Team Selection Test Selection Test 1
June 23, 2014
1:15 – 5:45pm

Problems:

1. Let $\leftarrow$ denote the left arrow[1] key on a standard keyboard. If one opens a text editor and types the keys "ab$\leftarrow$cd$\leftarrow\leftarrow$e$\leftarrow\leftarrow$f", the result is "faecdb". We say that a string B is *reachable* from a string A if it is possible to insert some amount of $\leftarrow$'s into A, such that typing the resulting characters produces B. So, our example shows that "faecdb" is reachable from "abcdef".

   Prove that for any two strings A and B, A is reachable from B if and only if B is reachable from A.

2. Consider a convex pentagon circumscribed about a circle. We name the lines that connect vertices of the pentagon with the opposite points of tangency with the circle *gergonnians*.

   (a) Prove that if four gergonnians are concurrent, then all five of them are concurrent.

   (b) Prove that if there is a triple of gergonnians that are concurrent, then you can find another triple of gergonnians that are concurrent.

3. Find all polynomial functions $P(x)$ with real coefficients that satisfy

$$P(x\sqrt{2}) = P(x + \sqrt{1 - x^2})$$

   for all real x with $|x| \leq 1$.

[1] Here is a short explanation of how the $\leftarrow$ key works. A computer's text editor always starts with an empty screen, and a cursor which we denote by "|". When you type a letter x, the cursor | is replaced by $x|$. So if the screen shows "m|th", and you press the "o" key, the result is "mo|th".

The $\leftarrow$ key moves the cursor one space backwards. That is, "mo|th" becomes "m|oth", and finally "|moth". If the cursor is already at the beginning of the string, the $\leftarrow$ key has no effect.

Note that the cursor is not considered to be a part of the final string. In the example above, after typing "ab$\leftarrow$cd$\leftarrow\leftarrow$e$\leftarrow\leftarrow$f", the screen displays "f|aecdb", so we take the result to be "faecdb".

Team Selection Test Selection Test 2
June 25, 2014
1:15 – 5:45pm

Problems:

4. Let $P(x)$ and $Q(x)$ be arbitrary polynomials with real coefficients, and let d be the degree of $P(x)$. Assume that $P(x)$ is not the zero polynomial. Prove that there exist polynomials $A(x)$ and $B(x)$ with real coefficients, such that:

   (i) both A and B have degree at most $d/2$, and

   (ii) at most one of A and B is the zero polynomial, and

   (iii) $\frac{A(x)+Q(x)B(x)}{P(x)}$ is a polynomial with real coefficients. That is, there is some polynomial $C(x)$ with real coefficients such that $A(x)+Q(x)B(x)=P(x)C(x)$.

5. Find the maximum number E such that the following holds: there is an edge-colored graph with 60 vertices and E edges, with each edge colored either red or blue, such that in that coloring, there are no monochromatic cycles of length 3 and no monochromatic cycles of length 5.

6. Suppose we have distinct positive integers a, b, c, d, and an odd prime p not dividing any of them, and an integer M such that if one considers the infinite sequence

$$
\begin{aligned}
&ca-db\\
&ca^2-db^2\\
&ca^3-db^3\\
&ca^4-db^4\\
&\cdots
\end{aligned}
$$

   and looks at the highest power of p that divides each of them, these powers are not all zero, and are all at most M. Prove that there exists some T (which may depend on a, b, c, d, p, M) such that whenever p divides an element of this sequence, the maximum power of p that divides that element is exactly p^T.