USAMO 2005 Solution Notes

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This is an compilation of solutions for the 2005 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

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1. Determine all composite positive integers n for which it is possible to arrange all divisors of n that are greater than 1 in a circle so that no two adjacent divisors are relatively prime.

2. Prove that the system of equations

$$x^6 + x^3 + x^3y + y = 147^{157}$$
$$x^3 + x^3y + y^2 + y + z^9 = 157^{147}$$

has no integer solutions.

3. Let ABC be an acute-angled triangle, and let P and Q be two points on side BC. Construct a point C_1 in such a way that the convex quadrilateral $APBC_1$ is cyclic, $\overline{QC_1} \parallel \overline{CA}$, and C_1 and Q lie on opposite sides of line AB. Construct a point B_1 in such a way that the convex quadrilateral $APCB_1$ is cyclic, $\overline{QB_1} \parallel \overline{BA}$, and B_1 and Q lie on opposite sides of line AC. Prove that the points B_1, C_1, P, and Q lie on a circle.

4. Legs L_1, L_2, L_3, L_4 of a square table each have length n, where n is a positive integer. For how many ordered 4-tuples (k_1, k_2, k_3, k_4) of nonnegative integers can we cut a piece of length k_i from the end of leg L_i and still have a stable table?

   (The table is *stable* if it can be placed so that all four of the leg ends touch the floor. Note that a cut leg of length 0 is permitted.)

5. Let $n > 1$ be an integer. Suppose $2n$ points are given in the plane, no three of which are collinear. Suppose n of the given $2n$ points are colored blue and the other n colored red. A line in the plane is called a *balancing line* if it passes through one blue and one red point and, for each side of the line, the number of blue points on that side is equal to the number of red points on the same side. Prove that there exist at least two balancing lines.

6. For a positive integer m, let $s(m)$ denote the sum of the decimal digits of m. A set S positive integers is k-*stable* if $s(\sum_{x \in X} x) = k$ for any nonempty subset $X \subseteq S$.

   For each integer $n \ge 2$ let $f(n)$ be the minimal k for which there exists a k-stable set with n integers. Prove that there are constants $0 < C_1 < C_2$ with

$$C_1 \log_{10} n \le f(n) \le C_2 \log_{10} n.$$

§1 USAMO 2005/1, proposed by Zuming Feng

Determine all composite positive integers n for which it is possible to arrange all divisors of n that are greater than 1 in a circle so that no two adjacent divisors are relatively prime.

The only bad ones are $n = pq$, products of two distinct primes. Clearly they can't be so arranged, so we show all others work.

- If n is a power of a prime, the result is obvious.
- If $n = p_1^{e_1} \dots p_k^{e_k}$ for some $k \geq 3$, then first situate p_1p_2, p_2p_3, $\dots$, p_kp_1 on the circle. Then we can arbitrarily place any multiples of p_i between $p_{i-1}p_i$ and p_ip_{i+1}. This finishes this case.
- Finally suppose $n = p^a q^b$. If $a > 1$, say, we can repeat the argument by first placing pq and p^2q and then placing multiples of p in one arc and multiples of q in the other arc. On the other hand the case $a = b = 1$ is seen to be impossible.

§2 USAMO 2005/2, proposed by Razvan Gelca

Prove that the system of equations

$$\begin{aligned} x^6 + x^3 + x^3 y + y &= 147^{157} \\ x^3 + x^3 y + y^2 + y + z^9 &= 157^{147} \end{aligned}$$

has no integer solutions.

Sum the equations and add 1 to both sides to get

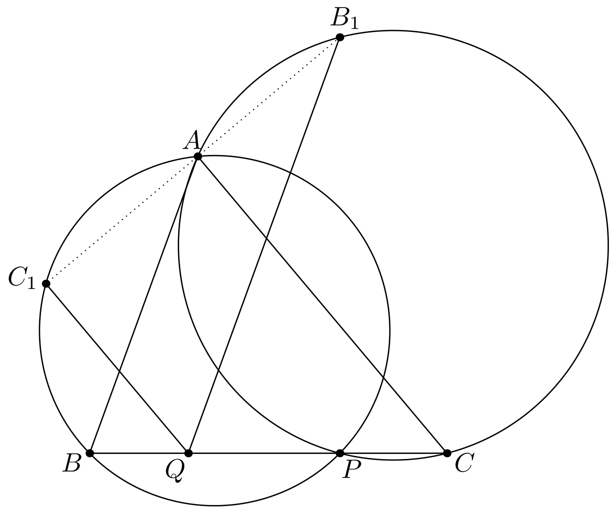
$$(x^3 + y + 1)^2 + z^9 = 147^{157} + 157^{147} + 1 \equiv 14 \pmod{19}$$

But $a^2 + b^9 \not\equiv 14 \pmod{19}$ for any integers a and b, since the ninth powers modulo 19 are 0, ± 1 and none of $\{13, 14, 15\}$ are squares modulo 19. Therefore, there are no integer solutions.

§3 USAMO 2005/3, proposed by Zuming Feng

Let ABC be an acute-angled triangle, and let P and Q be two points on side BC. Construct a point C_1 in such a way that the convex quadrilateral $APBC_1$ is cyclic, $\overline{QC_1} \parallel \overline{CA}$, and C_1 and Q lie on opposite sides of line AB. Construct a point B_1 in such a way that the convex quadrilateral $APCB_1$ is cyclic, $\overline{QB_1} \parallel \overline{BA}$, and B_1 and Q lie on opposite sides of line AC. Prove that the points B_1, C_1, P, and Q lie on a circle.

It is enough to prove that A, B_1, and C_1 are collinear, since then $\measuredangle C_1QP = \measuredangle ACP = \measuredangle AB_1P = \measuredangle C_1B_1P$.



First solution Let T be the second intersection of $\overline{AC_1}$ with (APC). Then readily $\triangle PC_1T \sim \triangle ABC$. Consequently, $\overline{QC_1} \parallel \overline{AC}$ implies TC_1QP cyclic. Finally, $\overline{TQ} \parallel \overline{AB}$ now follows from the cyclic condition, so $T = B_1$ as desired.

Second solution One may also use barycentric coordinates. Let $P = (0, m, n)$ and $Q = (0, r, s)$ with $m + n = r + s = 1$. Once again,

$$(APB) : -a^2yz - b^2zx - c^2xy + (x + y + z)(a^2m \cdot z) = 0.$$

Set $C_1 = (s - z, r, z)$, where $C_1Q \parallel AC$ follows by $(s - z) + r + z = 1$. We solve for this z.

$$\begin{aligned} 0 &= -a^2rz + (s - z)(-b^2z - c^2r) + a^2mz \\ &= b^2z^2 + (-sb^2 + rc^2)z - a^2rz + a^2mz - c^2rs \\ &= b^2z^2 + (-sb^2 + rc^2 + a^2(m - r))z - c^2rs \\ \implies 0 &= rb^2\left(\frac{z}{r}\right)^2 + (-sb^2 + rc^2 + a^2(m - r))\left(\frac{z}{r}\right) - c^2s. \end{aligned}$$

So the quotient of the z and y coordinates of C_1 satisfies this quadratic. Similarly, if $B_1 = (r - y, y, s)$ we obtain that

$$0 = sc^2\left(\frac{y}{s}\right)^2 + (-rc^2 + sb^2 + a^2(n - s))\left(\frac{y}{s}\right) - b^2r$$

Since these two quadratics are the same when one is written backwards (and negated), it follows that their roots are reciprocals. But the roots of the quadratics represent $\frac{z}{y}$ and $\frac{y}{z}$ for the points C_1 and B_1, respectively. This implies (with some configuration blah) that the points B_1 and C_1 are collinear with $A = (1, 0, 0)$ (in some line of the form $\frac{y}{z} = k$), as desired.

§4 USAMO 2005/4, proposed by Elgin Johnston

Legs L_1, L_2, L_3, L_4 of a square table each have length n, where n is a positive integer. For how many ordered 4-tuples (k_1, k_2, k_3, k_4) of nonnegative integers can we cut a piece of length k_i from the end of leg L_i and still have a stable table?

(The table is *stable* if it can be placed so that all four of the leg ends touch the floor. Note that a cut leg of length 0 is permitted.)

Flip the table upside-down so that that the tabele's surface rests on the floor. Then, we see that we want the truncated legs to have endpoints A, B, C, D which are coplanar (say).

Claim — This occurs if and only if $ABCD$ is a parallelogram.

Proof. Obviously $ABCD$ being a parallelogram is necessary. Conversely, if they are coplanar, we let D' be such that $ABCD'$ is a parallelogram. Then D' also lies in the same plane as $ABCD$, but is situated directly above D (since the table was a square). This implies $D' = D$, as needed. □

In still other words, we are counting the number of solutions to

$$(n - k_1) + (n - k_3) = (n - k_2) + (n - k_4) \iff k_1 + k_3 = k_2 + k_4.$$

Define

$$a_r = \#\{(a, b) \mid a + b = r,\ 0 \le a, b \le n\}$$

so that the number of solutions to $k_1 + k_3 = k_2 + k_4 = r$ is just given by a_r^2. We now just compute

$$\begin{aligned}\sum_{r=0}^{2n} a_r^2 &= 1^2 + 2^2 + \cdots + n^2 + (n+1)^2 + n^2 + \cdots + 1^2 \\ &= \frac{1}{3}(n+1)(2n^2 + 4n + 3).\end{aligned}$$

§5 USAMO 2005/5, proposed by Kiran Kedlaya

Let $n > 1$ be an integer. Suppose $2n$ points are given in the plane, no three of which are collinear. Suppose n of the given $2n$ points are colored blue and the other n colored red. A line in the plane is called a *balancing line* if it passes through one blue and one red point and, for each side of the line, the number of blue points on that side is equal to the number of red points on the same side. Prove that there exist at least two balancing lines.

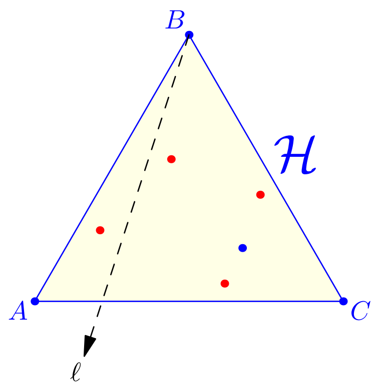
Consider the convex hull $\mathcal{H}$ of the polygon. There are two cases.

The easy case: if the convex hull $\mathcal{H}$ is not all the same color, there exist two edges of $\mathcal{H}$ (at least) which have differently colored endpoints. The extensions of those sides form balancing lines; indeed given any such line ℓ one side of ℓ has no points, the other has $n-1$ red and $n-1$ blue points.

So now assume $\mathcal{H}$ is all blue (WLOG). We will prove there are at least $|\mathcal{H}|$ balancing lines in the following way.

Claim — For any vertex B of $\mathcal{H}$ there is a balancing line through it.

Proof. Assume A, B, C are three consecutive blue vertices of $\mathcal{H}$. Imagine starting with line ℓ passing through B and A, then rotating it through B until it coincides with line BC, through the polygon.



During this process, we consider the set of points on the same side of ℓ as C, and let x be the number of such red points minus the number of such blue points. Note that:

- Every time ℓ touches a blue point, x increases by 1.
- Every time ℓ touches a red point, x decreases by 1.
- Initially, $x = +1$.
- Just before reaching the end we have $x = -1$.

So at the moment where x first equals zero, we have found our balancing line. □

§6 USAMO 2005/6, proposed by Titu Andreescu and Gabriel Dospinescu

For a positive integer m, let $s(m)$ denote the sum of the decimal digits of m. A set S positive integers is *k-stable* if $s(\sum_{x \in X} x) = k$ for any nonempty subset $X \subseteq S$.

For each integer $n \geq 2$ let $f(n)$ be the minimal k for which there exists a k-stable set with n integers. Prove that there are constants $0 < C_1 < C_2$ with

$$C_1 \log_{10} n \leq f(n) \leq C_2 \log_{10} n.$$

Lower bound: Let $n \geq 1$ and $r \geq 1$ be integers satisfying $1 + 2 + \cdots + n < 10^e$. Consider the set

$$S = \{10^e - 1,\ 2(10^e - 1), \dots,\ n(10^e - 1)\}.$$

For example, if $n = 6$ and $e = 3$, we have $S = \{999, 1998, 2997, 3996, 4995, 5994\}$.

The set S here is easily seen to be $9e$-good. Thus $f(n) \geq 9 \lceil \log_{10} n \rceil$, proving one direction.

> **Remark.** I think the problem is actually more natural with a multiset S rather than a vanilla set, in which case $S = \{10^e - 1, 10^e - 1, \dots 10^e - 1\}$ works fine, and is easier to think of.
>
> In some sense the actual construction is obtained by starting with this one, and then pushing together the terms together in order to get the terms to be distinct, hence the $1 + 2 + \cdots + n$ appearance.

Upper bound: we are going to prove the following, which obviously sufficient.

> **Claim —** Let r be a positive integer. In any (multi)set S of more than 12^k integers, there exists a subset whose sum of decimal digits exceeds k.

Proof. Imagine writing entries of S on a blackboard, while keeping a running sum Σ initially set to zero. For $i = 1, 2, \dots$ we will have a process such that at the end of the ith step all entries on the board are divisible by 10^i. It goes as follows:

- If the ith digit from the right of Σ is nonzero, then arbitrarily partition the numbers on the board into groups of 10, erasing any leftover numbers. Within each group of 10, we can find a nonempty subset with sum 0 mod 10^i; we then erase each group and replace it with that sum.

- If the ith digit from the right of Σ is zero, but some entry on the board is not divisible by 10^i, then we erase that entry and add it to Σ. Then we do the grouping as in the previous step.

- If the ith digit from the right of Σ is zero, and all entries on the board are divisible by 10^i, we do nothing and move on to the next step.

This process ends when no numbers remain on the blackboard. The first and second cases occur at least $k + 1$ times (the number of entries decreases by a factor of at most 12 each step), and each time Σ gets some nonzero digit, which is never changed at later steps. Therefore Σ has sum of digits at least $k + 1$ as needed. □

Remark. The official solutions contain a slicker proof: it turns out that any multiple of $10^e - 1$ has sum of decimal digits at least $9e$. However, if one does not know this lemma it seems hard to imagine coming up with it.