USAMO 2000 Solution Notes

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This is an compilation of solutions for the 2000 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

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§0 Problems

1. Call a real-valued function f *very convex* if

$$\frac{f(x)+f(y)}{2} \ge f\left(\frac{x+y}{2}\right) + |x-y|$$

holds for all real numbers x and y. Prove that no very convex function exists.

2. Let S be the set of all triangles ABC for which

$$5\left(\frac{1}{AP}+\frac{1}{BQ}+\frac{1}{CR}\right) - \frac{3}{\min\{AP, BQ, CR\}} = \frac{6}{r},$$

where r is the inradius and P, Q, R are the points of tangency of the incircle with sides AB, BC, CA respectively. Prove that all triangles in S are isosceles and similar to one another.

3. A game of solitaire is played with R red cards, W white cards, and B blue cards. A player plays all the cards one at a time. With each play he accumulates a penalty. If he plays a blue card, then he is charged a penalty which is the number of white cards still in his hand. If he plays a white card, then he is charged a penalty which is twice the number of red cards still in his hand. If he plays a red card, then he is charged a penalty which is three times the number of blue cards still in his hand.

Find, as a function of R, W, and B, the minimal total penalty a player can amass and the number of ways in which this minimum can be achieved.

4. Find the smallest positive integer n such that if n squares of a 1000×1000 chessboard are colored, then there will exist three colored squares whose centers form a right triangle with sides parallel to the edges of the board.

5. Let $A_1A_2A_3$ be a triangle, and let ω_1 be a circle in its plane passing through A_1 and A_2. Suppose there exists circles ω_2, ω_3, $\dots$, ω_7 such that for $k = 2, 3, \dots, 7$, circle ω_k is externally tangent to ω_{k-1} and passes through A_k and A_{k+1} (indices mod 3). Prove that $\omega_7 = \omega_1$.

6. Let $a_1, b_1, a_2, b_2, \dots, a_n, b_n$ be nonnegative real numbers. Prove that

$$\sum_{i,j=1}^{n} \min\{a_ia_j, b_ib_j\} \le \sum_{i,j=1}^{n} \min\{a_ib_j, a_jb_i\}.$$

§1 USAMO 2000/1

Call a real-valued function f *very convex* if

$$\frac{f(x)+f(y)}{2} \ge f\left(\frac{x+y}{2}\right) + |x-y|$$

holds for all real numbers x and y. Prove that no very convex function exists.

For $C \ge 0$, we say a function f is *C-convex*

$$\frac{f(x)+f(y)}{2} \ge f\left(\frac{x+y}{2}\right) + C\left|x-y\right|.$$

Suppose f is C-convex. Let $a < b < c < d < e$ be any arithmetic progression, such that $t = |e-a|$. Observe that

$$\begin{aligned} f(a)+f(c) &\ge 2f(b) + C \cdot \frac{1}{2}t \\ f(c)+f(e) &\ge 2f(d) + C \cdot \frac{1}{2}t \\ f(b)+f(d) &\ge 2f(c) + C \cdot \frac{1}{2}t \end{aligned}$$

Adding the first two to twice the third gives

$$f(a)+f(e) \ge 2f(c) + 2C \cdot t.$$

So we conclude C-convex function is also $2C$-convex. This is clearly not okay for $C > 0$.

§2 USAMO 2000/2

Let S be the set of all triangles ABC for which

$$5\left(\frac{1}{AP}+\frac{1}{BQ}+\frac{1}{CR}\right)-\frac{3}{\min\{AP,BQ,CR\}}=\frac{6}{r},$$

where r is the inradius and P, Q, R are the points of tangency of the incircle with sides AB, BC, CA respectively. Prove that all triangles in S are isosceles and similar to one another.

We will prove the inequality

$$\frac{2}{AP}+\frac{5}{BQ}+\frac{5}{CR}\geq\frac{6}{r}$$

with equality when $AP:BQ:CR=1:4:4$. This implies the problem statement.

Letting $x=AP$, $y=BQ$, $z=CR$, the inequality becomes

$$\frac{2}{x}+\frac{5}{y}+\frac{5}{z}\geq 6\sqrt{\frac{x+y+z}{xyz}}.$$

Squaring both sides and collecting terms gives

$$\frac{4}{x^2}+\frac{25}{y^2}+\frac{25}{z^2}+\frac{14}{yz}\geq\frac{16}{xy}+\frac{16}{xz}.$$

If we replace $x=1/a$, $y=4/b$, $z=4/c$, then it remains to prove the inequality

$$64a^2+25(b+c)^2\geq 64a(b+c)+36bc$$

where equality holds when $a=b=c$. This follows by two applications of AM-GM:

$$16\left(4a^2+(b+c)^2\right)\geq 64a(b+c)$$
$$9(b+c)^2\geq 36bc.$$

Again one can tell this is an inequality by counting degrees of freedom.

§3 USAMO 2000/3

A game of solitaire is played with R red cards, W white cards, and B blue cards. A player plays all the cards one at a time. With each play he accumulates a penalty. If he plays a blue card, then he is charged a penalty which is the number of white cards still in his hand. If he plays a white card, then he is charged a penalty which is twice the number of red cards still in his hand. If he plays a red card, then he is charged a penalty which is three times the number of blue cards still in his hand.

Find, as a function of R, W, and B, the minimal total penalty a player can amass and the number of ways in which this minimum can be achieved.

The minimum penalty is

$$f(B, W, R) = \min(BW, 2WR, 3RB)$$

or equivalently, the natural guess of "discard all cards of one color first" is actually optimal (though not necessarily unique).

This can be proven directly by induction. Indeed the base case $BWR = 0$ (in which case zero penalty is clearly achievable). The inductive step follows from

$$f(B, W, R) = \min \begin{cases} f(B-1, W, R) + W \\ f(B, W-1, R) + 2R \\ f(B, W, R-1) + 3B. \end{cases}$$

It remains to characterize the strategies. This is a routine calculation, so we just state the result.

- If any of the three quantities BW, $2WR$, $3RB$ is strictly smaller than the other three, there is one optimal strategy.
- If $BW = 2WR < 3RB$, there are $W + 1$ optimal strategies, namely discarding from 0 to W white cards, then discarding all blue cards. (Each white card discarded still preserves $BW = 2WR$.)
- If $2WR = 3RB < BW$, there are $R + 1$ optimal strategies, namely discarding from 0 to R red cards, and then discarding discarding all white cards.
- If $3WR = RB < 2WR$, there are $B + 1$ optimal strategies, namely discarding from 0 to B blue cards, and then discarding discarding all red cards.
- Now suppose $BW = 2WR = 3RB$. Discarding a card of one color ends up in exactly one of the previous three cases. This gives an answer of $R + W + B$ strategies.

§4 USAMO 2000/4

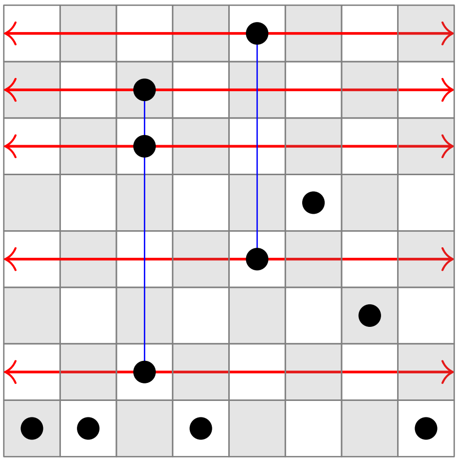
Find the smallest positive integer n such that if n squares of a 1000×1000 chessboard are colored, then there will exist three colored squares whose centers form a right triangle with sides parallel to the edges of the board.

The answer is $n = 1999$.

For a construction with $n = 1998$, take a punctured L as illustrated below (with 1000 replaced by 4):

$$\begin{bmatrix} 1 & & & \\ 1 & & & \\ 1 & & & \\ & 1 & 1 & 1 \end{bmatrix}.$$

We now show that if there is no right triangle, there are at most 1998 tokens (colored squares). In every column with more than two tokens, we have token emit a bidirectional horizontal death ray (laser) covering its entire row: the hypothesis is that the death ray won't hit any other tokens.

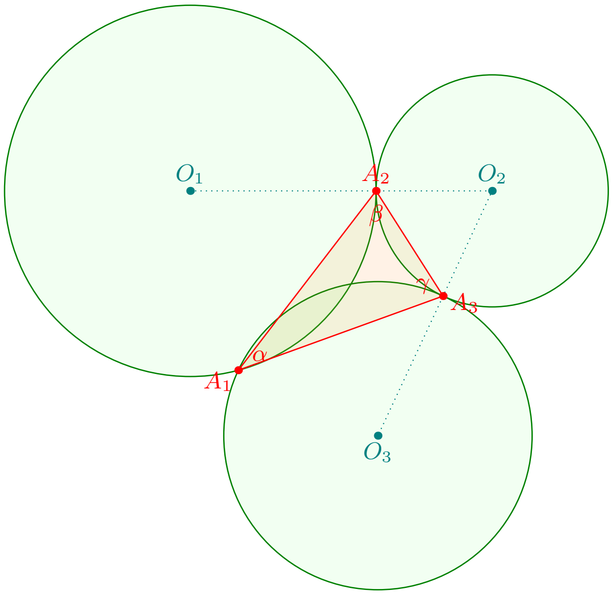


Assume there are n tokens and that $n > 1000$. Then obviously some column has more than two tokens, so at most 999 tokens don't emit a death ray (namely, any token in its own column). Thus there are at least $n - 999$ death rays. On the other hand, we can have at most 999 death rays total (since it would not be okay for the whole board to have death rays, as some row should have more than two tokens). Therefore, $n \le 999 + 999 = 1998$ as desired.

§5 USAMO 2000/5

Let $A_1A_2A_3$ be a triangle, and let ω_1 be a circle in its plane passing through A_1 and A_2. Suppose there exists circles ω_2, ω_3, ..., ω_7 such that for $k = 2, 3, \dots, 7$, circle ω_k is externally tangent to ω_{k-1} and passes through A_k and A_{k+1} (indices mod 3). Prove that $\omega_7 = \omega_1$.

The idea is to keep track of the subtended arc $\widehat{A_iA_{i+1}}$ of ω_i for each i. To this end, let $\beta = \measuredangle A_1A_2A_3$, $\gamma = \measuredangle A_2A_3A_1$ and $\alpha = \measuredangle A_1A_2A_3$.



Initially, we set $\theta = \measuredangle O_1A_2A_1$. Then we compute

$$\begin{aligned} \measuredangle O_1A_2A_1 &= \theta \\ \measuredangle O_2A_3A_2 &= -\beta - \theta \\ \measuredangle O_3A_1A_3 &= \beta - \gamma + \theta \\ \measuredangle O_4A_2A_1 &= (\gamma - \beta - \alpha) - \theta \end{aligned}$$

and repeating the same calculation another round gives

$$\measuredangle O_7A_2A_1 = k - (k - \theta) = \theta$$

with $k = \gamma - \beta - \alpha$. This implies $O_7 = O_1$, so $\omega_7 = \omega_1$.

§6 USAMO 2000/6, proposed by Gheorghita Zbaganu

Let $a_1, b_1, a_2, b_2, \dots, a_n, b_n$ be nonnegative real numbers. Prove that

$$\sum_{i,j=1}^{n} \min\{a_i a_j, b_i b_j\} \le \sum_{i,j=1}^{n} \min\{a_i b_j, a_j b_i\}.$$

We present two solutions.

First solution by creating a single min (Vincent Huang and Ravi Boppana) Let $b_i = r_i a_i$ for each i, and rewrite the inequality as

$$\sum_{i,j} a_i a_j \left[\min(r_i, r_j) - \min(1, r_i r_j)\right] \ge 0.$$

We now do the key manipulation to convert the double min into a separate single min. Let $\varepsilon_i = +1$ if $r_i \ge 1$, and $\varepsilon_i = -1$ otherwise, and let $s_i = |r_i - 1|$. Then we pass to absolute values:

$$\begin{aligned} 2\min(r_i, r_j) - 2\min(1, r_i r_j) &= |r_i r_j - 1| - |r_i - r_j| - (r_i - 1)(r_j - 1) \\ &= |r_i r_j - 1| - |r_i - r_j| - \varepsilon_i \varepsilon_j s_i s_j \\ &= \varepsilon_i \varepsilon_j \min\left(|1 - r_i r_j \pm (r_i - r_j)|\right) - \varepsilon_i \varepsilon_j s_i s_j \\ &= \varepsilon_i \varepsilon_j \min\left(s_i(r_j + 1), s_j(r_i + 1)\right) - \varepsilon_i \varepsilon_j s_i s_j \\ &= (\varepsilon_i s_i)(\varepsilon_j s_j) \min\left(\frac{r_j + 1}{s_j} - 1, \frac{r_i + 1}{s_i} - 1\right). \end{aligned}$$

So let us denote $x_i = a_i \varepsilon_i s_i \in \mathbb{R}$, and $t_i = \frac{r_i+1}{s_i} - 1 \in \mathbb{R}_{\ge 0}$. Thus it suffices to prove that:

> **Claim** — We have
> $$\sum_{i,j} x_i x_j \min(t_i, t_j) \ge 0$$
> for arbitrary $x_i \in \mathbb{R}$, $t_i \in \mathbb{R}_{\ge 0}$.

Proof. One can just check this "by hand" by assuming $t_1 \le t_2 \le \dots \le t_n$; then the left-hand side becomes

$$\sum_i t_i x_i^2 + 2 \sum_{i<j} 2 t_i x_i x_j = \sum_i (t_i - t_{i-1})(x_i + x_{i+1} + \dots + x_n)^2 \ge 0.$$

There is also a nice proof using the integral identity

$$\min(t_i, t_j) = \int_0^\infty \mathbf{1}(u \le t_i) \mathbf{1}(u \le t_j)\, du$$

where the $\mathbf{1}$ are indicator functions. Indeed,

$$\begin{aligned}\sum_{i,j} x_i x_j \min(t_i, t_j) &= \sum_{i,j} x_i x_j \int_0^\infty \mathbf{1}(u \le t_i)\mathbf{1}(u \le t_j)\, du \\ &= \int_0^\infty \sum_i x_i \mathbf{1}(u \le t_i) \sum_j x_j \mathbf{1}(u \le t_j)\, du \\ &= \int_0^\infty \left(\sum_i x_i \mathbf{1}(u \le t_i) \right)^2 du \\ &\ge 0.\end{aligned}$$

□

Second solution by smoothing (Alex Zhai) The case $n = 1$ is immediate, so we'll proceed by induction on $n \ge 2$.

Again, let $b_i = r_i a_i$ for each i, and write the inequality as

$$L_n(a_1, \dots, a_n, r_1, \dots, r_n) \overset{\text{def}}{=} \sum_{i,j} a_i a_j \left[\min(r_i, r_j) - \min(1, r_i r_j)\right] \ge 0.$$

First note that if $r_1 = r_2$ then

$$L_n(a_1, a_2, a_3, \dots, r_1, r_1, r_3 \dots) = L_{n-1}(a_1 + a_2, a_3, \dots, r_1, r_3, \dots)$$

and so our goal is to smooth to a situation where two of the r_i's are equal, so that we may apply induction.

On the other hand, L_n is a *piecewise linear* function in $r_1 \ge 0$. Let us smooth r_1 then. Note that if the minimum is attained at $r_1 = 0$, we can ignore a_1 and reduce to the $(n-1)$-variable case. On the other hand, the minimum must be achieved at a cusp which opens upward, which can only happen if $r_i r_j = 1$ for some j. (The $r_i = r_j$ cusps open downward, sadly.)

In this way, whenever some r_i is not equal to the reciprocal of any other $r_\bullet$, we can smooth it. This terminates; so we may smooth until we reach a situation for which

$$\{r_1, \dots, r_n\} = \{1/r_1, \dots, 1/r_n\}.$$

Now, assume WLOG that $r_1 = \max_i r_i$ and $r_2 = \min_i r_i$, hence $r_1 r_2 = 1$ and $r_1 \ge 1 \ge r_2$. We isolate the contributions from a_1, a_2, r_1 and r_2.

$$\begin{aligned}L_n(\dots) = a_1^2 &[r_1 - 1] + a_2^2 \left[r_2 - r_2^2\right] + 2a_1 a_2 [r_2 - 1] \\ &+ 2a_1 \left[(a_3 r_3 + \dots + a_n r_n) - (a_3 + \dots + a_n)\right] \\ &+ 2a_2 r_2 \left[(a_3 + \dots + a_n) - (a_3 r_3 + \dots + a_n r_n)\right] \\ &+ \sum_{i=3}^n \sum_{j=3}^n a_i a_j \left[\min(r_i, r_j) - \min(1, r_i r_j)\right].\end{aligned}$$

The idea now is to smooth via

$$(a_1, a_2, r_1, r_2) \longrightarrow \left(a_1, \frac{1}{t}a_2, \frac{1}{t}r_1, t r_2\right)$$

where $t \ge 1$ is such that $\frac{1}{t}r_1 \ge \max(1, r_3, \dots, r_n)$ holds. (This choice is such that a_1 and $a_2 r_2$ are unchanged, because we don't know the sign of $\sum_{i \ge 3}(1 - r_i)a_i$ and so the post-smoothing value is still at least the max.) Then,

$$\begin{aligned} & L_n(a_1, a_2, \dots, r_1, r_2, \dots) - L_n\left(a_1, \frac{1}{t}a_2, \dots, \frac{1}{t}r_1, tr_2\right) \\ &= a_1^2\left(r_1 - \frac{1}{t}r_1\right) + a_2^2\left(r_2 - \frac{1}{t}r_2\right) + 2a_1a_2\left(\frac{1}{t} - 1\right) \\ &= \left(1 - \frac{1}{t}\right)\left(r_1a_1^2 + r_2a_2^2 - 2a_1a_2\right) \ge 0 \end{aligned}$$

the last line by AM-GM. Now pick $t = \frac{r_1}{\max(1, r_3, \dots, r_n)}$, and at last we can induct down.