2015 USA Team Selection Test Selection Test Day 1
Carnegie Mellon University
June 23, 2015
1:15 – 5:45pm

1. Let $a_1, a_2, \ldots, a_n$ be a sequence of real numbers, and let m be a fixed positive integer less than n. We say an index k with $1 \leq k \leq n$ is *good* if there exists some ℓ with $1 \leq \ell \leq m$ such that
$$a_k + a_{k+1} + \cdots + a_{k+\ell-1} \geq 0,$$
where the indices are taken modulo n. Let T be the set of all good indices. Prove that
$$\sum_{k \in T} a_k \geq 0.$$

2. Let ABC be a scalene triangle. Let K_a, L_a, and M_a be the respective intersections with BC of the internal angle bisector, external angle bisector, and the median from A. The circumcircle of AK_aL_a intersects AM_a a second time at a point X_a different from A. Define X_b and X_c analogously. Prove that the circumcenter of $X_aX_bX_c$ lies on the Euler line of ABC.

   (The Euler line of ABC is the line passing through the circumcenter, centroid, and orthocenter of ABC.)

3. Let P be the set of all primes, and let M be a non-empty subset of P. Suppose that for any non-empty subset $\{p_1, p_2, \ldots, p_k\}$ of M, all prime factors of $p_1p_2\cdots p_k + 1$ are also in M. Prove that $M = P$.

2015 USA Team Selection Test Selection Test Day 2
Carnegie Mellon University
June 25, 2015
1:15 – 5:45pm

4. Let x, y, and z be real numbers (not necessarily positive) such that $x^4 + y^4 + z^4 + xyz = 4$. Show that
$$x \leq 2 \quad \text{and} \quad \sqrt{2 - x} \geq \frac{y + z}{2}.$$

5. Let $\varphi(n)$ denote the number of positive integers less than n that are relatively prime to n. Prove that there exists a positive integer m for which the equation $\varphi(n) = m$ has at least 2015 solutions in n.

6. A *Nim-style game* is defined as follows. Two positive integers k and n are specified, along with a finite set S of k-tuples of integers (not necessarily positive). At the start of the game, the k-tuple $(n, 0, 0, \ldots, 0)$ is written on the blackboard.

   A legal move consists of erasing the tuple $(a_1, a_2, \ldots, a_k)$ which is written on the blackboard and replacing it with $(a_1 + b_1, a_2 + b_2, \ldots, a_k + b_k)$, where $(b_1, b_2, \ldots, b_k)$ is an element of the set S. Two players take turns making legal moves, and the first to write a negative integer loses. In the event that neither player is ever forced to write a negative integer, the game is a draw.

   Prove that there is a choice of k and S with the following property: the first player has a winning strategy if n is a power of 2, and otherwise the second player has a winning strategy.