USAMO 2016 Solution Notes

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This is an unofficial solutions packet for the 2016 USAMO. In general, they are a combination of my own work, as well as the official solutions provided by the organizers (for which they hold any copyrights), and solutions found on the Art of Problem Solving forums.

Corrections and comments are welcome!

Contents

§0 Problems

1. Let $X_1, X_2, \dots, X_{100}$ be a sequence of mutually distinct nonempty subsets of a set S. Any two sets X_i and X_{i+1} are disjoint and their union is not the whole set S, that is, $X_i \cap X_{i+1} = \emptyset$ and $X_i \cup X_{i+1} \neq S$, for all $i \in \{1, \dots, 99\}$. Find the smallest possible number of elements in S.

2. Prove that for any positive integer k,
$$(k^2)! \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$$
is an integer.

3. Let ABC be an acute triangle and let I_B, I_C, and O denote its B-excenter, C-excenter, and circumcenter, respectively. Points E and Y are selected on $\overline{AC}$ such that $\angle ABY = \angle CBY$ and $\overline{BE} \perp \overline{AC}$. Similarly, points F and Z are selected on $\overline{AB}$ such that $\angle ACZ = \angle BCZ$ and $\overline{CF} \perp \overline{AB}$.

   Lines $I_B F$ and $I_C E$ meet at P. Prove that $\overline{PO}$ and $\overline{YZ}$ are perpendicular.

4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y,
$$(f(x) + xy) \cdot f(x - 3y) + (f(y) + xy) \cdot f(3x - y) = (f(x + y))^2.$$

5. An equilateral pentagon $AMNPQ$ is inscribed in triangle ABC such that $M \in \overline{AB}$, $Q \in \overline{AC}$, and $N, P \in \overline{BC}$. Let S be the intersection of $\overline{MN}$ and $\overline{PQ}$. Denote by ℓ the angle bisector of $\angle MSQ$.

   Prove that $\overline{OI}$ is parallel to ℓ, where O is the circumcenter of triangle ABC, and I is the incenter of triangle ABC.

6. Integers n and k are given, with $n \ge k \ge 2$. You play the following game against an evil wizard. The wizard has $2n$ cards; for each $i = 1, \dots, n$, there are two cards labeled i. Initially, the wizard places all cards face down in a row, in unknown order. You may repeatedly make moves of the following form: you point to any k of the cards. The wizard then turns those cards face up. If any two of the cards match, the game is over and you win. Otherwise, you must look away, while the wizard arbitrarily permutes the k chosen cards and then turns them back face-down. Then, it is your turn again.

   We say this game is *winnable* if there exist some positive integer m and some strategy that is guaranteed to win in at most m moves, no matter how the wizard responds. For which values of n and k is the game winnable?

§1 USAMO 2016/1, proposed by Iurie Boreico

Let $X_1, X_2, \dots, X_{100}$ be a sequence of mutually distinct nonempty subsets of a set S. Any two sets X_i and X_{i+1} are disjoint and their union is not the whole set S, that is, $X_i \cap X_{i+1} = \emptyset$ and $X_i \cup X_{i+1} \neq S$, for all $i \in \{1, \dots, 99\}$. Find the smallest possible number of elements in S.

Solution with Danielle Wang: the answer is that $|S| \ge 8$.

Since we must have $2^{|S|} \ge 100$, we must have $|S| \ge 7$. We will provide an inductive construction for a *chain* of subsets $X_1, X_2, \dots, X_{2^{n-1}+1}$ of $S = \{1, \dots, n\}$ satisfying $X_i \cap X_{i+1} = \emptyset$ and $X_i \cup X_{i+1} \neq S$ for each $n \ge 4$.

For $S = \{1, 2, 3, 4\}$, the following chain of length $2^3 + 1 = 9$ will work:

$$\{3,4\} \quad \{1\} \quad \{2,3\} \quad \{4\} \quad \{1,2\} \quad \{3\} \quad \{1,4\} \quad \{2\} \quad \{1,3\} \ .$$

Now, given a chain of subsets of $\{1, 2, \dots, n\}$ the following procedure produces a chain of subsets of $\{1, 2, \dots, n+1\}$:

1. take the original chain, delete any element, and make two copies of this chain, which now has even length;
2. glue the two copies together, joined by $\varnothing$ in between; and then
3. insert the element $n+1$ into the sets in alternating positions of the chain starting with the first.

For example, the first iteration of this construction gives:

$$\begin{array}{ccccccccc} 345 & 1 & 235 & 4 & 125 & 3 & 145 & 2 & 5 \\ 34 & 15 & 23 & 45 & 12 & 35 & 14 & 25 & \end{array}$$

It can be easily checked that if the original chain satisfies the requirements, then so does the new chain, and if the original chain has length $2^{n-1}+1$, then the new chain has length $2^n + 1$, as desired. This construction yields a chain of length 129 when $S = \{1, 2, \dots, 8\}$.

To see that $|S| = 8$ is the minimum possible size, consider a chain on the set $S = \{1, 2, \dots, 7\}$ satisfying $X_i \cap X_{i+1} = \emptyset$ and $X_i \cup X_{i+1} \neq S$. Because of these requirements any subset of size 4 or more can only be neighbored by sets of size 2 or less, of which there are $\binom{7}{1} + \binom{7}{2} = 28$ available. Thus, the chain can contain no more than 29 sets of size 4 or more and no more than 28 sets of size 2 or less. Finally, since there are only $\binom{7}{3} = 35$ sets of size 3 available, the total number of sets in such a chain can be at most $29 + 28 + 35 = 92 < 100$.

§2 USAMO 2016/2, proposed by Kiran Kedlaya

Prove that for any positive integer k,

$$(k^2)! \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$$

is an integer.

We show the exponent of any given prime p is nonnegative in the expression. Recall that the exponent of p in $n!$ is equal to $\sum_{i \ge 1} \lfloor n/p^i \rfloor$. In light of this, it suffices to show that for any prime power q, we have

$$\left\lfloor \frac{k^2}{q} \right\rfloor + \sum_{j=0}^{k-1} \left\lfloor \frac{j}{q} \right\rfloor \ge \sum_{j=0}^{k-1} \left\lfloor \frac{j+k}{q} \right\rfloor$$

Since both sides are integers, we show

$$\left\lfloor \frac{k^2}{q} \right\rfloor + \sum_{j=0}^{k-1} \left\lfloor \frac{j}{q} \right\rfloor > -1 + \sum_{j=0}^{k-1} \left\lfloor \frac{j+k}{q} \right\rfloor .$$

If we denote by $\{x\}$ the fractional part of x, then $\lfloor x \rfloor = x - \{x\}$ so it's equivalent to

$$\left\{ \frac{k^2}{q} \right\} + \sum_{j=0}^{k-1} \left\{ \frac{j}{q} \right\} < 1 + \sum_{j=0}^{k-1} \left\{ \frac{j+k}{q} \right\} .$$

However, the sum of remainders when $0, 1, \dots, k-1$ are taken modulo q is easily seen to be less than the sum of remainders when $k, k+1, \dots, 2k-1$ are taken modulo q. So

$$\sum_{j=0}^{k-1} \left\{ \frac{j}{q} \right\} \le \sum_{j=0}^{k-1} \left\{ \frac{j+k}{q} \right\}$$

follows, and we are done upon noting $\left\{ k^2/q \right\} < 1$.

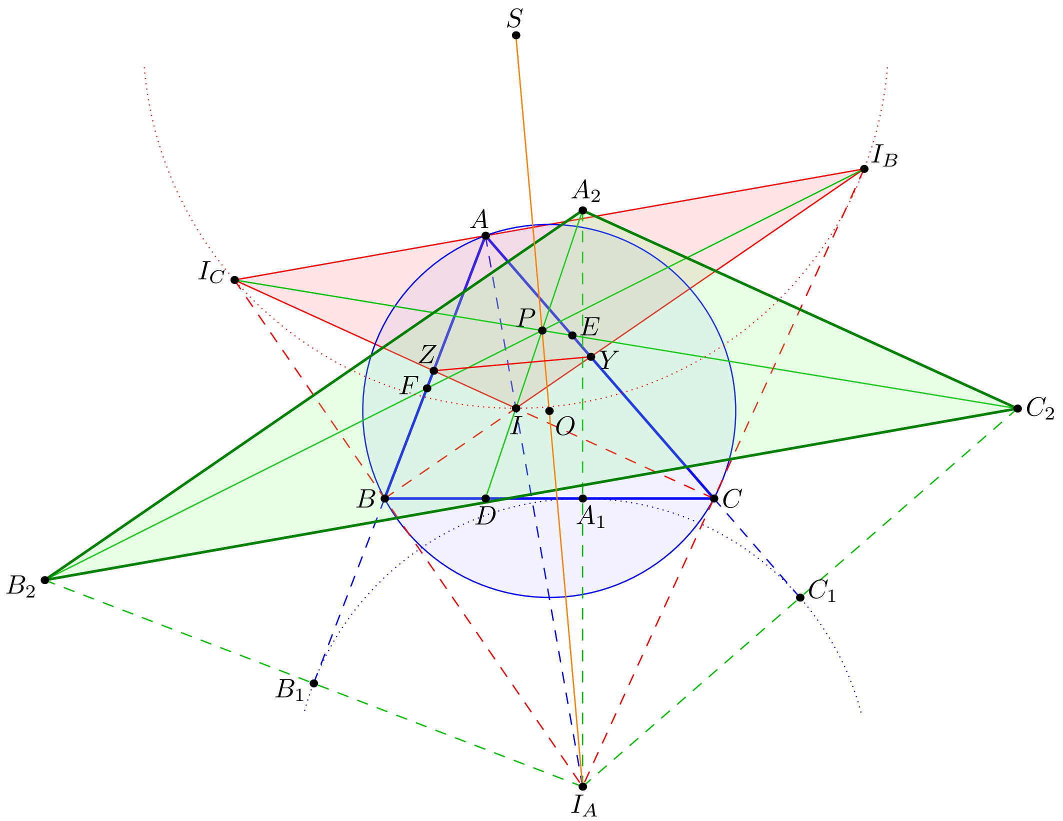
§3 USAMO 2016/3, proposed by Evan Chen and Telv Cohl

Let ABC be an acute triangle and let I_B, I_C, and O denote its B-excenter, C-excenter, and circumcenter, respectively. Points E and Y are selected on $\overline{AC}$ such that $\angle ABY = \angle CBY$ and $\overline{BE} \perp \overline{AC}$. Similarly, points F and Z are selected on $\overline{AB}$ such that $\angle ACZ = \angle BCZ$ and $\overline{CF} \perp \overline{AB}$.

Lines $I_B F$ and $I_C E$ meet at P. Prove that $\overline{PO}$ and $\overline{YZ}$ are perpendicular.

We present three solutions.

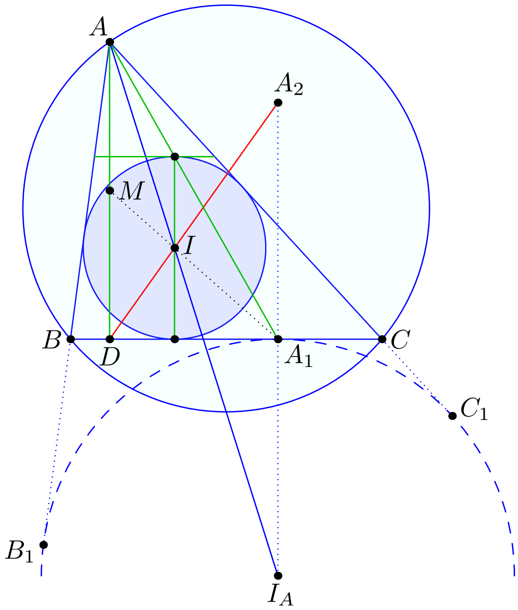
First solution Let I_A denote the A-excenter and I the incenter. Then let D denote the foot of the altitude from A. Suppose the A-excircle is tangent to $\overline{BC}$, $\overline{AB}$, $\overline{AC}$ at A_1, B_1, C_1 and let A_2, B_2, C_2 denote the reflections of I_A across these points. Let S denote the circumcenter of $\triangle II_BI_C$.



We begin with the following observation:

Claim — Points D, I, A_2 are collinear, as are points E, I_C, C_2 are collinear and points F, I_B, B_2 are collinear.

Proof. This basically follows from the "midpoints of altitudes" lemma. To see D, I, A_2 are collinear, recall first that $\overline{IA_1}$ passes through the midpoint M of $\overline{AD}$.



Now since $\overline{AD} \parallel \overline{I_A A_2}$, and M and A_1 are the midpoints of $\overline{AD}$ and $\overline{I_A A_2}$, it follows from the collinearity of A, I, I_A that D, I, A_2 are collinear as well.

The other two claims follow in a dual fashion. For example, using the homothety taking the A to C-excircle, we find that $\overline{C_1 I_C}$ bisects the altitude $\overline{BE}$, and since I_C, B, I_A are collinear the same argument now gives I_C, E, C_2 are collinear. The fact that I_B, F, B_2 are collinear is symmetric. $\square$

Observe that $\overline{B_2C_2} \parallel \overline{B_1C_1} \parallel \overline{I_BI_C}$. Proceeding similarly on the other sides, we discover $\triangle II_BI_C$ and $\triangle A_2B_2C_2$ are homothetic. Hence P is the center of this homothety (in particular, D, I, P, A_2 are collinear). Moreover, P lies on the line joining I_A to S, which is the Euler line of $\triangle II_BI_C$, so it passes through the nine-point center of $\triangle II_BI_C$, which is O. Consequently, P, O, I_A are collinear as well.

To finish, we need only prove that $\overline{OS} \perp \overline{YZ}$. In fact, we claim that $\overline{YZ}$ is the radical axis of the circumcircles of $\triangle ABC$ and $\triangle II_BI_C$. Actually, Y is the radical center of these two circumcircles and the circle with diameter $\overline{II_B}$ (which passes through A and C). Analogously Z is the radical center of the circumcircles and the circle with diameter $\overline{II_C}$, and the proof is complete.

Second solution (barycentric, outline, Colin Tang) we are going to use barycentric coordinates to show that the line through O perpendicular to $\overline{YZ}$ is concurrent with $\overline{I_BF}$ and $\overline{I_CE}$.

The displacement vector $\overrightarrow{YZ}$ is proportional to $(a(b-c) : -b(a+c) : c(a+b))$, and so by strong perpendicularity criterion and doing a calculation gives the line

$$x(b-c)bc(a+b+c) + y(a+c)ac(a+b-c) + z(a+b)ab(-a+b-c) = 0.$$

On the other hand, line I_CE has equation

$$0 = \det \begin{bmatrix} a & b & -c \\ S_C & 0 & S_A \\ x & y & z \end{bmatrix} = bS_a \cdot x + (-cS_C - aS_A) \cdot y + (-bS_C) \cdot z$$

and similarly for I_BF. Consequently, concurrence of these lines is equivalent to

$$\det \begin{bmatrix} bS_A & -cS_C - aS_A & -bS_C \\ cS_A & -cS_B & -aS_A - bS_B \\ (b-c)bc(a+b+c) & (a+c)ac(a+b-c) & (a+b)ab(-a+b-c) \end{bmatrix} = 0$$

which is a computation.

Authorship comments I was intrigued by a Taiwan TST problem which implied that, in the configuration above, $\angle I_B D I_C$ was bisected by $\overline{DA}$. This motivated me to draw all three properties above where I_A and P were isogonal conjugates with respect to DEF. After playing around with this picture for a long time, I finally noticed that O was on line PI_A. (So the original was to show that I_BF, I_CE, DA_2 concurrent). Eventually I finally noticed in the picture that PI_A actually passed through the circumcenter of ABC as well. This took me many hours to prove.

The final restatement (which follows quickly from P, O, I_A collinear) was discovered by Telv Cohl when I showed him the problem.

§4 USAMO 2016/4, proposed by Titu Andreescu

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y,

$$(f(x) + xy) \cdot f(x - 3y) + (f(y) + xy) \cdot f(3x - y) = (f(x + y))^2.$$

We claim that the only two functions satisfying the requirements are $f(x) \equiv 0$ and $f(x) \equiv x^2$. These work.

First, taking $x = y = 0$ in the given yields $f(0) = 0$, and then taking $x = 0$ gives $f(y)f(-y) = f(y)^2$. So also $f(-y)^2 = f(y)f(-y)$, from which we conclude f is even. Then taking $x = -y$ gives

$$\forall x \in \mathbb{R} : \qquad f(x) = x^2 \qquad \text{or} \qquad f(4x) = 0 \qquad (\bigstar)$$

for all x.

Now we claim

Claim — $f(z) = 0 \iff f(2z) = 0 \qquad (\spadesuit)$.

Proof. Let $(x, y) = (3t, t)$ in the given to get

$$\left(f(t) + 3t^2\right) f(8t) = f(4t)^2.$$

Now if $f(4t) \neq 0$ (in particular, $t \neq 0$), then $f(8t) \neq 0$. Thus we have $(\spadesuit)$ in the forwards direction.

Then $f(4t) \neq 0 \overset{(\bigstar)}{\Longrightarrow} f(t) \neq 0 \overset{(\spadesuit)}{\Longrightarrow} f(2t) \neq 0$ implies the reverse direction, the last step being the forward direction $(\spadesuit)$. $\square$

By putting together $(\bigstar)$ and $(\spadesuit)$ we finally get

$$\forall x \in \mathbb{R} : \qquad f(x) = x^2 \qquad \text{or} \qquad f(x) = 0 \qquad (\heartsuit)$$

We are now ready to approach the main problem. Assume there's an $a \neq 0$ for which $f(a) = 0$; we show that $f \equiv 0$.

Let $b \in \mathbb{R}$ be given. Since f is even, we can assume without loss of generality that $a, b > 0$. Also, note that $f(x) \geq 0$ for all x by $(\heartsuit)$. By using $(\spadesuit)$ we can generate $c > b$ such that $f(c) = 0$ by taking $c = 2^n a$ for a large enough integer n. Now, select $x, y > 0$ such that $x - 3y = b$ and $x + y = c$. That is,

$$(x, y) = \left(\frac{3c + b}{4}, \frac{c - b}{4}\right).$$

Substitution into the original equation gives

$$0 = (f(x) + xy)\, f(b) + (f(y) + xy)\, f(3x - y) = (f(x) + f(y) + 2xy)\, f(b)$$

Since $f(x) + f(y) + 2xy > 0$, if follows that $f(b) = 0$, as desired.

§5 USAMO 2016/5, proposed by Ivan Borsenco

An equilateral pentagon $AMNPQ$ is inscribed in triangle ABC such that $M \in \overline{AB}$, $Q \in \overline{AC}$, and $N, P \in \overline{BC}$. Let S be the intersection of $\overline{MN}$ and $\overline{PQ}$. Denote by ℓ the angle bisector of $\angle MSQ$.

Prove that $\overline{OI}$ is parallel to ℓ, where O is the circumcenter of triangle ABC, and I is the incenter of triangle ABC.

First solution (complex) In fact, we only need $AM = AQ = NP$ and $MN = QP$.

We use complex numbers with ABC the unit circle, assuming WLOG that A, B, C are labeled counterclockwise. Let x, y, z be the complex numbers corresponding to the arc midpoints of BC, CA, AB, respectively; thus $x + y + z$ is the incenter of $\triangle ABC$. Finally, let $s > 0$ be the side length of $AM = AQ = NP$.

Then, since $MA = s$ and $MA \perp OX$, it follows that

$$m - a = i \cdot sx.$$

Similarly, $n - p = i \cdot sy$ and $a - q = i \cdot sz$, so summing these up gives

$$i \cdot s(x + y + z) = (p - q) + (m - n) = (m - n) - (q - p).$$

Since $MN = PQ$, the argument of $(m - n) - (q - p)$ is along the external angle bisector of the angle formed, which is perpendicular to ℓ. On the other hand, $x + y + z$ is oriented in the same direction as OI, as desired.

Second solution (trig, Danielle Wang) Let δ and ϵ denote $\angle MNB$ and $\angle CPQ$. Also, assume $AMNPQ$ has side length 1.

In what follows, assume $AB < AC$. First, we note that

$$\begin{aligned} BN &= (c - 1)\cos B + \cos\delta, \\ CP &= (b - 1)\cos C + \cos\epsilon, \text{ and} \\ a &= 1 + BN + CP \end{aligned}$$

from which it follows that

$$\cos\delta + \cos\epsilon = \cos B + \cos C - 1$$

Also, by the Law of Sines, we have $\frac{c-1}{\sin\delta} = \frac{1}{\sin B}$ and similarly on triangle CPQ, and from this we deduce

$$\sin\epsilon - \sin\delta = \sin B - \sin C.$$

The sum-to-product formulas

$$\begin{aligned} \sin\epsilon - \sin\delta &= 2\cos\left(\frac{\epsilon + \delta}{2}\right)\sin\left(\frac{\epsilon - \delta}{2}\right) \\ \cos\epsilon - \cos\delta &= 2\cos\left(\frac{\epsilon + \delta}{2}\right)\cos\left(\frac{\epsilon - \delta}{2}\right) \end{aligned}$$

give us

$$\tan\left(\frac{\epsilon - \delta}{2}\right) = \frac{\sin\epsilon - \sin\delta}{\cos\epsilon - \cos\delta} = \frac{\sin B - \sin C}{\cos B + \cos C - 1}.$$

Now note that ℓ makes an angle of $\frac{1}{2}(\pi + \epsilon - \delta)$ with line BC. Moreover, if line OI intersects line BC with angle φ then

$$\tan \varphi = \frac{r - R\cos A}{\frac{1}{2}(b-c)}.$$

So in order to prove the result, we only need to check that

$$\frac{r - R\cos A}{\frac{1}{2}(b-c)} = \frac{\cos B + \cos C + 1}{\sin B - \sin C}.$$

Using the fact that $b = 2R\sin B$, $c = 2R\sin C$, this reduces to the fact that $r/R + 1 = \cos A + \cos B + \cos C$, which is the so-called Carnot theorem.

§6 USAMO 2016/6, proposed by Gabriel Carroll

Integers n and k are given, with $n \geq k \geq 2$. You play the following game against an evil wizard. The wizard has $2n$ cards; for each $i = 1, \dots, n$, there are two cards labeled i. Initially, the wizard places all cards face down in a row, in unknown order. You may repeatedly make moves of the following form: you point to any k of the cards. The wizard then turns those cards face up. If any two of the cards match, the game is over and you win. Otherwise, you must look away, while the wizard arbitrarily permutes the k chosen cards and then turns them back face-down. Then, it is your turn again.

We say this game is *winnable* if there exist some positive integer m and some strategy that is guaranteed to win in at most m moves, no matter how the wizard responds. For which values of n and k is the game winnable?

The game is winnable if and only if $k < n$.

First, suppose $2 \leq k < n$. Query the cards in positions $\{1, \dots, k\}$, then $\{2, \dots, k+1\}$, and so on, up to $\{2n-k+1, 2n\}$. By taking the difference of any two adjacent queries, we can deduce for certain the values on cards $1, 2, \dots, 2n-k$. If $k \leq n$, this is more than n cards, so we can find a matching pair.

For $k = n$ we remark the following: at each turn after the first, assuming one has not won, there are n cards representing each of the n values exactly once, such that the player has no information about the order of those n cards. We claim that consequently the player cannot guarantee victory. Indeed, let S denote this set of n cards, and $\overline{S}$ the other n cards. The player will never win by picking only cards in S or $\overline{S}$. Also, if the player selects some cards in S and some cards in $\overline{S}$, then it is possible that the choice of cards in S is exactly the complement of those selected from $\overline{S}$; the strategy cannot prevent this since the player has no information on S. This implies the result.