Solutions to the 2016 TST Selection Test

Evan Chen

July 4, 2016

§1 Solution to TSTST 1, proposed by Victor Wang

This is essentially an application of the division algorithm, but the details require significant care.

First, we claim that A/B can be written as a polynomial in x whose coefficients are rational functions in y. To see this, use the division algorithm to get

$$A = Q \cdot B + R$$

where Q and R are polynomials in x whose coefficients are rational functions in y, and moreover $\deg_x B > \deg_x R$.

Now, we claim that $R = 0$. Indeed, we have by hypothesis that for infinitely many values of y_0 that $B(x, y_0)$ divides A, which means $B(x, y_0) \mid R(x, y_0)$. Now, we have $\deg_x B(x, y_0) > \deg R(x, y_0)$ outside of finitely many values of y_0 (but not all of them!); since we have infinitely many, this implies $R(x, y) \equiv 0$.

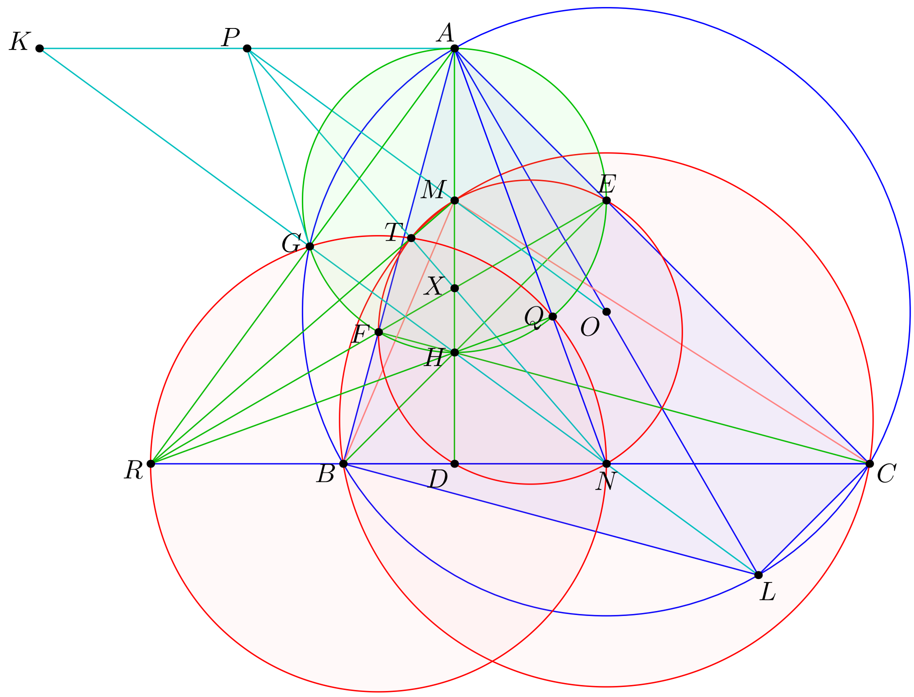
Consequently, we are able to write $A/B = F(x, y)/M(y)$ where F and M are each polynomials. Repeating the same argument now gives

$$\frac{A}{B} = \frac{F(x, y)}{M(y)} = \frac{G(x, y)}{N(x)}.$$

Now, by unique factorization of polynomials in $\mathbb{R}[x, y]$, we can discuss GCD's. So, we tacitly assume $\gcd(F, M) = \gcd(G, N) = (1)$. Also, we obviously have $\gcd(M, N) = (1)$. But $F \cdot N = G \cdot M$, so $M \mid F \cdot N$, thus we conclude M is the constant polynomial. This implies the result.

Remark. This fact does not generalize to arbitrary functions that are separately polynomial: see e.g. `http://artofproblemsolving.com/community/c6h523650p2978180`.

§2 Solution to TSTST 2, proposed by Evan Chen



First solution

Let L be diametrically opposite A on the circumcircle. Denote by $\triangle DEF$ the orthic triangle. Let $X = \overline{AH} \cap \overline{EF}$. Finally, let T be the second intersection of $(MFDNE)$ and (MBC).

We begin with a few easy observations. First, points H, G, N, L are collinear and $\angle AGL = 90^\circ$. Also, Q is the foot from H to $\overline{AN}$. Consequently, lines AG, EF, HQ, BC, TM concur at a point R (radical axis). Moreover, we already know $\angle MTN = 90^\circ$. This implies T lies on the circle with diameter $\overline{RN}$, which is exactly the circumcircle of $\triangle GQN$.

Note by Brokard's Theorem on $AFHE$, the point X is the orthocenter of $\triangle MBC$. But $\angle MTN = 90^\circ$ already, and N is the midpoint of $\overline{BC}$. Consequently, points T, X, N are collinear.

Finally, we claim P, X, N are collinear, which solves the problem. Note $P = \overline{GG} \cap \overline{AA}$. Set $K = \overline{HNL} \cap \overline{AP}$. Then by noting

$$-1 = (D, X; A, H) \stackrel{N}{=} (\infty, \overline{NX} \cap \overline{AK}; A, K)$$

we see that $\overline{NX}$ bisects segment $\overline{AK}$, as desired. (A more projective finish is to show that $\overline{PXN}$ is the polar of R to γ).

Second solution

Denote by $\triangle DEF$ the orthic triangle. Note that $\overline{AG}$, $\overline{EF}$, $\overline{BC}$ are concurrent at R by radical axis, and that $\overline{PA}$ and $\overline{PG}$ are tangents to γ.

Now, consider circles $(PAGM)$, $(MFDNE)$, and (MBC). They intersect at M but have radical center R, so are coaxial; assume they meet again at $T \in \overline{RM}$, say. Then $\angle PTM$ and $\angle MTN$ are both right angles, hence T lies on $\overline{PN}$.

Finally H is the orthocenter of $\triangle ARN$, and thus the circle with diameter $\overline{RN}$ passes through G, Q, N.

Remark. The original problem proposal was to show that the circumcircle of $\triangle MBC$ and the nine-point circle intersected at a point on line PN, where P was on line OM such that $\overline{PA} \parallel \overline{BC}$. The points G and Q were added to the picture later to prevent the problem from being immediate by coordinates.

§3 Solution to TSTST 3, proposed by Yang Liu

We claim that

$$Q(x) = 420(x^2-1)^2$$

works. Clearly, it suffices to prove the result when $n=4$ and when n is an odd prime p. The case $n=4$ is trivial, so assume now $n=p$ is an odd prime.

First, we prove the following easy claim.

Claim. For any odd prime p, there are at least $\frac12(p-3)$ values of a for which $\left(\frac{1-a^2}{p}\right)=+1$.

Proof. Note that if $k\neq 0$, $k\neq\pm1$, $k^2\neq -1$, then $a=2(k+k^{-1})^{-1}$ works. Also $a=0$ works. □

Let $F(x)=(x^2-1)^2$. The range of F modulo p is contained within the $\frac12(p+1)$ quadratic residues modulo p. On the other hand, if for some t neither of $1\pm t$ is a quadratic residue, then t^2 is omitted from the range of F as well. Call such a value of t *useful*, and let N be the number of useful residues. We aim to show $N \ge \frac14 p - 2$.

We compute a lower bound on the number N of useful t by writing

$$\begin{aligned} N &= \frac14\left(\sum_t\left[\left(1-\left(\frac{1-t}{p}\right)\right)\left(1-\left(\frac{1+t}{p}\right)\right)\right]-\left(1-\left(\frac{2}{p}\right)\right)-\left(1-\left(\frac{-2}{p}\right)\right)\right)\\ &\ge \frac14\sum_t\left[\left(1-\left(\frac{1-t}{p}\right)\right)\left(1-\left(\frac{1+t}{p}\right)\right)\right]-1\\ &= \frac14\left(p+\sum_t\left(\frac{1-t^2}{p}\right)\right)-1\\ &\ge \frac14\left(p+(+1)\cdot\tfrac12(p-3)+0\cdot 2+(-1)\cdot\left((p-2)-\tfrac12(p-3)\right)\right)-1\\ &\ge \frac14(p-5). \end{aligned}$$

Thus, the range of F has size at most

$$\frac12(p+1)-\frac12 N \le \frac38(p+3).$$

This is less than $0.499p$ for any $p\ge 11$.

Remark. In fact, the computation above is essentially an equality. There are only two points where terms are dropped: one, when $p\equiv 3 \pmod 4$ there are no $k^2=-1$ in the lemma, and secondly, the terms $1-(2/p)$ and $1-(-2/p)$ are dropped in the initial estimate for N. With suitable modifications, one can show that in fact, the range of F is exactly equal to

$$\frac12(p+1)-\frac12N=\begin{cases}\frac18(3p+5) & p\equiv 1 \pmod 8\\ \frac18(3p+7) & p\equiv 3 \pmod 8\\ \frac18(3p+9) & p\equiv 5 \pmod 8\\ \frac18(3p+3) & p\equiv 7 \pmod 8.\end{cases}$$

§4 Solution to TSTST 4, proposed by Linus Hamilton

The main observation is that the exponent of 2 decreases by at most 1 with each application of φ. This will give us the desired estimate.

Define the *weight* function w on positive integers as follows: it satisfies $w(ab) = w(a) + w(b)$ and $w(p) = w(p-1)$ for any prime p. By induction, we see that $w(n)$ counts the powers of 2 that are produced as φ is repeatedly applied to n. In particular, $k \ge w(n)$.

So, it suffices to prove that $w(p) \ge \log_3 p$ for every p. This is certainly true for $p = 2$. For any other p, we use strong induction and note that

$$w(p) = w(2) + w\left(\frac{p-1}{2}\right) \ge 1 + \log_3(p-1) - \log_3 2 \ge \log_3 p$$

for any $p > 2$. This solves the problem.

§5 Solution to TSTST 5, proposed by Linus Hamilton and David Stoner

We say a wall v is *above* another wall w if some point on v is directly above a point on w. (This relation is anti-symmetric, as walls do not intersect).

The critical claim is as follows:

Claim. There exists a lowest wall, i.e. a wall not above any other walls.

Proof. Assume not. Then we get a directed cycle of some length $n \ge 3$: it's possible to construct a series of points P_i, Q_i, for $i = 1, \dots, n$ (indices modulo n), such that the point Q_i is directly above P_{i+1} for each i, the segment $\overline{Q_i P_{i+1}}$ does not intersect any wall in its interior, and finally each segment $\overline{P_i Q_i}$ is contained inside a wall. This gives us a broken line on $2n$ vertices which is not self-intersecting.

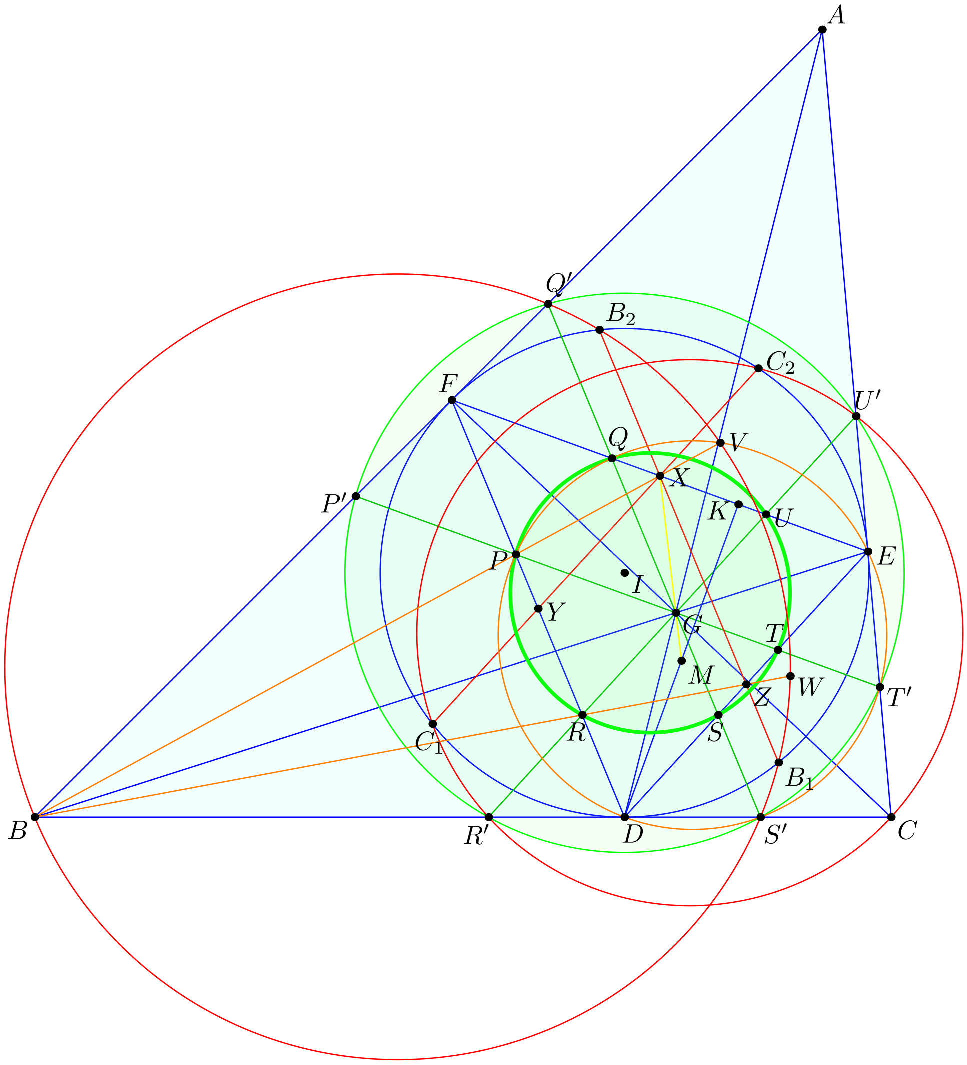
Now consider the leftmost vertical segment $\overline{Q_i P_{i+1}}$ and the rightmost vertical segment $\overline{Q_j P_{j+1}}$. The broken line gives a path from P_{i+1} to Q_j, as well as a path from P_{j+1} to Q_i. These clearly must intersect, contradiction. □

Remark. This claim is Iran TST 2010.

Thus if the bulldozer eventually moves upwards indefinitely, it may never hit the bottom side of the lowest wall. Similarly, if the bulldozer eventually moves downwards indefinitely, it may never hit the upper side of the highest wall.

§6 Solution to TSTST 6, proposed by Danielle Wang

First solution



Let X, Y, Z be midpoints of EF, FD, DE, and let G be the Gregonne point. By radical axis on $(AEIF)$, (DEF), (AIC) we see that B_1, X, B_2 are collinear. Likewise, B_1, Z, B_2 are collinear, so lines B_1B_2 and XZ coincide. Similarly, lines C_1C_2 and XY coincide. In particular lines B_1B_2 and C_1C_2 meet at X.

Note G is the symmedian point of DEF, so it is well-known that XG passes through the midpoint of DK. So we just have to prove G lies on the radical axis.

Construct parallelograms $GPFQ$, $GRDS$, $GTUE$ such that $P, R \in DF$, $S, T \in DE$, $Q, U \in EF$. As FG bisects PQ and is isogonal to FZ, we find $PQED$, hence $PQRU$, is cyclic. Repeating the same logic and noticing PR, ST, QU not concurrent, all six points $PQRSTU$ are cyclic. Moreover, since PQ bisects GF, we see that a dilation with factor 2 at G sends PQ to $P', Q' \in AB$, say, with F the midpoint of $P'Q'$. Define $R', S' \in BC$ similarly now and $T', U' \in CA$.

Note that $EQPDS'$ is in cyclic too, as $\measuredangle DS'Q = \measuredangle DRS = \measuredangle DEF$. By homothety through B, points B, P, X are collinear; assume they meet $(EQPDS')$ again at V. Thus $EVQPDS'$ is cyclic, and now

$$\measuredangle BVS' = \measuredangle PVS' = \measuredangle PQS = \measuredangle PTS = \measuredangle FED = \measuredangle XEZ = \measuredangle XVZ$$

hence V lies on $(BQ'S')$.

Since $FB \parallel QP$, we get $EVFB$ is cyclic too, so $XV \cdot XB = XE \cdot XF$ now; thus X lies on the radical axis of $(BS'Q')$ and (DEF). By the same argument with $W \in BZ$, we get Z lies on the radical axis too. Thus the radical axis of $(BS'Q')$ and (DEF) must be line XZ, which coincides with B_1B_2; so $(BB_1B_2) = (BS'Q')$.

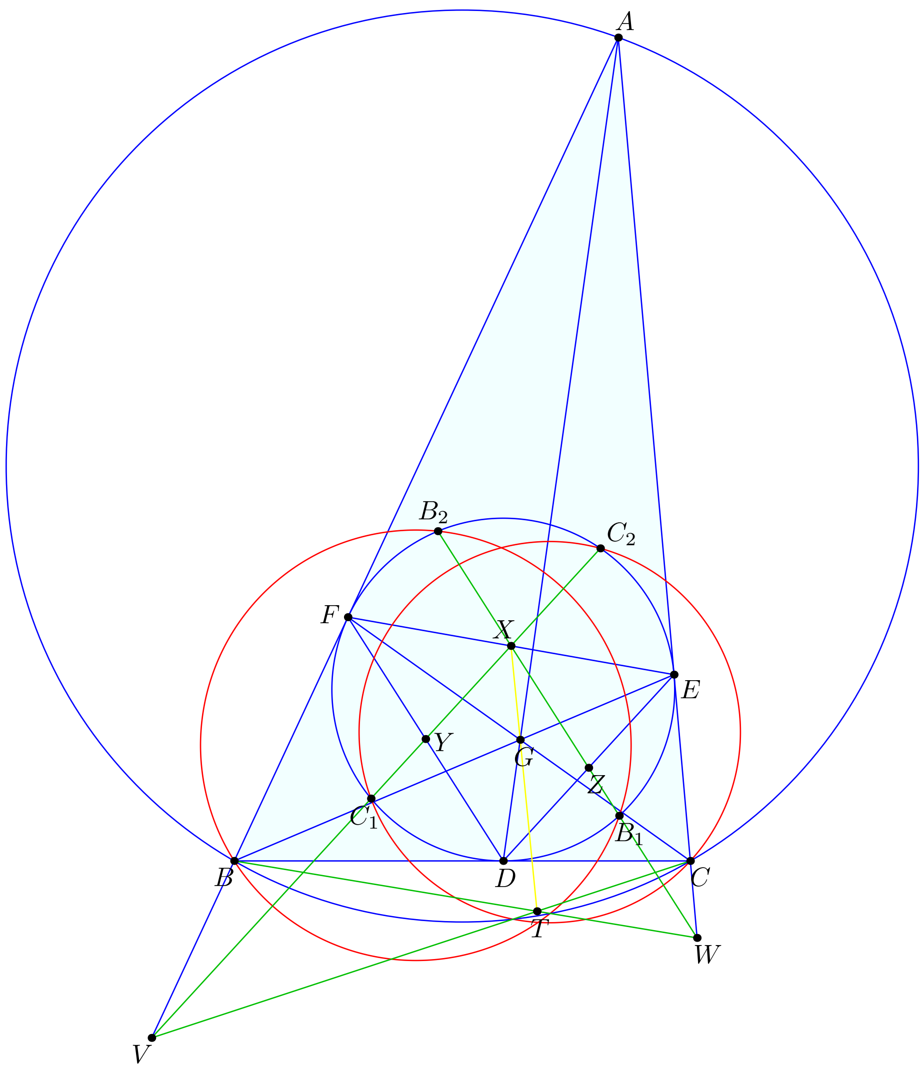
Analogously, $(CC_1C_2) = (CR'U')$. Since $G = Q'S' \cap R'U'$, we need only prove that $Q'R'S'U'$ is cyclic. But $QRSU$ is cyclic, so we are done.

Remark. The circle $(PQRSTU)$ is called the *Lemoine circle* of ABC.

Second solution

Here is another solution based on Allen Liu's paper. First, note that $\triangle DEF$ is the cevian triangle of the Gregonne point G. Set $V = \overline{XY} \cap \overline{AB}$, $W = \overline{XZ} \cap \overline{AC}$, and $T = \overline{BW} \cap \overline{CV}$.

First, we claim that X, G, T are collinear. View $\triangle XYZ$ as a cevian triangle of $\triangle DEF$. Then by Cevian Nest on $\triangle ABC$, it follows that $\overline{AX}$, $\overline{BY}$, $\overline{CZ}$ are concurrent, and hence $\triangle BYV$ and $\triangle CZW$ are perspective. So $\triangle BZW$ and $\triangle CYW$ are perspective too, and hence we deduce by Desargue theorem that T, X, and $\overline{BZ} \cap \overline{CY}$ are collinear. Finally, the Cevian Nest theorem applied on $\triangle GBC$ (which has cevian triangles $\triangle DEF$, $\triangle XYZ$) we deduce G, X, and $\overline{BZ} \cap \overline{CY}$, proving the claim.



Now, point V is the radical center (CC_1C_2), (ABC) and (DEF). To see this, let $V' = \overline{ED} \cap \overline{AB}$; then $(FV'; AB)$ is harmonic, and V is the midpoint FV', and thus $VA \cdot VB = VF^2 = VC_1 \cdot VC_2$. So in fact $\overline{CV}$ is the radical axis of (ABC) and (CC_1C_2).

Similarly, $\overline{BW}$ is the radical axis of (ABC) and (BB_1B_2) Thus T is the radical center of (ABC), (BB_1B_2), (CC_1C_2). But as in the first solution we know X, G, M are collinear, as needed.