USAMO 2018 Solution Notes

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This is an unofficial solutions packet for the 2018 USAMO. In general, they are a combination of my own work, as well as the official solutions provided by the organizers (for which they hold any copyrights), and solutions found on the Art of Problem Solving forums.

Corrections and comments are welcome!

Contents

§0 Problems

1. Let a, b, c be positive real numbers such that $a + b + c = 4\sqrt[3]{abc}$. Prove that

$$2(ab + bc + ca) + 4\min(a^2, b^2, c^2) \geq a^2 + b^2 + c^2.$$

2. Find all functions $f\colon (0, \infty) \to (0, \infty)$ such that

$$f\left(x + \frac{1}{y}\right) + f\left(y + \frac{1}{z}\right) + f\left(z + \frac{1}{x}\right) = 1$$

for all $x, y, z > 0$ with $xyz = 1$.

3. Let $n \geq 2$ be an integer, and let $\{a_1, \ldots, a_m\}$ denote the $m = \varphi(n)$ integers less than n and relatively prime to n. Assume that every prime divisor of m also divides n. Prove that m divides $a_1^k + \cdots + a_m^k$ for every positive integer k.

4. Let p be a prime, and let a_1, $\ldots$, a_p be integers. Show that there exists an integer k such that the numbers

$$a_1 + k, \; a_2 + 2k, \; \ldots, \; a_p + pk$$

produce at least $\frac{1}{2}p$ distinct remainders upon division by p.

5. Let $ABCD$ be a convex cyclic quadrilateral with $E = \overline{AC} \cap \overline{BD}$, $F = \overline{AB} \cap \overline{CD}$, $G = \overline{DA} \cap \overline{BC}$. The circumcircle of $\triangle ABE$ intersects line CB at B and P, and that the circumcircle of $\triangle ADE$ intersects line CD at D and Q. Assume C, B, P, G and C, Q, D, F are collinear in that order. Let $M = \overline{FP} \cap \overline{GQ}$. Prove that $\angle MAC = 90°$.

6. Let a_n be the number of permutations $(x_1, \ldots, x_n)$ of $(1, \ldots, n)$ such that the ratios x_k/k are all distinct. Prove that a_n is odd for all $n \geq 1$.

§1 USAMO 2018/1, proposed by Titu Andreescu

Let a, b, c be positive real numbers such that $a + b + c = 4\sqrt[3]{abc}$. Prove that

$$2(ab + bc + ca) + 4\min(a^2, b^2, c^2) \geq a^2 + b^2 + c^2.$$

WLOG let $c = \min(a, b, c) = 1$ by scaling. The given inequality becomes equivalent to

$$4ab + 2a + 2b + 3 \geq (a + b)^2 \qquad \forall a + b = 4(ab)^{1/3} - 1.$$

Now, let $t = (ab)^{1/3}$ and eliminate $a + b$ using the condition, to get

$$4t^3 + 2(4t - 1) + 3 \geq (4t - 1)^2 \iff 0 \leq 4t^3 - 16t^2 + 16t = 4t(t - 2)^2$$

which solves the problem.

Equality occurs only if $t = 2$, meaning $ab = 8$ and $a + b = 7$, which gives

$$\{a, b\} = \left\{ \frac{7 \pm \sqrt{17}}{2} \right\}$$

with the assumption $c = 1$. Scaling gives the curve of equality cases.

§1 USAMO 2018/1, proposed by Titu Andreescu

§2 USAMO 2018/2, proposed by Titu Andreescu and Nikolai Nikolov

Find all functions $f\colon (0,\infty) \to (0,\infty)$ such that

$$f\left(x + \frac{1}{y}\right) + f\left(y + \frac{1}{z}\right) + f\left(z + \frac{1}{x}\right) = 1$$

for all $x, y, z > 0$ with $xyz = 1$.

The main part of the problem is to show all solutions are linear. As always, let $x = b/c$, $y = c/a$, $z = a/b$ (classical inequality trick). Then the problem becomes

$$\sum_{\text{cyc}} f\left(\frac{b+c}{a}\right) = 1.$$

Let $f(t) = g(\frac{1}{t+1})$, equivalently $g(s) = f(1/s - 1)$. Thus $g\colon (0,1) \to (0,1)$ which satisfies $\sum_{\text{cyc}} g\left(\frac{a}{a+b+c}\right) = 1$, or equivalently

$$\boxed{g(a) + g(b) + g(c) = 1} \qquad \forall a + b + c = 1.$$

The rest of the solution is dedicated to solving this equivalent functional equation in g. It is a lot of technical details and I will only outline them (with apologies to the contestants who didn't have that luxury).

> **Claim —** The function g is linear.

Proof. This takes several steps, all of which are technical. We begin by proving g is linear over $[1/8, 3/8]$.

- First, whenever $a + b \le 1$ we have

$$1 - g(1 - (a+b)) = g(a) + g(b) = 2g\left(\frac{a+b}{2}\right).$$

  Hence g obeys Jensen's functional equation over $(0, 1/2)$.

- Define $h\colon [0,1] \to \mathbb{R}$ by $h(t) = g(\frac{2t+1}{8}) - (1-t) \cdot g(1/8) - t \cdot g(3/8)$, then h satisfies Jensen's functional equation too over $[0,1]$. We have also arranged that $h(0) = h(1) = 0$, hence $h(1/2) = 0$ as well.

- Since

$$h(t) = h(t) + h(1/2) = 2h(t/2 + 1/4) = h(t + 1/2) + h(0) = h(t + 1/2)$$

  for any $t < 1/2$, we find h is periodic modulo $1/2$. It follows one can extend $\widetilde{h}$ by

$$\widetilde{h}\colon \mathbb{R} \to \mathbb{R} \qquad \text{by} \qquad \widetilde{h}(t) = h(t - \lfloor t \rfloor)$$

  and still satisfy Jensen's functional equation. Because $\widetilde{h}(0) = 0$, it's well-known this implies $\widetilde{h}$ is additive (because $\widetilde{h}(x + y) = 2\widetilde{h}\left((x+y)/2\right) = \widetilde{h}(x) + \widetilde{h}(y)$ for any real numbers x to y).

But $\widetilde{h}$ is bounded below on $[0, 1]$ since $g \geq 0$, and since $\widetilde{h}$ is also additive, it follows (well-known) that $\widetilde{h}$ is linear. Thus h is the zero function. So, the function g is linear over $[1/8, 3/8]$; thus we may write $g(x) = kx + \ell$, valid for $1/8 \leq x \leq 3/8$.

Since $3g(1/3) = 1$, it follows $k + 3\ell = 1$.

For $0 < x < 1/8$ we have $g(x) = 2g(0.15) - g(0.3 - x) = 2(0.15k + \ell) - (k(0.3 - x) + \ell) = kx + \ell$, so g is linear over $(0, 3/8)$ as well. Finally, for $3/8 < x < 1$, we use the given equation

$$1 = g\left(\frac{1 - x}{2}\right) + g\left(\frac{1 - x}{2}\right) + g(x) \implies g(x) = 1 - 2\left(k \cdot \frac{1 - x}{2} + \ell\right) = kx + \ell$$

since $\frac{1-x}{2} < \frac{5}{16} < \frac{3}{8}$. Thus g is linear over all. $\square$

Putting this back in, we deduce that $g(x) = kx + \frac{1-k}{3}$ for some $k \in [-1/2, 1]$, and so

$$f(x) = \frac{k}{x + 1} + \frac{1 - k}{3}$$

for some $k \in [-1/2, 1]$. All such functions work.

§3 USAMO 2018/3, proposed by Ivan Borsenco

Let $n \geq 2$ be an integer, and let $\{a_1, \ldots, a_m\}$ denote the $m = \varphi(n)$ integers less than n and relatively prime to n. Assume that every prime divisor of m also divides n. Prove that m divides $a_1^k + \cdots + a_m^k$ for every positive integer k.

For brevity, given any n, we let $A(n) = \{1 \leq x \leq n, \gcd(x, n) = 1\}$ (thus $|A(n)| = \varphi(n)$). Also, let $S(n, k) = \sum_{a \in A(n)} a^k$.

We will prove the stronger statement (which eliminates the hypothesis on n).

> **Claim** — Let $n \geq 2$ be arbitrary (and $k \in \mathbb{Z}$). If $p \mid n$, then
> $$\nu_p(\varphi(n)) \leq \nu_p(S(n, k)).$$

We start with the special case where n is a prime power.

> **Lemma**
> We always have
> $$S(p^e) = \sum_{x \in A(p^e)} x^k \equiv 0 \pmod{p^{e-1}}.$$

Proof. For p odd, this follows by taking a primitive root modulo p^{e-1}. In the annoying case $p = 2$, the proof is broken into two cases: for k odd it follows by pairing x with $2^e - x$ and when k is even one can take 5 as a generator of all the quadratic residues as in the $p > 2$ case. $\square$

> **Corollary**
> We have $\nu_p(1^k + \cdots + t^k) \geq \nu_p(t) - 1$ for any k, t, p.

Proof. Assume $p \mid t$. Handle the terms in that sum divisible by p (by induction) and apply the lemma a bunch of times. $\square$

Now the idea is to add primes q one at a time to n, starting from the base case $n = p^e$. So, formally we proceed by induction on the number of prime divisors of n. We'll also assume $k \geq 1$ in what follows since the base case $k = 0$ is easy.

- First, suppose we want to go from n to nq where $q \nmid n$. In that case $\varphi(nq)$ gained $\nu_p(q - 1)$ factors of p and then we need to show $\nu_p(S(nq, k)) \geq \nu_p(\varphi(n)) + \nu_p(q - 1)$. The trick is to write
  $$A(nq) = \{a + nh \mid a \in A(n) \text{ and } h = 0, \ldots, q - 1\} \setminus qA(n)$$

and then expand using binomial theorem:

$$
\begin{aligned}
S(nq, k) &= \sum_{a \in A(n)} \sum_{h=0}^{q-1} (a + nh)^k - \sum_{a \in A(n)} (qa)^k \\
&= -q^k S(n, k) + \sum_{a \in A(n)} \sum_{h=0}^{q-1} \sum_{j=0}^{k} \left[\binom{k}{j} a^{k-j} n^j h^j \right] \\
&= -q^k S(n, k) + \sum_{j=0}^{k} \left[\binom{k}{j} n^j \left(\sum_{a \in A(n)} a^{k-j} \right) \left(\sum_{h=0}^{q-1} h^j \right) \right] \\
&= -q^k S(n, k) + \sum_{j=0}^{k} \left[\binom{k}{j} n^j S(n, k-j) \left(\sum_{h=1}^{q-1} h^j \right) \right] \\
&= (q - q^k) S(n, k) + \sum_{j=1}^{k} \left[\binom{k}{j} n^j S(n, k-j) \left(\sum_{h=1}^{q-1} h^j \right) \right].
\end{aligned}
$$

We claim every term here has enough powers of p. For the first term, $S(n, k)$ has at least $\nu_p(\varphi(n))$ factors of p; and we have the $q - q^k$ multiplier out there. For the other terms, we apply induction to $S(n, k-j)$; moreover $\sum_{h=1}^{q-1} h^j$ has at least $\nu_p(q-1) - 1$ factors of p by corollary, and we get one more factor of p (at least) from n^j.

- On the other hand, if q already divides n, then this time

$$
A(nq) = \{ a + nh \mid a \in A(n) \text{ and } h = 0, \ldots, q - 1 \}.
$$

and we have no additional burden of p to deal with; the same calculation gives

$$
S(nq, k) = q S(n, k) + \sum_{j=1}^{k} \left[\binom{k}{j} n^j S(n, k-j) \left(\sum_{h=1}^{q-1} h^j \right) \right].
$$

which certainly has enough factors of p already.

Remark. A curious bit about the problem is that $\nu_p(\varphi(n))$ can exceed $\nu_p(n)$, and so it is not true that the residues of $A(n)$ are well-behaved modulo $\varphi(n)$. For example, the official solutions give the following examples:

- Let $n = 7 \cdot 13$, so $\varphi(n) = 72$. In $A(91)$ contains nine elements which are $0 \pmod 9$, and only seven elements congruent to $7 \pmod 9$.

- Let $n = 3 \cdot 7 \cdot 13 = 273$, so $\varphi(n) = 144$. Then $A(273)$ contains 26 elements congruent to $1 \pmod 9$ and only 23 elements congruent to $4 \pmod 9$

§4 USAMO 2018/4, proposed by Ankan Bhattacharya

Let p be a prime, and let $a_1, \ldots, a_p$ be integers. Show that there exists an integer k such that the numbers

$$a_1 + k, \ a_2 + 2k, \ \ldots, \ a_p + pk$$

produce at least $\frac{1}{2}p$ distinct remainders upon division by p.

For each $k = 0, \ldots, p-1$ let G_k be the graph on on $\{1, \ldots, p\}$ where we join $\{i, j\}$ if and only if

$$a_i + ik \equiv a_j + jk \pmod{p} \iff k \equiv -\frac{a_i - a_j}{i - j} \pmod{p}.$$

So we want a graph G_k with at least $\frac{1}{2}p$ connected components.

However, each $\{i, j\}$ appears in exactly one graph G_k, so some graph has at most $\frac{1}{p}\binom{p}{2} = \frac{1}{2}(p-1)$ edges (by "pigeonhole"). This graph has at least $\frac{1}{2}(p+1)$ connected components, as desired.

> **Remark.** Here is an example for $p = 5$ showing equality can occur:
>
> $$\begin{bmatrix} 0 & 0 & 3 & 4 & 3 \\ 0 & 1 & 0 & 2 & 2 \\ 0 & 2 & 2 & 0 & 1 \\ 0 & 3 & 4 & 3 & 0 \\ 0 & 4 & 1 & 1 & 4 \end{bmatrix}.$$
>
> Ankan Bhattacharya points out more generally that $a_i = i^2$ is sharp in general.

§5 USAMO 2018/5, proposed by Kada Williams

Let $ABCD$ be a convex cyclic quadrilateral with $E = \overline{AC} \cap \overline{BD}$, $F = \overline{AB} \cap \overline{CD}$, $G = \overline{DA} \cap \overline{BC}$. The circumcircle of $\triangle ABE$ intersects line CB at B and P, and that the circumcircle of $\triangle ADE$ intersects line CD at D and Q. Assume C, B, P, G and C, Q, D, F are collinear in that order. Let $M = \overline{FP} \cap \overline{GQ}$. Prove that $\angle MAC = 90°$.

We present three general routes. (The second route, using the fact that $\overline{AC}$ is an angle bisector, has many possible variations.)

First solution (Miquel points)　　This is indeed a Miquel point problem, but the main idea is to focus on the self-intersecting cyclic quadrilateral $PBQD$ as the key player, rather than on the given $ABCD$.
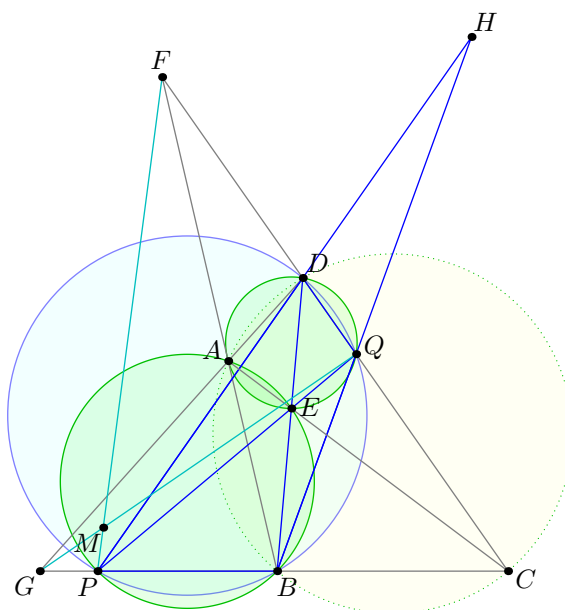
Indeed, we will prove that A is its Miquel point; this follows from the following two claims.

> **Claim —** The self-intersecting quadrilateral $PQDB$ is cyclic.

Proof. By power of a point from C: $CQ \cdot CD = CA \cdot CE = CB \cdot CP$. □

> **Claim —** Point E lies on line PQ.

Proof. $\angle AEP = \angle ABP = \angle ABC = \angle ADC = \angle ADQ = \angle AEQ$. □

To finish, let $H = \overline{PD} \cap \overline{BQ}$. By properties of the Miquel point, we have A is the foot from H to $\overline{CE}$. But also, points M, A, H are collinear by Pappus theorem on $\overline{BPG}$ and $\overline{DQF}$, as desired.

Second solution (projective)　　We start with a synthetic observation.

> **Claim** — The line $\overline{AC}$ bisects $\angle PAD$ and $\angle BAQ$.

Proof. Angle chase: $\angle PAC = \angle PAE = \angle PBE = \angle CBD = \angle CAD$. $\qquad\square$

There are three ways to finish from here:

- (Michael Kural) Suppose the external bisector of $\angle PAD$ and $\angle BAQ$ meet lines BC and DC at X and Y. Then

$$-1 = (GP; XC) = (FD; YC)$$

  which is enough to imply that $\overline{XY}$, $\overline{GQ}$, $\overline{PF}$ are concurrent.

- (Daniel Liu) Alternatively, apply the dual Desargues involution theorem to complete quadrilateral $GQFPCM$, through the point A. This gives that an involutive pairing of

$$(AC, AM)\ (AP, AQ)\ (AG, AF).$$

  This is easier to see if we project it onto the line ℓ through C perpendicular to $\overline{AC}$; if we let P', Q', G', F' be the images of the last four lines, we find the involution coincides with negative inversion through C with power $\sqrt{CB' \cdot CQ'}$ which implies that $\overline{AM} \cap \ell$ is an infinity point, as desired.

- (Kada Williams) The official solution instead shows the external angle bisector by a long trig calculation.

Third solution (inversion, Andrew Wu)   Noting that $CE \cdot CA = CP \cdot CB = CQ \cdot CD$, we perform an inversion at C swapping these pairs of points. The point G is mapped to a point G^* ray CB for which QEG^*C is cyclic, but then

$$\angle CG^*E = \angle CQE = \angle CQP = \angle DBC = \angle CBE$$

and so we conclude $EB = EG^*$. Similarly, $ED = EF^*$.

Finally, $M^* = (CG^*D) \cap (CF^*B) \neq C$, and we wish to show that $\angle EM^*C = 90°$.
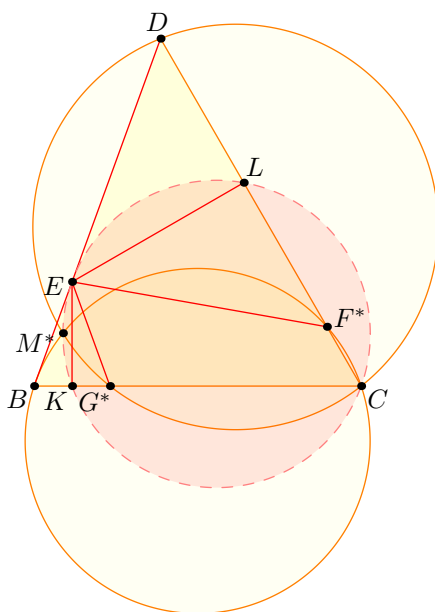
Note that M^* is the center of the spiral similarity sending $\overline{BG^*}$ to $\overline{F^*E}$. Hence it also maps the midpoint K of BG^* to the midpoint L of $\overline{F^*E}$. Consequently, M^* lies on the circumcircle KLC as well. In other words, $ELCKM^*$ is a cyclic pentagon with circumdiameter $\overline{CE}$, as desired.

§6 USAMO 2018/6, proposed by Richard Stong

Let a_n be the number of permutations $(x_1, \ldots, x_n)$ of $(1, \ldots, n)$ such that the ratios x_k/k are all distinct. Prove that a_n is odd for all $n \geq 1$.

This is the official solution; the proof has two main insights.
The first idea:

Lemma

If a permutation x works, so does the inverse permutation.

Thus it suffices to consider permutations x in which all cycles have length at most 2. Of course, there can be at most one fixed point (since that gives the ratio 1), and hence exactly one if n is odd, none if n is even.

We consider the graph K_n such that the edge $\{i, j\}$ is labeled with i/j (for $i < j$). The permutations we're considering are then equivalent to maximal matchings of this K_n. We call such a matching *fantastic* if it has an all of distinct edge labels.

Now the second insight is that if edges ab and cd have the same label for $a < b$ and $c < d$, then so do edges ac and bd. Thus:

Definition. Given a matching $\mathcal{M}$ as above we say the *neighbors* of $\mathcal{M}$ are those other matchings obtained as follows: for each label ℓ, we take some disjoint pairs of edges (possibly none) with label ℓ and apply the above switching operation (in which we replace ab and cd with ac and bd).

This neighborship relation is reflexive, and most importantly it is *symmetric* (because one can simply reverse the moves). But it is not transitive.
The second observation is that:

> **Claim** — The matching $\mathcal{M}$ has an odd number of neighbors (including itself) if and only if it is not fantastic.

Proof. Consider the label ℓ, and assume it appears $n_\ell \geq 1$ times.
If we pick k disjoint pairs and swap them, the number of ways to do this is $\binom{n_\ell}{2k}(2k-1)!!$, and so the total number of ways to perform operations on the edges labeled ℓ is

$$\sum_k \binom{n_\ell}{2k}(2k-1)!! \equiv \sum_k \binom{n_\ell}{2k} = 2^{n_\ell - 1} \pmod{2}.$$

This is even if and only if $n_\ell > 1$.
Finally, note that the number of neighbors of $\mathcal{M}$ is the product across all ℓ of the above. So it is odd if and only if each factor is odd, if and only if $n_\ell = 1$ for every ℓ. $\square$

To finish, consider a huge simple graph Γ on all the maximal matchings, with edge relations given by neighbor relation (we don't consider vertices to be connected to themselves). Observe that:

- Fantastic matchings correspond to isolated vertices (of degree zero, with no other neighbors) of Γ.

- The rest of the vertices of Γ have odd degrees (one less than the neighbor count)

- The graph Γ has an even number of vertices of odd degree (this is true for any simple graph, see "handshake lemma").

- The number of vertices of Γ is odd, namely $(2\lceil n/2 \rceil - 1)!!$.

This concludes the proof.