I)}{c}$ contains arbitrarily large integers, so there are infinitely many N with $cN \in T^{-1}(I)$ with $N > \frac{10m}{\varepsilon}$. We claim that for any such N, the sequence $F^{(N)}$ defined by

$$F_1^{(N)} = m, F_2^{(N)} = N$$

will be very multiplicatively similar to the normal Fibonacci numbers up to rescaling; indeed for $j=2, j=3$ we have $\frac{F_2^{(N)}}{f_2} = N$, $\frac{F_3^{(N)}}{f_3} = N+m$ and so by induction we will have

$$\frac{F_j^{(N)}}{f_j} \in [N, N+m] \subseteq [N, N(1+\varepsilon)]$$

for $j \ge 2$. Therefore, up to small multiplicative errors, we have

$$F_j^{(N)} \approx N f_j \approx cN\phi^j.$$

From this we see that for $j > M$ we have

$$T(F_j^{(N)}) \in T(cN) \cdot [1-2\varepsilon, 1+2\varepsilon].$$

In particular, since $T(cN) \in I$ and I is separated from each c_j by a factor of $(1+10\varepsilon)$, we get that $F_j^{(N)}$ is not in any of $F^1, F^2, \dots, F^k$.

Finishing is easy, since we now have a uniform estimate on how many terms we need to check for a new element before the exponential growth takes over. We will just use pigeonhole to argue that there are few possible collisions among those early terms, so we can easily pick a value of N which avoids them all. We write it out below.

For large L, the set

$$S_L = (I \cdot \phi^L) \cap \mathbb{Z}$$

contains at least $k_I\phi^L$ elements. As N ranges over S_L, for each fixed j, the value of $F_j^{(N)}$ varies by at most a factor of 1.1 because we imposed $I \subseteq [1.2, 1.3]$ and so this is true for the first two terms, hence for all subsequent terms by induction. Now suppose L is very large, and consider a fixed pair (i,j) with $i \le k$ and $j \le M$. We claim there is at most 1 possible value k such that the term F_k^i could equal $F_j^{(N)}$ for some $N \in S_L$; indeed, the terms of F^i are growing at exponential rate with factor $\phi > 1.1$, so at most one will be in a given interval of multiplicative width at most 1.1.

Hence, of these $k_I\phi^L$ values of N, at most kM could cause problems, one for each pair (i,j). However by monotonicity of $F_j^{(N)}$ in N, at most 1 value of N causes a collision for each pair (i,j). Hence for large L so that $k_I\phi^L > 10kM$ we can find a suitable $N \in S_L$ by pigeonhole and the sequence $F^{(N)}$ defined by $(m, N, N+m, \dots)$ works.