45th United States of America Junior Mathematical Olympiad

Day I 12:30PM — 5PM EDT

April 19, 2016

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper). Failure to meet this requirement will result in an automatic 1-point deduction.

USAJMO 1. The isosceles triangle $\triangle ABC$, with $AB = AC$, is inscribed in the circle ω. Let P be a variable point on the arc $\widehat{BC}$ that does not contain A, and let I_B and I_C denote the incenters of triangles $\triangle ABP$ and $\triangle ACP$, respectively.

Prove that as P varies, the circumcircle of triangle $\triangle PI_BI_C$ passes through a fixed point.

USAJMO 2. Prove that there exists a positive integer $n < 10^6$ such that 5^n has six consecutive zeros in its decimal representation.

USAJMO 3. Let $X_1, X_2, \ldots, X_{100}$ be a sequence of mutually distinct non-empty subsets of a set S. Any two sets X_i and X_{i+1} are disjoint and their union is not the whole set S, that is, $X_i \cap X_{i+1} = \emptyset$ and $X_i \cup X_{i+1} \neq S$, for all $i \in \{1, \ldots, 99\}$. Find the smallest possible number of elements in S.

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Day II 12:30PM — 5PM EDT

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USAJMO 4. Find, with proof, the least integer N such that if any 2016 elements are removed from the set $\{1, 2, \ldots, N\}$, one can still find 2016 distinct numbers among the remaining elements with sum N.

USAJMO 5. Let $\triangle ABC$ be an acute triangle, with O as its circumcenter. Point H is the foot of the perpendicular from A to line $\overleftrightarrow{BC}$, and points P and Q are the feet of the perpendiculars from H to the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$, respectively.

Given that

$$AH^2 = 2 \cdot AO^2,$$

prove that the points O, P, and Q are collinear.

USAJMO 6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y,

$$(f(x) + xy) \cdot f(x - 3y) + (f(y) + xy) \cdot f(3x - y) = (f(x + y))^2.$$