USAMO 2013 Solution Notes

Edited by Evan Chen

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This is an unofficial solutions packet for the 2013 USAMO. In general, they are a combination of my own work, as well as the official solutions provided by the organizers (for which they hold any copyrights), and solutions found on the Art of Problem Solving forums.

Corrections and comments are welcome!

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§0 Problems

1. In triangle ABC, points P, Q, R lie on sides BC, CA, AB, respectively. Let ω_A, ω_B, ω_C denote the circumcircles of triangles AQR, BRP, CPQ, respectively. Given the fact that segment AP intersects ω_A, ω_B, ω_C again at X, Y, Z respectively, prove that $YX/XZ = BP/PC$.

2. For a positive integer $n \ge 3$ plot n equally spaced points around a circle. Label one of them A, and place a marker at A. One may move the marker forward in a clockwise direction to either the next point or the point after that. Hence there are a total of $2n$ distinct moves available; two from each point. Let a_n count the number of ways to advance around the circle exactly twice, beginning and ending at A, without repeating a move. Prove that $a_{n-1} + a_n = 2^n$ for all $n \ge 4$.

3. Let n be a positive integer. There are $\frac{n(n+1)}{2}$ tokens, each with a black side and a white side, arranged into an equilateral triangle, with the biggest row containing n marks. Initially, each mark has the white side up. An operation is to choose a line parallel to the sides of the triangle, and flip all the marks on that line. A configuration is called admissible if it can be obtained from the initial configuration by performing a finite number of operations. For each admissible configuration C, let $f(C)$ denote the smallest number of operations required to obtain C from the initial configuration. Find the maximum value of $f(C)$, where C varies over all admissible configurations.

4. Find all real numbers $x, y, z \ge 1$ satisfying
$$\min\left(\sqrt{x+xyz}, \sqrt{y+xyz}, \sqrt{z+xyz}\right) = \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

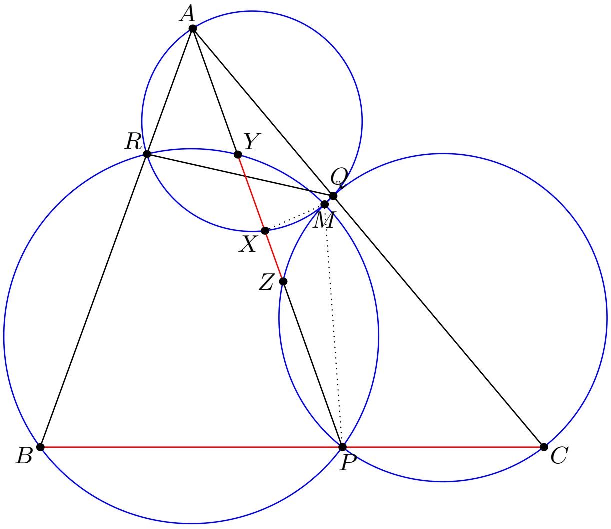
5. Let m and n be positive integers. Prove that there exists an integer c such that cm and cn have the same nonzero decimal digits.

6. Let ABC be a triangle. Find all points P on segment BC satisfying the following property: If X and Y are the intersections of line PA with the common external tangent lines of the circumcircles of triangles PAB and PAC, then
$$\left(\frac{PA}{XY}\right)^2 + \frac{PB \cdot PC}{AB \cdot AC} = 1.$$

§1 USAMO 2013/1, proposed by Zuming Feng

In triangle ABC, points P, Q, R lie on sides BC, CA, AB, respectively. Let ω_A, ω_B, ω_C denote the circumcircles of triangles AQR, BRP, CPQ, respectively. Given the fact that segment AP intersects ω_A, ω_B, ω_C again at X, Y, Z respectively, prove that $YX/XZ = BP/PC$.

Let M be the concurrence point of ω_A, ω_B, ω_C (by Miquel's theorem).



Then M is the center of a spiral similarity sending $\overline{YZ}$ to $\overline{BC}$. So it suffices to show that this spiral similarity also sends X to P, but

$$\measuredangle MXY = \measuredangle MXA = \measuredangle MRA = \measuredangle MRB = \measuredangle MPB$$

so this follows.

§2 USAMO 2013/2, proposed by Kiran Kedlaya

For a positive integer $n \ge 3$ plot n equally spaced points around a circle. Label one of them A, and place a marker at A. One may move the marker forward in a clockwise direction to either the next point or the point after that. Hence there are a total of $2n$ distinct moves available; two from each point. Let a_n count the number of ways to advance around the circle exactly twice, beginning and ending at A, without repeating a move. Prove that $a_{n-1} + a_n = 2^n$ for all $n \ge 4$.

Imagine the counter is moving along the set $S = \{0, 1, \dots, 2n\}$ instead, starting at 0 and ending at $2n$, in jumps of length 1 and 2. We can then record the sequence of moves as a matrix of the form

$$\begin{bmatrix} p_0 & p_1 & p_2 & \dots & p_{n-1} & p_n \\ p_n & p_{n+1} & p_{n+2} & \dots & p_{2n-1} & p_{2n} \end{bmatrix}$$

where $p_i = 1$ if the point i was visited by the counter, and $p_i = 0$ if the point was not visited by the counter. Note that $p_0 = p_{2n} = 1$ and the upper-right and lower-left entries are equal. Then, the problem amounts to finding the number of such matrices which avoid the contiguous submatrices

$$\begin{bmatrix} 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

which correspond to forbidding jumps of length greater than 2, repeating a length 2 jump and repeating a length 1 jump.

We will for now ignore the boundary conditions. Instead we say a $2 \times m$ matrix is *silver* ($m \ge 2$) if it avoids the three shapes above. We are interested in particular two types of silver matrices:

- *type B matrices*, of the shape $\begin{bmatrix} 1 & \dots & 1 \\ 0 & \dots & 0 \end{bmatrix}$, and
- *type C matrices*, of the shape $\begin{bmatrix} 1 & \dots & 0 \\ 0 & \dots & 1 \end{bmatrix}$.

We let b_m and c_m denote matrices of each type, of size $2 \times m$, and claim the following two recursions for $m \ge 4$:

$$\begin{aligned} b_m &= c_{m-1} + b_{m-2} + c_{m-2} \\ c_m &= b_{m-1} + b_{m-2} + c_{m-2}. \end{aligned}$$

We prove the first recursion since the second is analogous. Consider the second column of a type B matrix. If it is $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ then we find a type C matrix, giving c_{m-1}. If it is $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ then the third column must be either $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$, and we get a count of $b_{m-2} + c_{m-2}$.

Using this recursion, the first few values are

n	2	3	4	5	6	7	8
b_n	0	2	2	6	10	22	42
c_n	1	1	3	5	11	21	43

and a calculation gives $b_n = \frac{2^{n-1} - 2(-1)^{n-1}}{3}$, $c_n = \frac{2^{n-1} + (-1)^{n-1}}{3}$.

We now relate a_n to b_m and c_m. Observe that a matrix as described in the problem is a silver matrix of one of two forms:

$$\begin{bmatrix} 1 & p_1 & p_2 & \dots & p_{n-1} & 0 \\ 0 & p_{n+1} & p_{n+2} & \dots & p_{2n-1} & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & p_1 & p_2 & \dots & p_{n-1} & 1 \\ 1 & p_{n+1} & p_{n+2} & \dots & p_{2n-1} & 1 \end{bmatrix}.$$

There are c_{n+1} matrices of the first form and $2b_{n-1} + 2c_{n-1}$ matrices of the second form, thus calculation gives the result

$$a_n = c_{n+1} + 2b_{n-1} + 2c_{n-1} = \frac{2^{n+1} + (-1)^{n+1}}{3}.$$

§3 USAMO 2013/3, proposed by Warut Suksompong

Let n be a positive integer. There are $\frac{n(n+1)}{2}$ tokens, each with a black side and a white side, arranged into an equilateral triangle, with the biggest row containing n marks. Initially, each mark has the white side up. An operation is to choose a line parallel to the sides of the triangle, and flip all the marks on that line. A configuration is called admissible if it can be obtained from the initial configuration by performing a finite number of operations. For each admissible configuration C, let $f(C)$ denote the smallest number of operations required to obtain C from the initial configuration. Find the maximum value of $f(C)$, where C varies over all admissible configurations.

The answer is

$$\max_C f(C) = \begin{cases} 6k & n = 4k \\ 6k+1 & n = 4k+1 \\ 6k+2 & n = 4k+2 \\ 6k+3 & n = 4k+3. \end{cases}$$

The main point of the problem is actually to determine all linear dependencies among the $3n$ possible moves (since the moves commute and applying a move twice is the same as doing nothing). In what follows, assume $n > 1$ for convenience.

To this end, we consider sequences of operations as additive vectors in $v \in \mathbb{F}_2^{3n}$, with the linear map $T \colon \mathbb{F}_2^{3n} \to \mathbb{F}_2^{\frac{1}{2}n(n+1)}$ denoting the result of applying a vector v. We in particular focus on the following four vectors.

- Three vectors x, y, z are defined by choosing all n lines parallel to one axis. Note $T(x) = T(y) = T(z) = \mathbf{1}$ (i.e. these vectors flip all tokens).
- The vector θ which toggles all lines with an even number of tokens. One can check that $T(\theta) = \mathbf{0}$. (Easiest to guess from $n = 2$ and $n = 3$ case.) One amusing proof that this works is to use Vivani's theorem: in an equilateral triangle ABC, the sum of distances from an interior point P to the three sides is equal.

The main claim is:

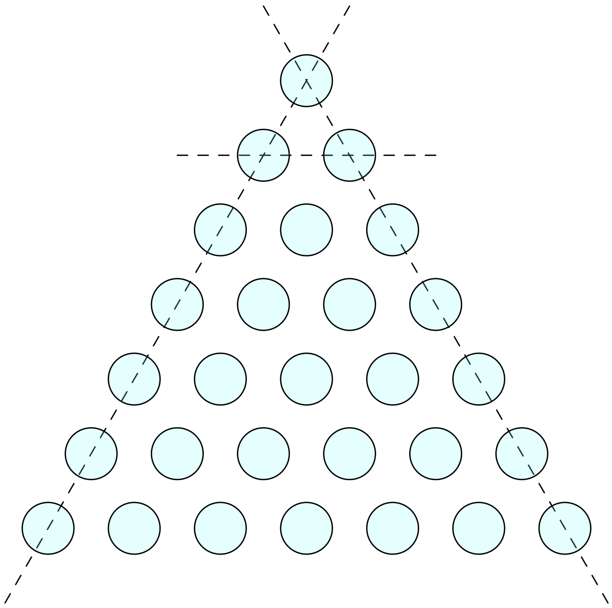
Claim — For $n \geq 2$, the kernel of T has exactly eight elements, namely $\{\mathbf{0}, x+y, y+z, z+x, \theta, \theta+x+y, \theta+y+z, \theta+z+x\}$.

Proof. Suppose $T(v) = 0$.

- If v uses the y-move of length n, then we replace v with $v + (x+y)$ to obtain a vector in the kernel not using the y-move of length n.
- If v uses the z-move of length n, then we replace v with $v + (x+z)$ to obtain a vector in the kernel not using the z-move of length n.
- If v uses the x-move of length 2, then
  - if n is odd, replace v with $v + \theta$;
  - if n is even, replace v with $v + (\theta + y + z)$

  to obtain a vector in the kernel not using the x-move of length 2.

A picture is shown below, with the unused rows being dotted.



Then, it is easy to check inductively that v must now be the zero vector, after the replacements. The idea is that for each token t, if two of the moves involving t are unused, so is the third, and in this way we can show all rows are unused. Thus the original v was in the kernel we described.

(An alternative proof by induction is feasible too; as a sequence of movings which does not affect the top n rows also does not affect the to $n-1$ rows.) ☐

Then problem is a coordinate bash, since given any v we now know exactly which vectors w have $T(v) = T(w)$, so given any admissible configuration C one can exactly compute $f(C)$ as the minimum of eight values.

§4 USAMO 2013/4, proposed by Titu Andreescu

Find all real numbers $x, y, z \ge 1$ satisfying

$$\min\left(\sqrt{x+xyz}, \sqrt{y+xyz}, \sqrt{z+xyz}\right) = \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

Set $x = 1+a$, $y = 1+b$, $z = 1+c$ which eliminates the $x, y, z \ge 1$ condition. Then the given equation rewrites as

$$\sqrt{(1+a)\left(1+(1+b)(1+c)\right)} = \sqrt{a} + \sqrt{b} + \sqrt{c}.$$

In fact, we are going to prove the left-hand side always exceeds the right-hand side, and then determine the equality cases. We have:

$$\begin{aligned}
(1+a)\left(1+(1+b)(1+c)\right) &= (a+1)\left(1+(b+1)(1+c)\right) \\
&\le (a+1)\left(1+\left(\sqrt{b}+\sqrt{c}\right)^2\right) \\
&\le \left(\sqrt{a}+\left(\sqrt{b}+\sqrt{c}\right)\right)
\end{aligned}$$

by two applications of Cauchy-Schwarz.

Equality holds if $bc = 1$ and $1/a = \sqrt{b} + \sqrt{c}$. Letting $c = t^2$ for $t \ge 1$, we recover $b = t^{-2} \le t^2$ and $a = \frac{1}{t+1/t} \le t^2$.

Hence the solution set is

$$(x, y, z) = \left(1 + \left(\frac{t}{t^2+1}\right)^2, 1 + \frac{1}{t^2}, 1 + t^2\right)$$

and permutations, for any $t > 0$.

§5 USAMO 2013/5, proposed by Richard Stong

Let m and n be positive integers. Prove that there exists an integer c such that cm and cn have the same nonzero decimal digits.

One-line spoiler: 142857.

To work out the details, we begin by noting:

Claim — Let m and n be fixed positive integers. Then there exist arbitrarily large primes p such that

$$p \mid 10^e m - n.$$

Proof. By Kobayashi theorem on the set $\{10^e m\} - n$. (Of course, more reasonable proofs exist...) $\square$

In that case, the periodic decimal expansions of $\frac{m}{p}$ and $\frac{n}{p}$ are cyclic shifts of each other. Thus if one looks at the repeating block of decimals in $\frac{1}{p}$, one may take c to be that resulting integer.

Remark. The official USAMO solutions propose using the fact that 10 is a primitive root modulo 7^e for each $e \ge 1$, by Hensel lifting lemma. This argument is *incorrect*, because it breaks if either m or n are divisible by 7.

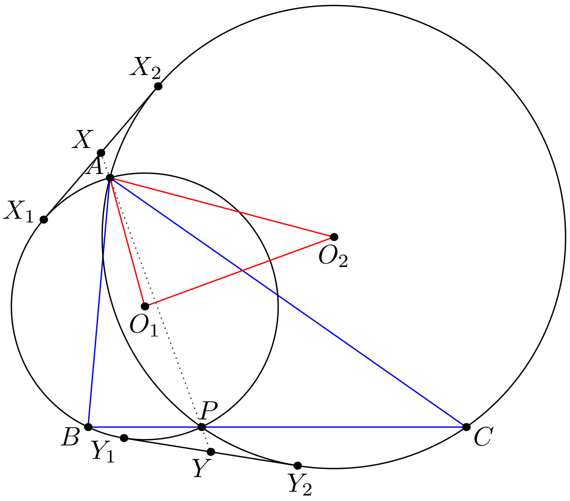
One may be tempted to resort to using large primes rather than powers of 7 to deal with this issue; it would certainly imply the claim, for instance. However it is an open conjecture (a special case of Artin's primitive root conjecture) whether or not $10 \pmod{p}$ is primitive infinitely often, which is the condition necessary for this argument to work.

§6 USAMO 2013/6, proposed by Titu and Cosmin

Let ABC be a triangle. Find all points P on segment BC satisfying the following property: If X and Y are the intersections of line PA with the common external tangent lines of the circumcircles of triangles PAB and PAC, then

$$\left(\frac{PA}{XY}\right)^2 + \frac{PB \cdot PC}{AB \cdot AC} = 1.$$

Let O_1 and O_2 denote the circumcenters of PAB and PAC. The main idea is to notice that $\triangle ABC$ and $\triangle AO_1O_2$ are spirally similar.



We first claim that $\triangle AO_1B \sim \triangle AO_2C$. Indeed, assume without loss of generality that $\angle APB \le 90^\circ$. Then

$$\angle AO_1B = 2\angle ABP$$

but

$$\angle AO_2C = 2\left(180 - \angle APC\right) = 2\angle ABP.$$

Hence $\angle AO_1B = \angle AO_2C$. Moreover, both triangles are isosceles, establishing the claim.

Since spiral similarities come in pairs, from $\triangle AO_1B \sim \triangle AO_2C$ we obtain $\triangle ABC \sim \triangle AO_1O_2$. But since $\frac{PA}{XY}$ is determined entirely by A, O_1, O_2, we find that $\frac{PA}{XY}$ does not depend on the choice of P!

The rest is basically just computation and several approaches are possible. Without loss of generality A is closer to X than Y, and let the common tangents be $\overline{X_1X_2}$ and $\overline{Y_1Y_2}$. Here are two approaches.

- One method is to first note that

  $$1 - \left(\frac{PA}{XY}\right)^2 = \frac{(XY - PA)(XY + PA)}{XY^2} = \frac{4XA \cdot XP}{XY^2} = \frac{X_1X_2^2}{XY^2}$$

  where the last equation follows by Power of a Point. Let $\theta = \angle X_2O_2O_1$. We have

  $$XY = \frac{X_1Y_1 + X_2Y_2}{2} = \frac{(r_1 + r_2)\sin\theta}{2} = (r_1 + r_2)\sin\theta$$

where $r_1 = O_1X_1$ and $r_2 = O_2X_2$, and $X_1Y_1 = O_1O_2 \sin\theta$. Hence the ratios in question is equal to

$$\frac{X_1X_2^2}{XY^2} = \left(\frac{O_1O_2}{r_1 + r_2}\right)^2 = \left(\frac{a}{b+c}\right)^2.$$

Hence the problem is equivalent to

$$PB \cdot PC = bc\left(\frac{a}{b+c}\right)^2.$$

If we write this as the quadratic $PB(a - PB) = bc\left(\frac{a}{b+c}\right)^2$, then it is clear there are at most two solutions, symmetric across the midpoint of $\overline{BC}$. If P is the foot of the angle bisector, then we have

$$PB \cdot PC = \left(\frac{b}{b+c} \cdot a\right)\left(\frac{c}{b+c} \cdot a\right) = bc\left(\frac{a}{b+c}\right)^2.$$

So the answer is the foot of the angle bisector and its reflection across the midpoint.

- Here's a solution that does the calculations with a, b, c alone.

  We'll perform the main calculation with the convenient scaling $O_BO_C = a$, $AO_C = b$, and $AO_B = c$. Let B_1 and C_1 be the tangency points of X, and let $h = AM$ be the height of $\triangle AO_BO_C$. Note that by Power of a Point, we have $X_1B^2 = X_1C^2 = XM^2 - h^2$. Also, by Pythag we easily obtain $B_1C_1 = a^2 - (b-c)^2$. Then

$$\begin{aligned}
1 - \left(\frac{MA}{MX}\right)^2 &= 1 - \frac{h^2}{h^2 + XB_1^2} \\
&= \frac{XB_1^2}{h^2 + XB_1^2} \\
&= \frac{B_1C_1^2}{(2h)^2 + B_1C_1^2} \\
&= \frac{a^2 - (b-c)^2}{\left(\frac{4[AO_BO_C]}{a}\right)^2 + a^2 - (b-c)^2} \\
&= \frac{a^2(a^2 - (b-c)^2)}{(a+b+c)(-a+b+c)(a-b+c)(a+b-c) + a^2(a^2 - (b-c)^2)} \\
&= \frac{a^2}{(a+b+c)(-a+b+c) + a^2} \\
&= \left(\frac{a}{b+c}\right)^2.
\end{aligned}$$

  Now, for P the foot of the angle bisector or its reflection, we have

$$\frac{PB \cdot PC}{AB \cdot AC} = \frac{\left(\frac{b}{b+c}a\right)\left(\frac{c}{b+c}a\right)}{bc} = \left(\frac{a}{b+c}\right)^2.$$

  Since there are clearly at most two solutions as $\frac{PA}{XY}$ is fixed, these are the only two solutions.