USAMO 2004 Solution Notes

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This is an compilation of solutions for the 2004 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

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§0 Problems

1. Let $ABCD$ be a quadrilateral circumscribed about a circle, whose interior and exterior angles are at least 60 degrees. Prove that

$$\frac{1}{3}|AB^3 - AD^3| \le |BC^3 - CD^3| \le 3|AB^3 - AD^3|.$$

When does equality hold?

2. Let $a_1, a_2, \dots, a_n$ be integers whose greatest common divisor is 1. Let S be a set of integers with the following properties:

(a) $a_i \in S$ for $i = 1, \dots, n$.

(b) $a_i - a_j \in S$ for $i, j = 1, \dots, n$, not necessarily distinct.

(c) If $x, y \in S$ and $x + y \in S$, then $x - y \in S$ too.

Prove that $S = \mathbb{Z}$.

3. For what real values of $k > 0$ is it possible to dissect a $1 \times k$ rectangle into two similar but noncongruent polygons?

4. Alice and Bob play a game on a 6 by 6 grid. On his turn, a player chooses a rational number not yet appearing in the grid and writes it in an empty square of the grid. Alice goes first and then the players alternate. When all squares have numbers written in them, in each row, the square with the greatest number in that row is colored black. Alice wins if he can then draw a line from the top of the grid to the bottom of the grid that stays in black squares, and Bob wins if he can't. (If two squares share a vertex, Alice can draw a line from one to the other that stays in those two squares.) Find, with proof, a winning strategy for one of the players.

5. Let a, b, c be positive reals. Prove that

$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \ge (a + b + c)^3.$$

6. A circle ω is inscribed in a quadrilateral $ABCD$. Let I be the center of ω. Suppose that

$$(AI + DI)^2 + (BI + CI)^2 = (AB + CD)^2.$$

Prove that $ABCD$ is an isosceles trapezoid.

§1 USAMO 2004/1, proposed by Titu Andreescu

Let $ABCD$ be a quadrilateral circumscribed about a circle, whose interior and exterior angles are at least 60 degrees. Prove that

$$\frac{1}{3}|AB^3 - AD^3| \le |BC^3 - CD^3| \le 3|AB^3 - AD^3|.$$

When does equality hold?

Clearly it suffices to show the left inequality. Since $AB + CD = BC + AD \implies |AB - AD| = |BC - CD|$, it suffices to prove

$$\frac{1}{3}(AB^2 + AB \cdot AD + AD^2) \le BC^2 + BC \cdot CD + CD^2.$$

This follows by noting that

$$\begin{aligned} BC^2 + BC \cdot CD + CD^2 &\ge BC^2 + CD^2 - 2(BC)(CD)\cos(\angle BCD) \\ &= BD^2 \\ &= AB^2 + AD^2 - 2(AB)(AD)\cos(\angle BAD) \\ &\ge AB^2 + AD^2 - AB \cdot AD \\ &\ge \tfrac{1}{3}(AB^2 + AD^2 + AB \cdot AD) \end{aligned}$$

the last line following by AM-GM.

The equality holds iff $ABCD$ is a cite with $AB = AD$, $CB = CD$.

§2 USAMO 2004/2, proposed by Kiran Kedlaya

Let a_1, a_2, $\dots$, a_n be integers whose greatest common divisor is 1. Let S be a set of integers with the following properties:

(a) $a_i \in S$ for $i = 1, \dots, n$.

(b) $a_i - a_j \in S$ for $i, j = 1, \dots, n$, not necessarily distinct.

(c) If $x, y \in S$ and $x + y \in S$, then $x - y \in S$ too.

Prove that $S = \mathbb{Z}$.

The idea is to show any linear combination of the a_i are in S, which implies (by Bezout) that $S = \mathbb{Z}$. This is pretty intuitive, but the details require some care (in particular there is a parity obstruction at the second lemma).

First, we make the following simple observations:

- $0 \in S$, by putting $i = j = 1$ in (b).
- $s \in S \iff -s \in S$, by putting $x = 0$ in (c).

Now, we prove that:

Lemma

For any integers c, d, and indices i, j, we have $ca_i + da_j \in S$.

Proof. We will assume $c, d > 0$ since the other cases are anologous. In that case it follows by induction on $c + d$ for example $ca_i + (d-1)a_j$, a_j, $ca_i + da_j$ in S implies $ca_i + (d+1)a_j \in S$. □

Lemma

For any nonzero integers c_1, c_2, $\dots$, c_m, and any distinct indices $\{i_1, i_2, \dots, i_m\}$, we have

$$\sum_k c_k a_{i_k} \in S.$$

Proof. By induction on m, with base case $m \le 2$ already done.

For the inductive step, we will assume that $i_1 = 1$, $i_2 = 2$, et cetera, for notational convenience. The proof is then split into two cases.

First Case: some c_i is even. WLOG $c_1 \neq 0$ is even and note that

$$x \overset{\text{def}}{=} \frac{1}{2}c_1 a_1 + \sum_{k \ge 3} c_k a_k \in S$$

$$y \overset{\text{def}}{=} -\frac{1}{2}c_1 a_1 - c_2 a_2 \in S$$

$$x + y = -c_2 a_2 + \sum_{k \ge 3} c_k a_k \in S$$

$$\implies x - y = \sum_{k \ge 1} c_k a_k \in S.$$

Second Case: all c_i are odd. We reduce this to the first case as follows. Let $u = \frac{a_1}{\gcd(a_1,a_2)}$ and $v = \frac{a_2}{\gcd(a_1,a_2)}$. Then $\gcd(u, v) = 1$ and so WLOG u is odd. Then

$$c_1a_1 + c_2a_2 = (c_1 + v)a_1 + (c_2 - u)a_2$$

and so we can replace our given combination by $(c_1+v)a_1+(c_2-u)a_2+c_3a_3+\dots$ which now has an even coefficient for a_2. $\square$

We then apply the lemma at $m = n$; this implies the result since Bezout's lemma implies that $\sum c_ia_i = 1$ for some choice of $c_i \in \mathbb{Z}$.

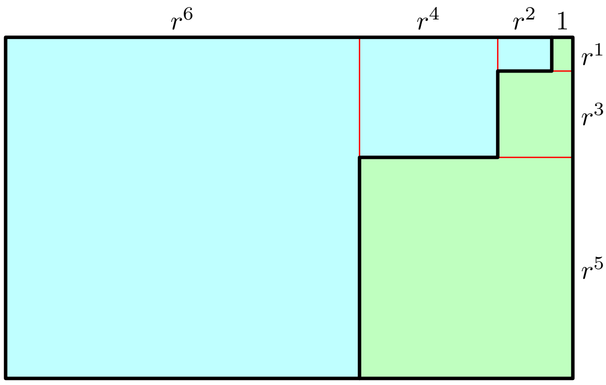
§3 USAMO 2004/3, proposed by Ricky Liu

For what real values of $k > 0$ is it possible to dissect a $1 \times k$ rectangle into two similar but noncongruent polygons?

The answer is all $k \neq 1$. Here a "dissection" does not need to be a single cut.

The construction for $k > 1$ is not so hard to find (my friends and I got it over dinner at Legal Seafoods). Pick $2n$ and $r \in (1, \infty)$ arbitrary. Consider n rectangles of dimensions $1 \times r$, $r \times r^2$, $(1 + r^2) \times r$, $(r + r^3) \times r^4$, and so on, gluing them together in this order in the obvious fashion.

For example, the case of $n = 3$ is shown below.



Now, the aspect ratio of the completed rectangle is

$$\frac{1 + r^2 + \cdots + r^{2n}}{r + r^3 + \cdots + r^{2n-1}}$$

which for a fixed n can goes from $(1 + \frac{1}{n}, \infty)$ as $r > 1$. Thus by taking n large, we can get any $k > 1$. Of course by symmetry we can also get any $k < 1$.

Now we show that $k = 1$ is impossible (the tricky part). Consider a dissection into two similar polygons $\mathcal{P} \sim \mathcal{Q}$, and let $\mathcal{B}$ be their common boundary. By counting the number of sides of $\mathcal{P}$ and $\mathcal{Q}$ we see $\mathcal{B}$ must run from one side of the square to an opposite side. From this we can show that the *longest* side of $\mathcal{P}$ and $\mathcal{Q}$ are either the sides of the square (of length 1), or a side shared in $\mathcal{B}$. In particular, they are equal, which means in fact $\mathcal{P} \cong \mathcal{Q}$.

§4 USAMO 2004/4, proposed by Melanie Wood

Alice and Bob play a game on a 6 by 6 grid. On his turn, a player chooses a rational number not yet appearing in the grid and writes it in an empty square of the grid. Alice goes first and then the players alternate. When all squares have numbers written in them, in each row, the square with the greatest number in that row is colored black. Alice wins if he can then draw a line from the top of the grid to the bottom of the grid that stays in black squares, and Bob wins if he can't. (If two squares share a vertex, Alice can draw a line from one to the other that stays in those two squares.) Find, with proof, a winning strategy for one of the players.

Bob can win. When Alice writes a number in row 1 or 2 and column c, Bob writes the same number in row 2 or 1 and column $c+3 \pmod 6$, respectively. Thus the first two rows will look like

$$\begin{bmatrix} a & b & c & d & e & f \\ d & e & f & a & b & c \end{bmatrix}$$

When Alice writes a number in any other row, Bob writes a number in the same row (or anything else in rows 3 to 6). Under this strategy the black squares in the first two rows will not touch, as desired.

§5 USAMO 2004/5, proposed by Titu Andreescu

Let a, b, c be positive reals. Prove that

$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \ge (a + b + c)^3 .$$

Observe that for all real numbers a, the inequality

$$a^5 - a^2 + 3 \ge a^3 + 2$$

holds. Then the problem follows by Hölder in the form

$$(a^3 + 1 + 1)(1 + b^3 + 1)(1 + 1 + c^3) \ge (a + b + c)^3.$$

§6 USAMO 2004/6, proposed by Zuming Feng

A circle ω is inscribed in a quadrilateral $ABCD$. Let I be the center of ω. Suppose that

$$(AI + DI)^2 + (BI + CI)^2 = (AB + CD)^2.$$

Prove that $ABCD$ is an isosceles trapezoid.

Here's a completely algebraic solution. WLOG ω has radius 1, and let a, b, c, d be the lengths of the tangents from A, B, C, D to ω. It follows that $a+b+c+d = abc+bcd+cda+dab$ $(\star)$ as many people noted. Then, the content of the problem is to show that

$$(\sqrt{a^2+1}+\sqrt{d^2+1})^2 + (\sqrt{b^2+1}+\sqrt{c^2+1})^2 \le (a+b+c+d)^2$$

subject to $(\star)$, with equality only when $a = d = \frac{1}{b} = \frac{1}{c}$.

Let $S = ab+bc+cd+da+ac+bd$. Then the inequality is

$$\sqrt{(a^2+1)(d^2+1)} + \sqrt{(b^2+1)(c^2+1)} \le S - 2.$$

Now, by **USAMO 2014 Problem 1** and the condition $(\star)$, we have that $(a^2+1)(b^2+1)(c^2+1)(d^2+1) = (S - abcd - 1)^2$. So squaring both sides, the inequality becomes

$$(ad)^2 + (bc)^2 + a^2 + b^2 + c^2 + d^2 \le S^2 - 6S + 2abcd + 4.$$

To simplify this, we use the identities

$$\begin{aligned} S^2 &= 6abcd + \sum_{\text{sym}} a^2bc + \frac{1}{4}\sum_{\text{sym}} a^2b^2 \\ (a+b+c+d)^2 &= (abc+bcd+cda+dab)(a+b+c+d) \\ &= 4abcd + \frac{1}{2}\sum_{\text{sym}} a^2bc \end{aligned}$$

So $S^2 + 2abcd = \frac{1}{4}\sum_{\text{sym}} a^2b^2 + 2(a^2+b^2+c^2+d^2) + 4S$ and the inequality we want to prove reduces to

$$2S \le (ab)^2 + (ac)^2 + (bd)^2 + (cd)^2 + 4 + a^2 + b^2 + c^2 + d^2.$$

This is immediate by repeated applications of AM-GM, and the equality case is when $ab = ac = bd = cd = 1$, $a = d$, $b = c$.

Note that a priori one expects an inequality. Indeed,

- Quadrilaterals with incircles have four degrees of freedom.
- There is one condition imposed.
- Isosceles trapezoid with incircles have two degrees of freedom.