USAMO 2012 Solution Notes

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This is an unofficial solutions packet for the 2012 USAMO. In general, they are a combination of my own work, as well as the official solutions provided by the organizers (for which they hold any copyrights), and solutions found on the Art of Problem Solving forums.

Corrections and comments are welcome!

Contents

§0 Problems

1. Find all integers $n \geq 3$ such that among any n positive real numbers $a_1, a_2, \dots, a_n$ with
$$\max(a_1, a_2, \dots, a_n) \leq n \cdot \min(a_1, a_2, \dots, a_n),$$
there exist three that are the side lengths of an acute triangle.

2. A circle is divided into congruent arcs by 432 points. The points are colored in four colors such that some 108 points are colored red, some 108 points are colored green, some 108 points are colored blue, and the remaining 108 points are colored yellow. Prove that one can choose three points of each color in such a way that the four triangles formed by the chosen points of the same color are congruent.

3. Determine which integers $n > 1$ have the property that there exists an infinite sequence a_1, a_2, a_3, ... of nonzero integers such that the equality
$$a_k + 2a_{2k} + \cdots + na_{nk} = 0$$
holds for every positive integer k.

4. Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $f(n!) = f(n)!$ for all positive integers n and such that $m - n$ divides $f(m) - f(n)$ for all distinct positive integers m, n.

5. Let P be a point in the plane of $\triangle ABC$, and γ a line through P. Let A', B', C' be the points where the reflections of lines PA, PB, PC with respect to γ intersect lines BC, CA, AB respectively. Prove that A', B', C' are collinear.

6. For integer $n \geq 2$, let x_1, x_2, ..., x_n be real numbers satisfying
$$x_1 + x_2 + \cdots + x_n = 0 \quad \text{and} \quad x_1^2 + x_2^2 + \cdots + x_n^2 = 1.$$
For each subset $A \subseteq \{1, 2, \dots, n\}$, define $S_A = \sum_{i \in A} x_i$. (If A is the empty set, then $S_A = 0$.) Prove that for any positive number λ, the number of sets A satisfying $S_A \geq \lambda$ is at most $2^{n-3}/\lambda^2$. For which choices of x_1, x_2, ..., x_n, λ does equality hold?

§1 USAMO 2012/1, proposed by Titu Andreescu

Find all integers $n \geq 3$ such that among any n positive real numbers $a_1, a_2, \dots, a_n$ with

$$\max(a_1, a_2, \dots, a_n) \leq n \cdot \min(a_1, a_2, \dots, a_n),$$

there exist three that are the side lengths of an acute triangle.

The answer is $n > 12$; a counterexample for $n \leq 12$ is to let $F_1 = 1$, $F_2 = 1$, $F_3 = 2$ etc denote the Fibonacci numbers and let $a_i = \sqrt{F_i}$ for each $1 \leq i \leq n$.

To show that all $n > 12$ work, assume $1 = a_1 \leq a_2 \leq \dots \leq a_n$ without loss of generality. Assume for contradiction there are no acute triangles. Then $a_2 \geq 1$ and henceforth

$$a_{n+2}^2 \geq a_{n+1}^2 + a_n^2$$

and so in the same way we derive $a_n^2 \geq F_n$. But for $n > 12$ we have $F_n^2 > n^2$, a contradiction.

§2 USAMO 2012/2, proposed by Gregory Galperin

A circle is divided into congruent arcs by 432 points. The points are colored in four colors such that some 108 points are colored red, some 108 points are colored green, some 108 points are colored blue, and the remaining 108 points are colored yellow. Prove that one can choose three points of each color in such a way that the four triangles formed by the chosen points of the same color are congruent.

First, consider the 431 possible non-identity rotations of the red points, and count overlaps with green points. If we select a rotation randomly, then each red point lies over a green point with probability $\frac{108}{431}$; hence the expected number of red-green incidences is

$$\frac{108}{431} \cdot 108 > 27$$

and so by pigeonhole, we can find a red 28-gon and a green 28-gon which are rotations of each other.

Now, look at the 430 rotations of this 28-gon (that do not give the all-red or all-green configuration) and compare it with the blue points. The same approach gives

$$\frac{108}{430} \cdot 28 > 7$$

incidences, so we can find red, green, blue 8-gons which are similar under rotation.

Finally, the 429 nontrivial rotations of this 8-gon expect

$$\frac{108}{429} \cdot 8 > 2$$

incidences with yellow. So finally we have four monochromatic 3-gons, one of each color, which are rotations of each other.

§3 USAMO 2012/3, proposed by Gabriel Carroll

Determine which integers $n > 1$ have the property that there exists an infinite sequence a_1, a_2, a_3, ... of nonzero integers such that the equality

$$a_k + 2a_{2k} + \cdots + na_{nk} = 0$$

holds for every positive integer k.

Answer: all $n > 2$.

For $n = 2$, we have $a_k + 2a_{2k} = 0$, which is clearly not possible, since it implies $a_{2^k} = \frac{a_1}{2^k}$ for all k.

For $n \geq 3$ we will construct a *completely multiplicative* sequence (meaning $a_{ij} = a_i a_j$ for all i and j). Thus (a_i) is determined by its value on primes, and satisfies the condition as long as $a_1 + 2a_2 + \cdots + na_n = 0$. The idea is to take two large primes and use Bezout's theorem, but the details require significant care.

We start by solving the case where $n \geq 9$. In that case, by Bertrand postulate there exists primes p and q such that

$$\lceil n/2 \rceil < q < 2\lceil n/2 \rceil. \quad \text{and} \quad \frac{1}{2}(q-1) < p < q-1$$

Clearly $p \neq q$, and $q \geq 7$, so $p > 3$. Also, $p < q < n$ but $2q > n$, and $4p \geq 4\left(\frac{1}{2}(q+1)\right) > n$. We now stipulate that $a_r = 1$ for any prime $r \neq p, q$ (in particular including $r = 2$ and $r = 3$). There are now three cases, identical in substance.

- If $p, 2p, 3p \in [1, n]$ then we would like to choose nonzero a_p and a_q such that

$$6p \cdot a_p + q \cdot a_q = 6p + q - \frac{1}{2}n(n+1)$$

which is possible by Bézout lemma, since $\gcd(6p, q) = 1$.

- Else if $p, 2p \in [1, n]$ then we would like to choose nonzero a_p and a_q such that

$$3p \cdot a_p + q \cdot a_q = 3p + q - \frac{1}{2}n(n+1)$$

which is possible by Bézout lemma, since $\gcd(3p, q) = 1$.

- Else if $p \in [1, n]$ then we would like to choose nonzero a_p and a_q such that

$$p \cdot a_p + q \cdot a_q = p + q - \frac{1}{2}n(n+1)$$

which is possible by Bézout lemma, since $\gcd(p, q) = 1$. (This case is actually possible in a few edge cases, for example when $n = 9$, $q = 7$, $p = 5$.)

It remains to resolve the cases where $3 \leq n \leq 8$. We enumerate these cases manually:

- For $n = 3$, let $a_n = (-1)^{\nu_3(n)}$.
- For $n = 4$, let $a_n = (-1)^{\nu_2(n)+\nu_3(n)}$.
- For $n = 5$, let $a_n = (-2)^{\nu_5(n)}$.
- For $n = 6$, let $a_n = 5^{\nu_2(n)} \cdot 3^{\nu_3(n)} \cdot (-42)^{\nu_5(n)}$.
- For $n = 7$, let $a_n = (-3)^{\nu_7(n)}$.
- For $n = 8$, we can choose $(p, q) = (5, 7)$ in the prior construction.

This completes the constructions for all $n > 2$.

§4 USAMO 2012/4, proposed by Gabriel Dospinescu

Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $f(n!) = f(n)!$ for all positive integers n and such that $m - n$ divides $f(m) - f(n)$ for all distinct positive integers m, n.

By putting $n = 1$ and $n = 2$ we give $f(1), f(2) \in \{1, 2\}$. Also, we will use the condition

$$m! - n! \text{ divides } f(m)! - f(n)!.$$

We consider four cases on $f(1)$ and $f(2)$, and dispense with three of them.

- If $f(2) = 1$ then for all $m \ge 3$ we have $m! - 2$ divides $f(m)! - 1$, so $f(m) = 1$ for modulo 2 reasons. Then clearly $f(1) = 1$.
- If $f(1) = f(2) = 2$ we first obtain $3! - 1 \mid f(3)! - 2$, which implies $f(3) = 2$. Then $m! - 3 \mid f(m)! - 2$ for $m \ge 4$ implies $f(m) = 2$ for modulo 3 reasons.

Hence we are left with the case where $f(1) = 1$ and $f(2) = 2$.

Continuing by induction, suppose $f(1) = 1, \dots, f(k) = k$.

$$k! \cdot k = (k+1)! - k! \mid f(k+1)! - k!$$

and thus we deduce that $f(k+1) \ge k$, and hence

$$k \mid \frac{f(k+1)!}{k!} - 1.$$

Then plainly $f(k+1) \le 2k$ for mod k reasons, but also $f(k+1) \equiv 1 \pmod{k}$ so we conclude $f(k) = k + 1$.

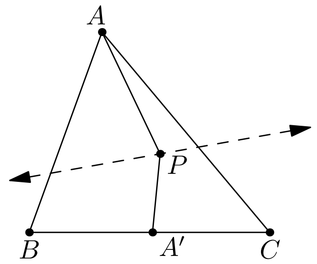
§5 USAMO 2012/5, proposed by Titu and Cosmin

Let P be a point in the plane of $\triangle ABC$, and γ a line through P. Let A', B', C' be the points where the reflections of lines PA, PB, PC with respect to γ intersect lines BC, CA, AB respectively. Prove that A', B', C' are collinear.

We present two solutions.

First solution (complex numbers) Let $p = 0$ and set γ as the real line. Then A' is the intersection of bc and $p\overline{a}$. So, we get

$$a' = \frac{\overline{a}(\overline{b}c - b\overline{c})}{(\overline{b} - \overline{c})\overline{a} - (b - c)a}.$$



Note that

$$\overline{a}' = \frac{a(b\overline{c} - \overline{b}c)}{(b - c)a - (\overline{b} - \overline{c})\overline{a}}.$$

Thus it suffices to prove

$$0 = \begin{vmatrix} \frac{\overline{a}(\overline{b}c - b\overline{c})}{(\overline{b}-\overline{c})\overline{a}-(b-c)a} & \frac{a(b\overline{c}-\overline{b}c)}{(b-c)a-(\overline{b}-\overline{c})\overline{a}} & 1 \\ \frac{\overline{b}(\overline{c}a-c\overline{a})}{(\overline{c}-\overline{a})\overline{b}-(c-a)b} & \frac{b(c\overline{a}-\overline{c}a)}{(c-a)b-(\overline{c}-\overline{a})\overline{b}} & 1 \\ \frac{\overline{c}(\overline{a}b-a\overline{b})}{(\overline{a}-\overline{b})\overline{c}-(a-b)c} & \frac{c(a\overline{b}-\overline{a}b)}{(a-b)c-(\overline{a}-\overline{b})\overline{c}} & 1 \end{vmatrix}.$$

This is equivalent to

$$0 = \begin{vmatrix} \overline{a}(\overline{b}c - b\overline{c}) & a(\overline{b}c - b\overline{c}) & (\overline{b} - \overline{c})\overline{a} - (b - c)a \\ \overline{b}(\overline{c}a - c\overline{a}) & b(\overline{c}a - c\overline{a}) & (\overline{c} - \overline{a})\overline{b} - (c - a)b \\ \overline{c}(\overline{a}b - a\overline{b}) & c(\overline{a}b - a\overline{b}) & (\overline{a} - \overline{b})\overline{c} - (a - b)c \end{vmatrix}.$$

Evaluating the determinant gives

$$\sum_{\text{cyc}} ((\overline{b} - \overline{c})\overline{a} - (b - c)a) \cdot - \begin{vmatrix} b & \overline{b} \\ c & \overline{c} \end{vmatrix} \cdot (\overline{c}a - c\overline{a}) \left(\overline{a}b - a\overline{b}\right)$$

or, noting the determinant is $b\overline{c} - \overline{b}c$ and factoring it out,

$$(\overline{b}c - c\overline{b})(\overline{c}a - c\overline{a})(\overline{a}b - a\overline{b}) \sum_{\text{cyc}} \left(ab - ac + \overline{ca} - \overline{ba}\right) = 0.$$

Second solution (Desargues involution) We let $C'' = \overline{A'B'} \cap \overline{AB}$. Consider complete quadrilateral $ABCA'B'C''C$. We see that there is an involutive pairing τ at P swapping (PA, PA'), (PB, PB'), (PC, PC''). From the first two, we see τ coincides with reflection about ℓ, hence conclude $C'' = C$.

§6 USAMO 2012/6, proposed by Gabriel Carroll

For integer $n \geq 2$, let $x_1, x_2, \ldots, x_n$ be real numbers satisfying

$$x_1 + x_2 + \cdots + x_n = 0 \quad \text{and} \quad x_1^2 + x_2^2 + \cdots + x_n^2 = 1.$$

For each subset $A \subseteq \{1, 2, \ldots, n\}$, define $S_A = \sum_{i \in A} x_i$. (If A is the empty set, then $S_A = 0$.) Prove that for any positive number λ, the number of sets A satisfying $S_A \geq \lambda$ is at most $2^{n-3}/\lambda^2$. For which choices of $x_1, x_2, \ldots, x_n, \lambda$ does equality hold?

Let ε_i be a coin flip of 0 or 1. Then we have

$$\begin{aligned} \mathbb{E}[S_A^2] &= \mathbb{E}\left[\left(\sum \varepsilon_i x_i\right)^2\right] = \sum_i \mathbb{E}[\varepsilon_i^2] x_i^2 + \sum_{i<j} \mathbb{E}[\varepsilon_i \varepsilon_j] 2 x_i x_j \\ &= \frac{1}{2} \sum x_i^2 + \frac{1}{2} \sum x_i x_j = \frac{1}{2} + \frac{1}{2} \sum_{i<j} x_i x_j = \frac{1}{2} + \frac{1}{2}\left(-\frac{1}{2}\right) = \frac{1}{4}. \end{aligned}$$

In other words, $\sum_A S_A^2 = 2^{n-2}$. Since can always pair A with its complement, we conclude

$$\sum_{S_A > 0} S_A^2 = 2^{n-3}.$$

Equality holds iff $S_A \in \{\pm\lambda, 0\}$ for every A. This occurs when $x_1 = 1/\sqrt{2}$, $x_2 = -1/\sqrt{2}$, $x_3 = \cdots = 0$ (or permutations), and $\lambda = 1/\sqrt{2}$.